J. Phys. A: Math. Gen. **30** (1997) 1629–1657. Printed in the UK PII: S0305-4470(97)69891-1

Quantum motion on the three-dimensional sphere: the ellipso-cylindrical bases

C Grosche†, Kh H Karayan‡, G S Pogosyan§ and A N Sissakian§

† II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany

‡ Yerevan State University, Alec Manukyan Street 1, 375049 Yerevan, Armenia

§ Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research (Dubna), 141980 Dubna, Moscow Region, Russia

Received 27 November 1995, in final form 30 September 1996

Abstract. We study the free quantum motion on the three-dimensional sphere $S^{(3)}$ in ellipso-cylindrical coordinates, where we distinguish between prolate elliptic and oblate elliptic coordinates. The oblate and prolate elliptic wavefunctions are constructed in terms of the hyper-spherical and cylindrical wavefunctions. A perturbation theory for the elliptic wavefunctions in powers of the elliptic parameter is established. Finally, the interbasis expansions are used to set up path integral identities for the two ellipso-cylindrical coordinate systems on $S^{(3)}$.

1. Introduction

The investigation of separation of variables in the Helmholtz (Schrödinger) equation has a long history, and the case of spaces of non-vanishing constant curvature has been thoroughly discussed by Olevskiĭ [1]. This includes the two- and three-dimensional sphere with two and six orthogonal coordinate systems, and the two- and three-dimensional hyperboloid with nine and 34 orthogonal coordinate systems, respectively. The corresponding cases for the flat spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ with four and eleven orthogonal coordinate systems can be found in [2]. However, many of these coordinate systems are not well known, and the usual studies of physical systems, potential problems, the incorporation of magnetic fields, monopole problems, perturbation theory, or scattering theory, are studied in the more familiar systems, e.g., in Cartesian, spherical or parabolic coordinates, depending which coordinate system is best suited to match the symmetry properties of the problem under consideration.

The choice of a coordinate system emphasizes which observables are considered to be the most appropriate for a particular investigation. Actually, the number of coordinate systems which can be found in a homogeneous space equals the number of sets of functionally independent observables in this space. For instance, in a problem with spherical symmetry, the total angular momentum operator $\boldsymbol{L}^2$, and the L_z operator are sufficient for a comprehensive understanding of classical and quantum motion in $\mathbb{R}^3$ with spherical symmetry and a preferred axis.

The incorporation of interaction, i.e. potentials or magnetic fields, generally reduces the number of coordinate systems in which the Schrödinger equation is separable. If it turns out that the problem is separable in more that one coordinate system, it follows that there are additional observables exceeding the numbers of degrees of freedom. If more than one

0305-4470/97/051629+29$19.50

such coordinate system exist, there will be different representations of the solutions of the Schrödinger equation, and to each of them corresponds a set of commuting observables, which may be simultaneously assigned definite eigenvalues. These systems are called super-integrable, cf [3–6]. The most important super-integrable systems are the oscillator and the Kepler–Coulomb problem, which are said to be maximally super-integrable, and in three dimensions have five functionally independent observables. A comprehensive discussion of super-integrable systems in two and three dimensions in spaces of constant curvature can be found in [7] for the Euclidean space, in [8] for the two- and three-dimensional sphere, and in [9, 10] for the two- and three-dimensional hyperboloid.

However, the investigation of separation of variables goes beyond the search of super-integrable systems and the corresponding observables. Perturbing a super-integrable system often leads to a problem which may be separable only in one coordinate system, and it may even not be exactly solvable. Then, the unperturbed system may serve as a starting point for a perturbative investigation, and the energy-levels of a perturbed system may be classified according to the dynamical symmetry group of the unperturbed system. Because the algebra of the angular momentum operator $\boldsymbol{L}$ and the Pauli–Lenz–Runge operator $\boldsymbol{K}$ (suitably rescaled) closes for constant energy, the dynamical group O(4), where the corresponding homogeneous space is $S^{(3)}$, describes the discrete spectrum, and the Lorentz group O(3, 1), where the corresponding homogeneous space is the three-dimensional hyperboloid $\Lambda^{(3)}$, the continuous spectrum [11, 17]. For instance, the ellipso-cylindrical system is of interest because it enables one to set up by means of the elliptic observable Λ, cf equation (2.20), a complete classification of the energy levels of the diamagnetic Kepler problem, i.e. the quadratic Zeeman effect (cf Herrick [13], Lakshmann and Hasegawa [12], Solov'ev [14], Brown and Solov'ev [15], and Gutzwiller [16]). We will make some further remarks concerning the quadratic Zeeman effect in section 6.

Furthermore, the ellipso-cylindrical bases as investigated in this paper are very useful for physicists, because we can on the one hand study the global properties of the bases, and not only the separating equation with the corresponding wavefunction as expansions over sn, cn or dn as in [17]. On the other hand, the ellipso-cylindrical bases in the form of an expansion over the spherical or cylindrical basis is useful for the calculation of matrix elements for the Kepler problem in external (magnetic) fields with the approximate third observable, as mentioned above.

The ellipso-cylindrical coordinate systems are also useful for the investigation of the two-centre Kepler problem on the three-dimensional sphere, because this problem separates only in these coordinates [18, 19]; our investigations are also eventually useful in the theory of special functions.

