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# Electromagnetic duality for anyons

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**Abstract**

Electromagnetic duality was established for anyons: the 3D oscillator with coordinates confined by the 2D half-up cone for an angle of $\pi/6$ is dual to the 2D charge-dyon bound system obeying fractional statistics. © 1997 Published by Elsevier Science B.V.

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In modern field theory a central role is attributed to the idea of electromagnetic duality [1], according to which a strongly coupled theory can be formulated in terms of weakly coupled magnetic monopoles.

Electromagnetic duality has a beautiful analog in non-relativistic quantum mechanics [2–4]: oscillators in $R^8$, $R^4$ and $R^2$ can be transformed into the charge-dyon bound systems in $R^5$, $R^3$ and $R^2$, respectively. These magic dimensions follow from the Hurwitz theorem [5]. In particular, the reduction of the circular oscillator by the $Z_2$ group action results in two systems, a two-dimensional hydrogen atom and a two-dimensional spin-1/2 charge-dyon bound system [4].

In two space dimensions, however, we not only have bosons and fermions but also anyons, i.e. objects with fractional statistics [6]. These possibilities arise from the peculiar topological properties of the configuration space of the collection of two-dimensional identical particles [7]. Hence, the question arises as to whether it is possible to construct an oscillator model which can be transformed into a charge-dyon bound system with fractional statistics, i.e. a system continuously interpolating between the bosonic and fermionic systems considered in Ref. [4].

The answer we have found in this paper is positive. We show that the role of the desired oscillator model is played by the $C_2^+$ oscillator, i.e. the 3D oscillator with coordinates confined by the 2D half-up cone $C_2^+$ for an angle of $\pi/6$. [3]

Thus, we have an important extension of electromagnetic duality for anyons. At the same time, our result opens up a wide range of applications for oscillator physics because anyons are important for massive gauge fields and soluble quantum gravity [10], the fractional quantum Hall effect [11] and high-$T_c$ superconductivity [12].

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[3] The analogy between anyon ($\approx$ Aharonov–Bohm) problems and free motion on a cone ($\approx 2+1$-dimensional gravity) has been previously noted in Refs. [8,9].


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Let $u_\mu$ be the Cartesian coordinates in $R^3$ and $C_2^+ \subset R^3$. By definition, $u_\mu \in C_2^+$ if $3(u_1^2+u_2^2)-u_3^2=0$, and $u_3>0$. It is convenient to describe the cone $C_2^+$ by the parametrization $u_1+\mathrm{i}u_2 = u\,\mathrm{e}^{2\mathrm{i}\phi}/2$, $u_3=\sqrt{3}u/2$, where $\phi\in[0,\pi]$. It should be emphasized that $u\in(0,\infty)$, i.e. the tip of the cone is not included. As a consequence, the space $C_2^+$ is infinitely connected, i.e. loops with different winding numbers are homotopically inequivalent. After each winding, the wavefunction $\Psi$ acquires the phase $\mathrm{e}^{\mathrm{i}\pi\nu}$ with $\nu\in[0,1]$. Thus, the function $\Psi^\nu=\mathrm{e}^{-\mathrm{i}\pi\phi}\Psi$ satisfies the usual condition $\Psi(u,\phi+\pi)=\mathrm{e}^{\mathrm{i}\nu\pi}\Psi(u,\phi)$. Since $\mathrm{d}l^2 = \mathrm{d}u_1^2+\mathrm{d}u_2^2+\mathrm{d}u_3^2=\mathrm{d}u^2+u^2\mathrm{d}\phi^2$, we have

$$\frac{\partial^2\Psi^\nu}{\partial u^2}+\frac{1}{u}\frac{\partial\Psi^\nu}{\partial u}+\frac{1}{u^2}\left(\frac{\partial}{\partial\phi}+\mathrm{i}\nu\right)^2\Psi^\nu + \frac{2M}{\hbar^2}\left(E-\frac{M\omega^2u^2}{2}\right)\Psi^\nu=0. \quad (1)$$

The change of variables $r=u^2$, $\varphi=2\phi$ transforms the $C_2^+$ oscillator into the system described by the equation

$$\frac{1}{2M}\left(-\mathrm{i}\hbar\frac{\partial}{\partial x_\mu}+\frac{e}{c}A_\mu\right)^2\Psi^s-\frac{\alpha}{r}\Psi^s=\epsilon\Psi^s. \quad (2)$$

