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Chern–Simons term at finite density

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Abstract

The Chern–Simons topological term coefficient is derived at arbitrary finite density. As it happens $\mu^2 = m^2$ is the crucial point for Chern–Simons. So when $\mu^2 < m^2$ the μ-influence disappears and we get the usual Chern–Simons term. On the other hand when $\mu^2 > m^2$ the Chern–Simons term vanishes because of the non-zero density of background fermions. In particular for the massless case the parity anomaly is absent at any finite density. This result holds in any odd dimension both in the abelian and in the nonabelian case. © 1997 Published by Elsevier Science B.V.

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A great number of papers devoted to the Chern–Simons topological term has appeared since the publication of [1]. This interest is based on the variety of significant physical effects caused by the Chern–Simons secondary characteristic class. These are, for example, gauge particles mass generation in quantum field theory, applications to condensed matter physics, such as the fractional quantum Hall effect and high-T_c superconductivity, the possibility of a free of metric tensor theory construction, and so on.

In a conventional zero density gauge theory it was shown [2–4] that the Chern–Simons term is generated in the Euler–Heisenberg effective action by quantum corrections. The main goal of this paper is to explore the dynamical generation of the parity anomalous Chern–Simons term at finite density in odd dimensional gauge theory.

In the excellent paper by Niemi [5] it was emphasized that the charge density at $\mu \neq 0$ becomes a nontopological object, i.e. it contains both a topological part and a nontopological part. The charge density at $\mu \neq 0$ (nontopological, neither parity-odd nor parity-even object) [3] in QED_3 at finite density was calculated and exploited in [6]. Here we are interested in the effect of a finite density influence on the covariant parity-odd form in the action leading to the gauge field mass generation – the Chern–Simons topological term. Deep insight into this phenomenon at small densities was obtained in [5,7]. The result for the Chern–Simons term coefficient in QED_3 is $\left[\,\mathrm{sgn}(m-\mu) + \mathrm{sgn}(m+\mu)\right]$ (see [7], formulas (10.19)). However, to get this result

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it was heuristically assumed that at small densities the index theorem could still be used and only an odd in energy part of the spectral density is responsible for the parity nonconserving effect. Because of this in [7] it had been stressed that the result holds only for small μ. However, as we shall see below, this result holds for any value of the chemical potential. Thus, to obtain a reliable result at any value of μ one has to use a transparent procedure which is free of any restriction on μ and which would allow to perform the calculations with arbitrary nonabelian background gauge fields. That is why we will use here the perturbative technique for the Chern–Simons term calculation. In addition this approach is completely covariant.

Since the chemical potential term $\mu\bar{\psi}\gamma^0\psi$ is odd under charge conjugation we can expect that it would contribute to a P and CP nonconserving quantity - the Chern–Simons term. As we will see, this expectation is completely justified.

The zero density approach usually is a good quantum field approximation when the value of the chemical potential is small as compared with the characteristic energy scale of physical processes. Nevertheless, for an investigation of topological effects this is not the case. As we will see below, even the presence of a small density could lead to principal effects.

Introduction of a chemical potential μ in the theory corresponds to the presence of a nonvanishing background fermion density. It must be emphasized that the formal addition of a chemical potential looks like a simple gauge transformation with gauge function μt. However, it does not only shift the time component of the vector potential but also gives the corresponding prescription for handling Green's function poles. The correct introduction of a chemical potential redefines the ground state (Fermi energy), which leads to a new spinor propagator with the proper ϵ-prescription for poles. So, for the free spinor propagator we have (see, for example, [8,9])

$$S(p;\mu) = \frac{\tilde{\not p} + m}{(\tilde{p}_0 + i\epsilon \,\mathrm{sgn}\, p_0)^2 - \boldsymbol{p}^2 - m^2}, \tag{1}$$

where $\tilde{p} = (p_0 + \mu, \boldsymbol{p})$. It is easy to see that in the $\mu = 0$ case one at once gets the usual ϵ-prescription because of the positivity of $p_0 \,\mathrm{sgn}\, p_0$.

Let us first consider nonabelian 3-dimensional gauge theory. The only graphs whose P-odd parts contribute to the parity anomalous Chern–Simons term are shown in Fig. 1.

So, the part of the effective action containing the Chern–Simons term is

$$I_{\rm eff}^{\rm CS} = \frac{1}{2}\int\limits_x A_\mu(x) \int\limits_p \mathrm{e}^{-ixp} A_\nu(p)\Pi^{\mu\nu}(p) + \frac{1}{3}\int\limits_x A_\mu(x)\int\limits_{p,r} \mathrm{e}^{-ix(p+r)}A_\nu(p)A_\alpha(r)\Pi^{\mu\nu\alpha}(p,r), \tag{2}$$

where the polarization operator and the vertices have the standard form

$$\Pi^{\mu\nu}(p) = g^2 \int\limits_k \mathrm{tr}\left[\gamma^\mu S(p+k;\mu)\gamma^\nu S(k;\mu)\right],$$