Furthermore, the spherical, the sphero-conical and a rotated version of the prolate elliptic coordinate system [8] separate the Kepler–Coulomb problem on $S^{(3)}$, and the ellipsoidal system separates the Neumann model [20, 21].

A classification of the coordinate systems on $S^{(3)}$, including an enumeration of the corresponding observables has been already performed in [17, 22]. The simpler case of the two-dimensional sphere is due to [23, 24]. Whereas on the two-dimensional sphere the solution of the Schrödinger equation in spherical coordinates in terms of the spherical functions $Y_l^m(\vartheta, \varphi)$, and in elliptic coordinates in terms of Lamé polynomials [23, 24] is sufficiently understood, including the corresponding interbasis expansions, not very much is known in the three-dimensional case.

In order to solve the Schrödinger equation in different coordinate systems there are two approaches. The first consists of the separation of the Schrödinger equation and a direct solution of the emerging differential equations in terms of powers of the variables, say.

In [17] a first attempt was made to characterize and classify the free quantum motion on $S^{(3)}$, and the authors determined the ellipso-cylindrical wavefunctions by making an ansatz in terms of powers of Jacobi elliptic functions, and called the emerging polynomial solutions *associated Lamé polynomials*.

The second is based on knowledge of the complete set of observables. The first investigation of this kind was done by Coulson and Joseph [25] for the basis of the hydrogen atom in spheroidal coordinates by constructing an expansion in terms of the spherical basis. This method was also used by Patera and Winternitz [24] for the sphero-conical wavefunctions on $S^{(2)}$. Mardoyan *et al* [26] used it for constructing the spheroidal basis in terms of the parabolic basis for the Coulomb–Kepler problem, and Kibler *et al* [27] analysed a generalized Coulomb–Kepler problem by this method.

The second approach is superior to the first because not only does it allow one to calculate the wavefunctions, at least recursively, but it also determines at the same time the coefficients of the interbasis expansion with respect to another basis. After obtaining recurrence relations for the interbasis coefficients, they can serve as a starting point for an algebraic perturbation description [28] of the construction of the wavefunctions.

The paper is organized as follows. In section 2 we discuss the two ellipso-cylindrical coordinate systems on $S^{(3)}$, and define the relevant observables. In section 3, we construct the ellipso-cylindrical bases in terms of the hyper-spherical and cylindrical bases. In section 4 we consider a perturbation theory for the ellipso-cylindrical bases. The cases of the elliptic parameter $0 < |a| \ll 1$ and $a \gg 1$ are discussed separately. In section 5 the interbasis expansions are used to establish path integral identities in the ellipso-cylindrical coordinate systems. A first attempt to such a formulation was tried in [29], however only on a heuristic level. In section 6 we give a summary and discussion of our results. Some results concerning the matrix elements of operators with respect to the cylindrical and hyper-spherical bases are compiled in two appendices.

2. The Schrödinger equation and the integrals of motion

The Schrödinger (Helmholtz) equation for the free motion on the three-dimensional unit sphere $S^{(3)}$

$$s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1 \tag{2.1}$$

embedded in four-dimensional Euclidean space has the form

$$-\frac{\hbar^2}{2M}\Delta\Psi = \frac{\hbar^2}{2M}J(J+2)\Psi \qquad J = 0, 1, 2, \ldots\ . \tag{2.2}$$

Δ is the Laplace operator which is given by

$$\Delta = -\frac{1}{\hbar^2}(\boldsymbol{L}^2 + \boldsymbol{M}^2). \tag{2.3}$$

L_i and M_i, $i = 1, 2, 3$, are the six generators of the group O(4) which have the following form:

$$L_i = \frac{\hbar}{\mathrm{i}}\epsilon_{ijk}s_i\frac{\partial}{\partial s_k} \qquad M_i = \frac{\hbar}{\mathrm{i}}\left(s_i\frac{\partial}{\partial s_4} - s_4\frac{\partial}{\partial s_i}\right) \qquad i = 1, 2, 3. \tag{2.4}$$

They satisfy the commutation relations

$$[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_j \qquad [M_i, M_j] = \mathrm{i}\hbar\epsilon_{ijk}L_j \qquad [L_i, M_j] = \mathrm{i}\hbar\epsilon_{ijk}M_j\ . \tag{2.5}$$

Equation (2.2) is separable in six coordinate systems which we list in table 1, together with their corresponding characteristic observables.

Table 1. Coordinate systems on the three-dimensional sphere.