Here $x_1+\mathrm{i}x_2=r\,\mathrm{e}^{\mathrm{i}\varphi}$, $A_j=-(g/2\pi)\epsilon_{jk}x_k/r^2$, $g=hcs/e$, $h=\hbar/2\pi$, $\Psi^s(r,\varphi)=\Psi^\nu(u,\phi)$ and

$$\epsilon=-\frac{M\omega^2}{8},\quad \alpha=\frac{E}{4},\quad s=\nu/2\in[0,1/2]. \quad (3)$$

Since $B=\nabla\wedge A=g\delta^{(2)}(\boldsymbol{x})$, Eq. (2) describes the motion of a particle of mass $M$ and electric charge $e=-\sqrt{E}/2$ in the fields of the dyon of electric charge $Q=+\sqrt{E}/2$ and magnetic charge $g$, localized at the origin. It follows from (2) that $s=\nu/2$ has the meaning of the spin of the dyon. Since $s\in[0,1/2]$, the $C_2^+$ oscillator is dual to the fractional charge-dyon bound system which continuously interpolates between the bosonic ($s=0$) and fermionic ($s=1/2$) systems considered in Ref. [6]. Eq. (2) leads to the bounded solution

$$\Psi^s(r,\varphi)=C\,\mathrm{e}^{-z/2}z^{|j|} \times F(-n_r,2|j|+1,z)\mathrm{e}^{\mathrm{i}m\varphi}/\sqrt{2\pi}$$

with $z=2\lambda r/r_0$, $r_0=\hbar^2/\mu\alpha$, $\lambda=(-2\epsilon\hbar^2/\mu\alpha^2)^{1/2}$, $n_r=0,1,2,\ldots,m\in Z, j=m+s$, and the eigenvalues

$$\epsilon=-\frac{M\alpha^2}{2\hbar^2(n_r+|m+s|+\frac{1}{2})^2}. \quad (4)$$

It follows from (4) that for $s=0$ and $s=1/2$, $\epsilon$ describes the $(2n_r+2|m|+1)$- and $(2n_r+2|m|+1)$-fold energy levels of the two-dimensional hydrogen atom and spin-1/2 charge-dyon system, respectively. For $s\in(0,1/2)$, the energy levels (4) are decomposed into sublevels with $(n_r+|m|+1)$ and $(n_r+|m|)$ degrees of degeneracy [4]. It follows from $r=u^2$ and the accepted parametrization that $C_2^+\Rightarrow R^2(\boldsymbol{x})$ can be interpreted as the mapping of the half-up cone $C_2^+$ on the paraboloid $\mathcal{R}(r=u^2)$ with the following projection $\mathcal{R}\Rightarrow R^2(\boldsymbol{x})$. This composite transformation can be written as

$$x_1+\mathrm{i}x_2=4(u_1^2+u_2^2)^{1/2}(u_1+\mathrm{i}u_2). \quad (5)$$

Eq. (5) together with the ansatz (3) forms the duality transformation mapping the $C_2^+$ oscillator onto the fractional charge-dyon system. Let us stress what we mean by the term duality. Consider the system $S$ with energy $E$ and coupling constant $\omega$. The system $S^{\mathrm{d}}$ is called dual to $S$ if $S^{\mathrm{d}}$ is governed by the same dynamical equation as $S$ but with fixed $E$ and quantized $\omega$. The transformation $T^{\mathrm{d}}$ which maps $S$ onto the system $S^{\mathrm{d}}$ with energy $\epsilon$ and coupling constant $\alpha$, such that $\epsilon=\epsilon(\omega)$ and $\alpha=\alpha(E)$, is called the transformation of duality. In our case, $S$, $S^{\mathrm{d}}$ and $T^{\mathrm{d}}$ mean the $C_2^+$ oscillator, fractional charge-dyon bound system and paraboloidal projection, respectively. The electromagnetic duality for anyons is conserved even after the arbitrary scalar potential $V(u^2)$ is excluded. The dual partner of the $C_2^+$ oscillator modified in this way is the fractional charge-dyon system placed in the potential $V(r)/4r$.

Finally we explain why we considered the half-up cone for an angle of $\theta=\pi/6$ only, but not for arbitrary $\theta$. The answer is that only for $\theta=\pi/6$ the paraboloidal projection transforms $C_2^+$ into the Euclidean plane. For other values of $\theta$, after such a projection, one obtains a plane with non-Euclidean metric $\mathrm{d}l^2=\mathrm{d}r^2+4r^2\sin^2\theta\mathrm{d}\varphi^2$. It is to be emphasized that for $\theta=\pi/6$ $C_2^+$ has the topology of the manifold describing the relative coordinate of two identical parti-

[4] The dynamical symmetry of the magnetic vortex was established by Jackiw in his beautiful article [13].

cles placed in the Euclidean plane [7], and therefore, the $C_2^+$ oscillator is also a prototype of the anyon.

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