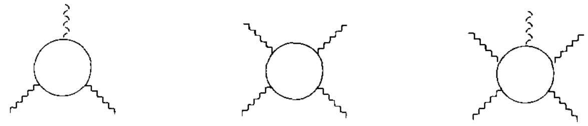
$$\Pi^{\mu\nu\alpha}(p,r) = g^3 \int\limits_k \mathrm{tr}\left[\gamma^\mu S(p+r+k;\mu)\gamma^\nu S(r+k;\mu)\gamma^\alpha S(k;\mu)\right]. \tag{3}$$

First consider the second order term (Fig. 1, graph (a)). It is well-known that the only object giving us the possibility to construct a P and T odd form in the action is the Levi-Civita tensor[4]. Thus, we will drop all terms not containing a Levi-Civita tensor. The signal for the mass generation (Chern–Simons term) is $\Pi^{\mu\nu}(p^2 = 0) \neq 0$. So we get

[4] In three dimensions it arises as a trace of three γ-matrices (Pauli matrices)

(a) (b)

Fig. 1. Graphs whose P-odd parts contribute to the Chern–Simons term in nonabelian 3D gauge theory.



(a) (b) (c)

Fig. 2. Graphs whose P-odd parts contribute to the Chern–Simons term in nonabelian 5D theory.

$$\Pi^{\mu\nu} = g^2 \int_k (-i2me^{\mu\nu\alpha}p_\alpha)\frac{1}{(\tilde{k}^2 - m^2 + i\epsilon(k_0+\mu)\ \mathrm{sgn}(k_0))^2}. \tag{4}$$

After some simple algebra one obtains

$$\Pi^{\mu\nu} = -i2mg^2 e^{\mu\nu\alpha}p_\alpha\Bigg[\int\frac{d^3k}{(2\pi)^3}\frac{1}{(\tilde{k}^2 - m^2 + i\epsilon)^2} - \int\frac{dk_0}{2\pi}\theta\left(-(k_0+\mu)\ \mathrm{sgn}(k_0)\right)\int\frac{d^2k}{(2\pi)^2}\left(\frac{1}{(\tilde{k}^2-m^2+i\epsilon)^2} - \frac{1}{(\tilde{k}^2-m^2-i\epsilon)^2}\right)\Bigg]. \tag{5}$$

Using the well-known identity

$$\Im m\frac{1}{a\pm i\varepsilon} = \mp i\pi\delta(a)$$

and evaluating the integrals one immediately comes to

$$\Pi^{\mu\nu} = -i\frac{m}{|m|}\frac{g^2}{4\pi}e^{\mu\nu\alpha}p_\alpha\theta(m^2-\mu^2). \tag{6}$$

In the same manner handling the third order contribution (Fig. 1b) one gets

$$\begin{aligned}\Pi^{\mu\nu\alpha} &= -2g^3 ie^{\mu\nu\alpha}\int\frac{d^3k}{(2\pi)^3}\left[\frac{m(k^2-m^2)}{(k^2-m^2+i\varepsilon\ \mathrm{sgn}(k_0)(k_0+\mu))^3}\right]\\ &= -i2mg^3e^{\mu\nu\alpha}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{(k^2-m^2+i\varepsilon\ \mathrm{sgn}(k_0)(k_0+\mu))^2}\right]\end{aligned} \tag{7}$$

and further all calculations are identical to those the second order

$$\Pi^{\mu\nu\alpha} = -i\frac{m}{|m|}\frac{g^3}{4\pi}e^{\mu\nu\alpha}\theta(m^2-\mu^2). \tag{8}$$

Substituting (6), (8) in the effective action (2) we get eventually

$$I_{\mathrm{eff}}^{\mathrm{CS}} = \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^2}{8\pi}\int d^3x e^{\mu\nu\alpha}\ \mathrm{tr}\left(A_\mu\partial_\nu A_\alpha - \tfrac{2}{3}gA_\mu A_\nu A_\alpha\right). \tag{9}$$

Thus, we get the Chern–Simons term with a μ dependent coefficient.

Let us now consider 5-dimensional gauge theory. Here the Levi-Civita tensor is the 5-dimensional $e^{\mu\nu\alpha\beta\gamma}$ and the relevant graphs are shown in Fig. 2.

The part of the effective action containing the Chern–Simons term reads

$$I_{\text{eff}}^{\text{CS}} = \frac{1}{3}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r)}A_\nu(p)A_\alpha(r)\Pi^{\mu\nu\alpha}(p,r)$$
$$+\frac{1}{4}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r+s)}A_\nu(p)A_\alpha(r)A_\beta(s)\Pi^{\mu\nu\alpha\beta}(p,r,s)$$
$$+\frac{1}{5}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r+s+q)}A_\nu(p)A_\alpha(r)A_\beta(s)A_\gamma(s)\Pi^{\mu\nu\alpha\beta\gamma}(p,r,s,q). \tag{10}$$