Coordinate system	Coordinates	Observables I_1, I_2
I. Cylindrical	$s_1 = \cos\gamma\cos\varphi_1$	$I_1 = M_3^2$
$\gamma \in [0, \pi/2]$	$s_2 = \cos\gamma\sin\varphi_1$	$I_2 = L_3^2$
$\varphi_{1,2} \in [0, 2\pi)$	$s_3 = \sin\gamma\sin\varphi_2$	
	$s_4 = \sin\gamma\cos\varphi_2$	
II. Sphero-Elliptic	$s_1 = \sin\chi\,\mathrm{sn}(\alpha, k)\,\mathrm{dn}(\beta, k')$	$I_1 = \boldsymbol{L}^2$
$\chi \in [0, \pi]$	$s_2 = \sin\chi\,\mathrm{cn}(\alpha, k)\,\mathrm{cn}(\beta, k')$	$I_2 = L_1^2 + k'^2 L_2^2$
$\alpha \in [-K, K]$	$s_3 = \sin\chi\,\mathrm{dn}(\alpha, k)\,\mathrm{sn}(\beta, k')$	
$\beta \in [-2K', 2K']$	$s_4 = \cos\chi$	
III. Spherical	$s_1 = \sin\chi\sin\vartheta\cos\varphi$	$I_1 = \boldsymbol{L}^2$
$\chi \in [0, \pi]$	$s_2 = \sin\chi\sin\vartheta\sin\varphi$	$I_2 = L_3^2$
$\vartheta \in [0, \pi]$	$s_3 = \sin\chi\cos\vartheta$	
$\varphi \in [0, 2\pi)$	$s_4 = \cos\chi$	
IV. Oblate elliptic	$s_1 = \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')\cos\varphi$	$I_1 = (1-k^2)\boldsymbol{L}^2 - k^2 M_3^2$
$\mu \in [0, 2K]$	$s_2 = \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')\sin\varphi$	$I_2 = L_3^2$
$\nu \in [-K', K']$	$s_3 = \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')$	
$\varphi \in [0, 2\pi)$	$s_4 = \mathrm{dn}(\mu, k)\,\mathrm{sn}(\nu, k')$	
V. Prolate elliptic	$s_1 = \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\cos\varphi$	$I_1 = \boldsymbol{L}^2 + k^2 M_3^2$
$\mu \in [-K, K]$	$s_2 = \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\sin\varphi$	$I_2 = L_3^2$
$\nu \in [-K', K']$	$s_3 = \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')$	
$\varphi \in [0, 2\pi)$	$s_4 = \mathrm{dn}(\mu, k)\,\mathrm{sn}(\nu, k')$	
VI. Ellipsoidal	$s_1^2 = \frac{(\varrho_1 - a_1)(\varrho_2 - a_1)(\varrho_3 - a_1)}{(a_4 - a_1)(b_3 - a_1)(a_2 - a_1)}$	$I_1 = (a_1 + a_4)L_1^2 + (a_2 + a_4)L_2^2 + (a_3 + a_4)L_3^2$
$a_1 < \varrho_1 < a_2 < \varrho_3$	$s_2^2 = \frac{(\varrho_1 - a_2)(\varrho_2 - a_2)(\varrho_3 - a_2)}{(a_4 - a_2)(a_3 - a_2)(a_1 - a_2)}$	$+(a_2 + a_3)M_1^2 + (a_1 + a_3)M_2^2 + (a_1 + a_2)M_3^2$
$< a_3 < \varrho_4 < a_4$	$s_3^2 = \frac{(\varrho_1 - a_3)(\varrho_2 - a_3)(\varrho_3 - a_3)}{(a_1 - a_3)(a_2 - a_3)(a_4 - a_3)}$	$I_2 = a_1 a_4 L_1^2 + a_2 a_4 L_2^2 + a_3 a_4 L_3^2$
$a_{1,2,3,4} \in \mathbb{R}$	$s_4^2 = \frac{(\varrho_1 - a_4)(\varrho_2 - a_4)(\varrho_3 - a_4)}{(a_1 - a_4)(a_2 - a_4)(a_3 - a_4)}$	$+a_2 a_3 M_1^2 + a_1 a_3 M_2^2 + a_1 a_2 M_3^2$

2.1. The ellipso-cylindrical coordinate systems

On the three-dimensional sphere there are two ellipso-cylindrical coordinate systems, the *oblate elliptic system*, also called the elliptic-cylindrical 1 system, and the *prolate elliptic system*, also called the elliptic-cylindrical 2 system.

2.1.1. Oblate elliptic coordinates. Let us start with the oblate elliptic system. In terms of the Jacobi elliptic functions cn, sn and dn [41, p 910] it has the form [8]

$$
\begin{aligned}
s_1 &= \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')\cos\varphi \\
s_2 &= \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')\sin\varphi \\
s_3 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k') \\
s_4 &= \mathrm{dn}(\mu, k)\,\mathrm{sn}(\nu, k')
\end{aligned}
\tag{2.6}
$$

where $0 \leqslant \mu \leqslant 2K$, $-K' \leqslant \nu \leqslant K'$, $0 \leqslant \varphi < 2\pi$. k is the modulus of the elliptic functions, with $k'^2 = 1 - k^2$. K and K' are the complete elliptic integrals. In what follows, if not explicitly stated otherwise, we usually omit in the Jacobi elliptic functions the explicit dependence on the modulus k. The coordinates (2.6) are a one-parametric coordinate system depending on k. In the limits $k \to 0$ and $k \to 1$ simpler coordinate systems emerge.

In the limiting case $k \to 0$, i.e. $k' \to 1$ we obtain

$$\begin{aligned} &\mathrm{cn}(\mu, k) \to \cos\vartheta && 0 \leqslant \vartheta \leqslant \pi \\ &\mathrm{sn}(\nu, k') \to \tan\nu \equiv \cos\chi && 0 \leqslant \chi \leqslant \pi. \end{aligned} \tag{2.7}$$

Therefore the oblate elliptic coordinate system in this limit yields spherical coordinates:

$$\begin{aligned} s_1 &= \sin\chi \sin\vartheta \cos\varphi \\ s_2 &= \sin\chi \sin\vartheta \sin\varphi \\ s_3 &= \sin\chi \cos\vartheta \\ s_4 &= \cos\chi. \end{aligned} \tag{2.8}$$

Alternatively we can consider $k \to 1$, i.e. $k' \to 0$, and for the Jacobi elliptic functions we obtain

$$\begin{aligned} &\mathrm{sn}(\mu, k) \to \tan\mu \equiv \sin\gamma && 0 \leqslant \gamma \leqslant \pi \\ &\mathrm{cn}(\nu, k') \to \sin\varphi_2 && 0 \leqslant \varphi_2 \leqslant \pi. \end{aligned} \tag{2.9}$$

Hence oblate elliptic coordinates yield cylindrical coordinates on $S^{(3)}$ ($\varphi \equiv \varphi_1$):

$$\begin{aligned} s_1 &= \sin\gamma \cos\varphi_1 \\ s_2 &= \sin\gamma \sin\varphi_1 \\ s_3 &= \cos\gamma \sin\varphi_2 \\ s_4 &= \cos\gamma \cos\varphi_2. \end{aligned} \tag{2.10}$$

Finally, the Laplace operator in oblate elliptic coordinates is given by

$$\begin{aligned} \Delta = {} & \frac{1}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\mathrm{cn}\mu\,\mathrm{dn}\mu}{\mathrm{sn}\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} - k'^2\frac{\mathrm{sn}\nu\,\mathrm{cn}\nu}{\mathrm{dn}\nu}\frac{\partial}{\partial\nu}\right) \\ & + \frac{1}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\frac{\partial^2}{\partial\varphi^2}. \end{aligned} \tag{2.11}$$

2.1.2. Prolate elliptic coordinates. Next, we consider prolate elliptic coordinates which have in terms of the Jacobi elliptic functions the form

$$\begin{aligned} s_1 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\cos\varphi \\ s_2 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\sin\varphi \\ s_3 &= \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k') \\ s_4 &= \mathrm{dn}(\mu, k)\,\mathrm{sn}(\nu, k'). \end{aligned} \tag{2.12}$$

Here $-K \leqslant \mu \leqslant K$, $-K' \leqslant \nu \leqslant K'$, $0 \leqslant \varphi \leqslant 2\pi$. In comparison with the oblate elliptic coordinate system, the prolate elliptic system has the same limiting coordinate system for the two limiting cases $k \to 0$ and $k \to 1$ (for which $k' \to 1$ and $k' \to 0$, respectively), namely the spherical system. In the limit $k \to 0$ we find the coordinates (2.8). For the limit $k \to 1$ we find that

$$\begin{aligned} &\mathrm{sn}(\mu, k) \to \tan\mu \equiv \cos\chi && 0 \leqslant \chi \leqslant \pi \\ &\mathrm{sn}(\nu, k') \to \cos\vartheta && 0 \leqslant \vartheta \leqslant \pi \end{aligned} \tag{2.13}$$

and we therefore obtain the coordinate system representation

$$\begin{aligned} s_1 &= \sin\chi\,\sin\vartheta\,\cos\varphi \\ s_2 &= \sin\chi\,\sin\vartheta\,\sin\varphi \\ s_3 &= \cos\chi \\ s_4 &= \sin\chi\,\cos\vartheta \end{aligned} \tag{2.14}$$

where the coordinates s_3 and s_4 are interchanged compared with (2.8).

We finally note that both elliptic coordinate systems can be transformed into each other by the substitution

$$k \to \frac{\mathrm{i}k}{k'} \qquad k' \to \frac{1}{k'} \qquad \mu \to K - \mu. \tag{2.15}$$

The Laplacian in prolate elliptic coordinates has the form

$$\Delta = \frac{1}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} - \frac{\mathrm{sn}\mu\,\mathrm{dn}\mu}{\mathrm{cn}\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} - \frac{\mathrm{sn}\nu\,\mathrm{dn}\nu}{\mathrm{cn}\nu}\frac{\partial}{\partial\nu}\right) + \frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\frac{\partial^2}{\partial\varphi^2}. \tag{2.16}$$

2.2. *The observables*

We consider the Schrödinger equation of the free motion on $S^{(3)}$ in oblate spheroidal coordinates (2.6). We make the following ansatz for the wavefunctions:

$$\Psi(\mu,\nu,\varphi;k) = \psi_1(\mu;k)\psi_2(\nu;k)\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}} \qquad m \in \mathbb{Z} \tag{2.17}$$

and obtain the two coupled ordinary differential equations

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}\mu^2} + \frac{\mathrm{cn}\mu\,\mathrm{dn}\mu}{\mathrm{sn}\mu}\frac{\mathrm{d}\psi_1}{\mathrm{d}\mu} - \left[k^2 J(J+2)\,\mathrm{sn}^2\mu + \frac{m^2}{\mathrm{sn}^2\mu}\right]\psi_1 = -\lambda_q(k)\psi_1 \tag{2.18}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}\nu^2} - k'^2\frac{\mathrm{sn}\nu\,\mathrm{cn}\nu}{\mathrm{dn}\nu}\frac{\mathrm{d}\psi_2}{\mathrm{d}\nu} + \left[J(J+2)\,\mathrm{dn}^2\nu + k^2\frac{m^2}{\mathrm{dn}^2\nu}\right]\psi_2 = +\lambda_q(k)\psi_2. \tag{2.19}$$