All calculations are similar to the 3-dimensional case. First consider the third order contribution (Fig. 2a)

$$\Pi^{\mu\nu\alpha}(p,r) = g^3\int_k \operatorname{tr}\left[\gamma^\mu S(p+r+k;\mu)\gamma^\nu S(r+k;\mu)\gamma^\alpha S(k;\mu)\right]. \tag{11}$$

Taking into account that the trace of the five γ-matrices in 5-dimensions is

$$\operatorname{tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\rho\right] = 4ie^{\mu\nu\alpha\beta\rho},$$

we extract the parity-odd part of the vertices

$$\Pi^{\mu\nu\alpha} = g^3\int\frac{d^5k}{(2\pi)^5}(i4me^{\mu\nu\alpha\beta\sigma}p_\beta r_\sigma)\frac{1}{(\tilde{k}^2-m^2+i\epsilon(k_0+\mu)\ \operatorname{sgn}(k_0))^3}, \tag{12}$$

or, in a more transparent way,

$$\Pi^{\mu\nu\alpha} = i4mg^3e^{\mu\nu\alpha\beta\sigma}p_\alpha r_\sigma\left[\int\frac{d^5k}{(2\pi)^5}\frac{1}{(\tilde{k}^2-m^2+i\epsilon)^3}\right.$$
$$\left.-\int\frac{dk_0}{2\pi}\theta(-(k_0+\mu)\ \operatorname{sgn}(k_0))\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{(\tilde{k}^2-m^2+i\epsilon)^3}-\frac{1}{(\tilde{k}^2-m^2-i\epsilon)^3}\right)\right]. \tag{13}$$

Evaluating these integrals one comes to

$$\Pi^{\mu\nu\alpha} = i\frac{m}{|m|}\frac{g^3}{16\pi^2}e^{\mu\nu\alpha\beta\sigma}p_\alpha r_\sigma\theta(m^2-\mu^2). \tag{14}$$

In the same way operating graphs (b) and (c) in Fig. 2 one will obtain

$$\Pi^{\mu\nu\alpha\beta} = i\frac{m}{|m|}\frac{g^4}{8\pi^2}e^{\mu\nu\alpha\beta\sigma}s_\sigma\theta(m^2-\mu^2) \tag{15}$$

and

$$\Pi^{\mu\nu\alpha\beta\gamma} = i\frac{m}{|m|}\frac{g^5}{16\pi^2}e^{\mu\nu\alpha\beta\sigma}\theta(m^2-\mu^2). \tag{16}$$

Substituting (14)–(16) into the effective action (10) we get the final result for Chern–Simons in 5-dimensional theory:

$$I_{\text{eff}}^{\text{CS}} = \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^3}{48\pi^2}\int d^5x e^{\mu\nu\alpha\beta\gamma}$$
$$\times\ \operatorname{tr}\left(A_\mu\partial_\nu A_\alpha\partial_\beta A_\gamma+\tfrac{3}{2}gA_\mu A_\nu A_\alpha\partial_\beta A_\gamma+\tfrac{3}{5}g^2A_\mu A_\nu A_\alpha A_\beta A_\gamma\right). \tag{17}$$

It is remarkable that all parity-odd contributions are finite both in the 3-dimensional case as well as in the 5-dimensional case. Thus, all values in the effective action are renormalized in a standard way, i.e. the renormalizations are determined by the conventional (parity-even) parts of the vertices.

From the above direct calculations it is clearly seen that the chemical potential dependent coefficient is the same for all parity-odd parts of the diagrams and does not depend on the space dimension. So, the influence of a finite density on the generation of the Chern–Simons term is the same in any odd dimension:

$$I_{\rm eff}^{\rm C.S} = \frac{m}{|m|}\theta(m^2 - \mu^2)\pi W[A], \tag{18}$$

where $W[A]$ is the Chern–Simons secondary characteristic class in any odd dimension. Since only the lowest orders of the perturbative series contribute to the CS term at finite density (the same situation is well-known at zero density), the result obtained by using formally a perturbative technique appears to be nonperturbative. Thus, the μ-dependent CS term coefficient reveals the amazing property of universality. Namely, it does not depend on either the dimension of the theory or on the abelian or nonabelian gauge theory studied.

The arbitrariness of μ gives us the possibility to see Chern–Simons coefficient behavior at any mass. It is very interesting that $\mu^2 = m^2$ is the crucial point for Chern–Simons. Indeed, it is clearly seen from (18) that when $\mu^2 < m^2$ the μ-influence disappears and we get the usual Chern–Simons term $I_{\rm eff}^{\rm CS} = \pi W[A]$. On the other hand, when $\mu^2 > m^2$ the situation is absolutely different. One can see that here the Chern–Simons term disappears because of the non-zero density of background fermions. We would like to emphasize the important massless case $m = 0$ considered in [7]. Then even a negligible density, which always take place in any physical process, leads to vanishing of the parity anomaly.

In conclusion, let us stress again that we nowhere have used any restrictions on μ. Thus we not only confirm the result in [7] for Chern–Simons in QED_3 at small density, but also expand it to the arbitrary μ, nonabelian case with arbitrary odd dimension.

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