The quantum number q labels the eigenvalues of the observables of the oblate elliptic system. In order to determine the operator Λ corresponding to the eigenvalue λ we eliminate the quantum number J from the equations (2.18), (2.19). This yields

$$\begin{aligned} \Lambda = \frac{1}{k^2\,\mathrm{sn}^2\mu - \mathrm{dn}^2\nu}\Bigg(&\mathrm{dn}^2\nu\frac{\partial^2}{\partial\mu^2} + k^2\,\mathrm{sn}^2\mu\frac{\partial^2}{\partial\nu^2} + \frac{\mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{dn}^2\nu}{\mathrm{sn}\mu}\frac{\partial}{\partial\mu} \\ &-k^2k'^2\frac{\mathrm{sn}\nu\,\mathrm{cn}\nu\,\mathrm{sn}^2\mu}{\mathrm{dn}\nu}\frac{\partial}{\partial\nu}\Bigg) - \frac{k^2\,\mathrm{sn}^2\mu + \mathrm{dn}^2\nu}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\frac{\partial^2}{\partial\varphi^2}. \end{aligned} \tag{2.20}$$

If we re-insert the coordinate $\boldsymbol{s}$ in (2.20) it follows that the operator Λ can be represented as

$$\hbar^2\Lambda = (1-k^2)\boldsymbol{L}^2 + k^2(L_3^2 - M_3^2) + k^2\mathcal{L}_1. \tag{2.21}$$

Here $\mathcal{L}_1 = -\hbar^2\Delta$. We can introduce another operator $\mathcal{L}_2$ by defining

$$\begin{aligned} \hbar^2\mathcal{L}_2^{\mathrm{oblate}} &= \frac{1}{1-k^2}\left[\hbar^2\Lambda - k^2\mathcal{L}_1 + k^2L_3^2\right] \\ &= \boldsymbol{L}^2 - dM_3^2 \qquad d = \frac{k^2}{1-k^2} \in [0,\infty). \end{aligned} \tag{2.22}$$

After the transformation (2.15) we obtain for the prolate coordinate system the corresponding operator

$$\hbar^2 \mathcal{L}_2^{\text{prolate}} = \boldsymbol{L}^2 + k^2 M_3^2. \tag{2.23}$$

Both operators (2.22), (2.23) can be cast as one operator, yielding

$$\hbar^2 \mathcal{L}_2 = \boldsymbol{L}^2 - a M_3^2 \tag{2.24}$$

where now $a \in [-1, \infty)$. For a positive we have oblate elliptic coordinates, for $a \in [-1, 0]$ prolate elliptic coordinates. Therefore the operator $\mathcal{L}_2 = \mathcal{L}_2(a)$ is a continuous function of the parameter a. It is possible to analytically continue the dependence on a to the entire real line by means of the mapping $k^2 \to 1/k^2$ which maps the domain $[0, \infty) \to (-\infty, -1]$.

The Laplace operator $\mathcal{L}_1 = -\hbar^2 \Delta$, the elliptic operator $\mathcal{L}_2$, $\mathcal{L}_3 = -\hbar^2 \partial^2/\partial\varphi^2$ and the parity operators $\Pi_3 : \ \Pi_3(s_3) = -s_3$, $\Pi_4 : \ \Pi_4(s_4) = -s_4$, form a complete set of commuting operators and fix the elliptic basis uniquely.

The limiting cases of $a \to \infty$, $a \to 0$ and $a \to -1$ give us the corresponding observables for the cylindrical and hyper-spherical systems, i.e.

$$\begin{aligned} &\lim_{a\to 0} \mathcal{L}_2 = \boldsymbol{L}^2 \\ &\lim_{a\to -1} \mathcal{L}_2 = \boldsymbol{L}^2 + M_3^2 \\ &\lim_{a\to\infty} \frac{\mathcal{L}_2}{a} = -M_3^2. \end{aligned} \tag{2.25}$$

3. The ellipso-cylindrical bases

3.1. Expansion of the elliptic basis with respect to the cylindrical basis

Let us construct the elliptic basis in terms of the cylindrical basis according to

$$\Psi_{Jqm}(\mu, \nu, \varphi; a) = \sum_{m_2=-(J-|m|)}^{J-|m|} T_{Jqm_2}(a) \Psi_{Jmm_2}(\vartheta, \varphi_1, \varphi_2) \tag{3.1}$$

with the wavefunctions $\Psi_{Jmm_2}(\vartheta, \varphi_1, \varphi_2)$ as defined in (A.1), where m_2 is even or odd depending on the parity of $J - |m|$, and we use $m \equiv m_1$ in the cylindrical basis. Inserting the expansion (3.1) into the operator equation

$$\mathcal{L}_2 \Psi_{Jqm} = \lambda_q \hbar^2 \Psi_{Jqm} \tag{3.2}$$

taking into account the orthogonality of the cylindrical wavefunctions Ψ_{Jmm_2}, and using equations (2.24) and (A.6), we obtain

$$(\lambda_q + a m_2^2) T_{Jqm_2}(a) = \frac{1}{\hbar^2} \sum_{m_2'=-(J-|m|)}^{J-|m|} T_{Jqm_2'}(a) (\boldsymbol{L}^2)_{m_2 m_2'} \tag{3.3}$$

where

$$(\boldsymbol{L}^2)_{m_2 m_2'} = \int \mathrm{d}\boldsymbol{\Omega} \ \Psi^*_{Jmm_2}(\gamma, \varphi_1, \varphi_2) \boldsymbol{L}^2 \Psi_{Jmm_2'}(\gamma, \varphi_1, \varphi_2) \tag{3.4}$$

and $\mathrm{d}\boldsymbol{\Omega}$ denotes the invariant volume element on $S^{(3)}$. For example, using

$$\boldsymbol{L}^2 = \tfrac{1}{2}(L_+ L_- + L_- L_+) + L_3^2 \tag{3.5}$$

and the formulae in appendix A, we have the following expression for the matrix elements:

$$\frac{1}{\hbar^2}(\boldsymbol{L}^2)_{m_2 m_2'} = -A_{m_2-2}\delta_{m_2,m_2'-2} + B_{m_2}\delta_{m_2 m_2'} - A_{m_2}\delta_{m_2,m_2'+2} \tag{3.6}$$

with the coefficients A_{m_2} and B_{m_2} being given by

$$A_{m_2} = \tfrac{1}{4}\sqrt{(J-m_2-m)(J-m_2+m)(J+m_2+m+2)(J+m_2-m+2)} \tag{3.7}$$

$$B_{m_2} = \tfrac{1}{2}[J(J+2) + (m^2 - m_2^2)]. \tag{3.8}$$

The coefficients A_{m_2} and B_{m_2} satisfy the recurrence relations

$$A_{-m_2} = A_{m_2-2} \qquad B_{-m_2} = B_{m_2}. \tag{3.9}$$

Inserting equation (3.6) in (3.3) we obtain a three-term recurrence relation for the coefficients $T_{qm_2} \equiv T_{Jqm_2}(a)$:

$$A_{m_2}T_{q,m_2+2} + \left(am_2^2 + \lambda_q - B_{m_2}\right)T_{qm_2} + A_{m_2-2}T_{q,m_2-2} = 0. \tag{3.10}$$

These recurrence relations can be rewritten in the form of two sets of homogeneous linear equations, one for each parity class, depending on whether the quantum number m_2 is even or odd, respectively. On the one hand, for $J - |m| = 2k$ we have

$$\begin{aligned}
0 &= (B_{2k} - 4ak^2 - \lambda_q)T_{q,2k} - A_{2k-2}T_{q,2k-2}\\
0 &= A_{2k-2}T_{q,2k} - \left(B_{2k-2} - 4a(k-1)^2 - \lambda_q\right)T_{q,2k-2} + A_{2k-4}T_{q,2k-4}\\
&\dots\\
0 &= A_0T_0 - (B_0 - \lambda_q)T_0 + A_{-2}T_{q,-2}\\
&\dots\\
0 &= A_{-2k}T_{q,-2k+2} - (B_{-2k} - 4ak^2 - \lambda_q)T_{q,-2k}.
\end{aligned} \tag{3.11}$$

On the other, for $J - |m| = 2k+1$ we have

$$\begin{aligned}
0 &= \left(B_{2k+1} - (2k+1)^2a - \lambda_q\right)T_{q,2k+1} - A_{2k-1}T_{q,2k-1}\\
0 &= A_{2k-1}T_{q,2k+1} - \left(B_{2k-1} - (2k-1)^2a - \lambda_q\right)T_{q,2k-1} + A_{2k-3}T_{q,2k-3}\\
&\dots\\
0 &= A_1T_3 - (B_1 - a - \lambda_q)T_1 + A_{-1}T_{q,-1}\\
0 &= A_{-1}T_{q1} - (B_{-1} - a - \lambda_q)T_{q,-1} + A_{-3}T_{q,-3}\\
&\dots\\
0 &= A_{-2k-1}T_{q,-2k+1} - \left(B_{-2k-2} - (2k-1)^2a - \lambda_q\right)T_{q,-2k-1}.
\end{aligned} \tag{3.12}$$

The homogeneous systems (3.11), (3.12) satisfy the symmetry conditions

$$T_{q,-k} = pT_{qk} \qquad p = \pm 1. \tag{3.13}$$

We can determine the classes $(r, p) = (\pm, \pm)$, where $p = \pm 1$ and $r = (-1)^{|m_2|} = (-1)^{J-|m|}$, and we see that the recurrence relations for the T_{qm_2} can be put into four classes:

(i) $J - |m| = 2k, m_2 = 2n$:

$$\sqrt{(k-n)(k+n+1)(J-k-n)(J-k+n+1)}\,(1 \pm \delta_{n0})\,T^{(+,\pm)}_{q,2n+2}$$
$$+\left\{\lambda_q + 4an^2 - \tfrac{1}{2}[J(J+2) + (J-2k-2n)(J-2k+2n)]\right\}T^{(+,\pm)}_{q,2n}$$

$$+\sqrt{(k+n)(k-n+1)(J-k+n)(J-k-n+1)}\,T^{(+,\pm)}_{q,2n-2}=0 \qquad (3.14)$$

(ii) $J-|m|=2k+1, m_2=2n+1$:

$$\sqrt{(k-n)(k+n+2)(J-k-n-1)(J-k+n+1)}\,T^{(-,\pm)}_{q,2n+3}+\big\{\lambda_q+a(2n+1)^2$$
$$-[J(J+2)+(2k+1)J+2(k-n)(k-n-1)]\big\}\,T^{(-,\pm)}_{q,2n+1}$$
$$+\sqrt{(k-n+1)(k+n+1)(J-k-n)(J-k+n)}\,T^{(-,\pm)}_{q,2n-1}$$
$$\pm(J-k)(k+1)\delta_{n0}T^{(-,\pm)}_{q,1}=0. \qquad (3.15)$$

The homogeneous equations corresponding to the three-term recurrence relations (3.14), (3.15) have non-trivial solutions if their determinants are equal to zero. These conditions lead to four algebraic equations and give us the eigenvalue $\lambda_q \equiv \lambda^{(r,p)}_{Jqm}(a)$ of the elliptic observable Λ. For each class we have for the quantum number q

$J-|m|=2k$:

$$\begin{array}{lll} q=0,2,\ldots,2k & k+1 \text{ states of class } (+,+) \\ q=1,3,\ldots,2k-1 & k \text{ states of class } (+,-) \end{array} \qquad (3.16)$$

$J-|m|=2k+1$:

$$\begin{array}{lll} q=0,2,\ldots,2k & k+1 \text{ states of class } (-,+) \\ q=1,3,\ldots,2k+1 & k+1 \text{ states of class } (-,-). \end{array} \qquad (3.17)$$

The quantum number q labels the elliptic observable and counts according to the oscillation theorem the number of zeros of the 'angular' elliptic wavefunction $\psi_1(\mu,k)$, and the multiplicity of the degeneracy for fixed J is $(J+1)^2$.

Therefore, we can write the interbasis expansions of the elliptic wavefunctions $\Psi^{(r,p)}_{Jqm}$ in the following form:

$$\Psi^{(+,+)}_{Jqm}(\mu,\nu,\varphi;a)=\sum_{n=0,1}^{(J-|m|)/2}\left(1-\tfrac{1}{2}\delta_{n0}\right)T^{(+,+)}_{q,2n}\Psi^{(+,+)}_{J,m,2n}(\gamma,\varphi_1,\varphi_2) \qquad (3.18)$$

$$\Psi^{(+,-)}_{Jqm}(\mu,\nu,\varphi;a)=\sum_{n=1,2}^{(J-|m|)/2}T^{(+,-)}_{q,2n}\Psi^{(+,-)}_{J,m,2n}(\gamma,\varphi_1,\varphi_2) \qquad (3.19)$$

$$\Psi^{(-,+)}_{Jqm}(\mu,\nu,\varphi;a)=\sum_{n=0,1}^{(J-|m|-1)/2}T^{(-,+)}_{q,2n+1}\Psi^{(-,+)}_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2) \qquad (3.20)$$

$$\Psi^{(-,-)}_{Jqm}(\mu,\nu,\varphi;a)=\sum_{n=0,1}^{(J-|m|-1)/2}T^{(-,-)}_{q,2n+1}\Psi^{(-,-)}_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2) \qquad (3.21)$$

where

$$\Psi^{(+,\pm)}_{J,m,2n}(\gamma,\varphi_1,\varphi_2)=\frac{1}{\sqrt{2}}\big[\Psi_{J,m,2n}(\gamma,\varphi_1,\varphi_2)\pm\Psi_{J,m,-2n}(\gamma,\varphi_1,\varphi_2)\big] \qquad (3.22)$$

$$\Psi^{(-,\pm)}_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2)=\frac{1}{\sqrt{2}}\big[\Psi_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2)\pm\Psi_{J,m,-2n-1}(\gamma,\varphi_1,\varphi_2)\big]. \qquad (3.23)$$

The ellipso-cylindrical wavefunctions are subject to the orthogonality relation

$$\int \mathrm{d}\boldsymbol{\Omega}\ \Psi^{(r,p)*}_{Jqm}(\mu,\nu,\varphi;a)\Psi^{(r,p)}_{J'q'm'}(\mu,\nu,\varphi;a) = \delta_{JJ'}\delta_{qq'}\delta_{mm'}\delta_{rr'}\delta_{pp'} \tag{3.24}$$

and the following orthonormalization conditions hold:

$$\sum_{n=0,1}^{(J-|m|)/2} \left(1-\tfrac{1}{2}\delta_{n0}\right)T^{(+,+)*}_{q,2n}T^{(+,+)}_{q',2n} = \sum_{n=1,2}^{(J-|m|)/2} T^{(+,-)*}_{q,2n}T^{(+,-)}_{q',2n} = \sum_{n=0,1}^{(J-|m|-1)/2} T^{(-,+)*}_{q,2n}T^{(-,+)}_{q',2n}$$

$$= \sum_{n=0,1}^{(J-|m|-1)/2} T^{(-,-)*}_{q,2n}T^{(-,-)}_{q',2n} = \delta_{q,q'}. \tag{3.25}$$

The expansions (3.18)–(3.21) can be inverted due to their unitarity. Using, for instance, the orthogonality conditions (3.24) one can show that

$$\left(T^{(r,p)}_{qm_2}\right)^{-1} = T^{(r,p)*}_{qm_2}. \tag{3.26}$$

Hence we can write the inverse expansion of the cylindrical basis over the elliptic basis in the following form:

$$\Psi^{(r,p)}_{Jmm_2}(\vartheta,\varphi_1,\varphi_2) = \sum_q T^{(r,p)*}_{qm_2}(a)\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a) \tag{3.27}$$

where q is given by (3.16), (3.17) and by taking into account the proper parity classes it follows that

$$\sum_{q=0,2}^{J-|m|} \left(1-\tfrac{1}{2}\delta_{n0}\right)T^{(+,+)*}_{q,2n}T^{(+,+)}_{q,2n'} = \sum_{q=1,3}^{J-|m|-1} T^{(+,-)*}_{q,2n}T^{(+,-)}_{q,2n'} = \sum_{q=0,2}^{J-|m|} T^{(-,+)*}_{q,2n+1}T^{(-,+)}_{q,2n'+1}$$

$$= \sum_{q=1,3}^{J-|m|-1} T^{(-,-)*}_{q,2n+1}T^{(-,-)}_{q,2n'+1} = \delta_{n,n'}. \tag{3.28}$$

Some results for the coefficients $T^{(r,p)}_{q,2n}$ and $\lambda^{(r,p)}_q$ are listed in table 2.

3.2. *Expansion of the elliptic basis with respect to the hyper-spherical basis*

Let us consider the interbasis expansion of the elliptic basis with respect to the hyper-spherical basis

$$\Psi_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|}^{J} N_{Jql}(a)\Psi_{Jlm}(\chi,\vartheta,\varphi) \tag{3.29}$$

with the wavefunctions $\Psi_{Jlm}(\chi,\vartheta,\varphi)$ as defined in (B.1). Performing the same manipulations as in the previous subsection we obtain

$$\frac{1}{a}[-\lambda_q + l(l+1)]N_{ql}(a) = \frac{1}{\hbar^2}\sum_{l'=|m|}^{J} N_{Jql'}(a)(M_3^2)_{ll'} \tag{3.30}$$

where

$$(M_3^2)_{ll'} = \int \mathrm{d}\boldsymbol{\Omega}\ \Psi^*_{Jlm}(\chi,\vartheta,\varphi)M_3^2\Psi_{Jl'm}(\chi,\vartheta,\varphi). \tag{3.31}$$

Using the formulae of appendix B we have

$$\frac{1}{\hbar^2}(M_3^2)_{ll'} = -C_{l-2}\delta_{l,l'-2} + E_l\delta_{ll'} - C_l\delta_{l,l'+2} \tag{3.32}$$

Table 2. Coefficients for the interbasis expansion (3.1).

J	m	n	$T_{q,2n}^{(+,+)}$	$\lambda_q^{(+,+)}$
For $(r,p)=(+,+)$				
0	0	0	$1/\sqrt{2}$	$\lambda_q^{(+,+)}=0$
1	±1	0	1	$\lambda_q^{(+,+)}=2$
2	0	0	$\sqrt{\dfrac{\lambda_q^{(+,+)}+4a-2}{\lambda_q^{(+,+)}+2a-3}}$	$\left[2-4a-\lambda_q^{(+,+)}\right]\left[4-\lambda_q^{(+,+)}\right]=8$
2	0	1	$-\sqrt{\dfrac{\lambda_q^{(+,+)}-4}{2(\lambda_q^{(+,+)}+2a-3)}}$	$\left[2-4a-\lambda_q^{(+,+)}\right]\left[4-\lambda_q^{(+,+)}\right]=8$
2	±2	0	$1/\sqrt{2}$	$\lambda_q^{(+,+)}=6$
3	±1	0	$\sqrt{\dfrac{\lambda_q^{(+,+)}+4a-6}{\lambda_q^{(+,+)}+2a-7}}$	$\left[6-4a-\lambda_q^{(+,+)}\right]\left[8-\lambda_q^{(+,+)}\right]=24$
3	±1	1	$-\sqrt{\dfrac{\lambda_q^{(+,+)}-8}{2(\lambda_q^{(+,+)}+2a-7)}}$	$\left[6-4a-\lambda_q^{(+,+)}\right]\left[8-\lambda_q^{(+,+)}\right]=24$
3	±3	0	$1/\sqrt{2}$	$\lambda_q^{(+,+)}=12$
For $(r,p)=(+,-)$				
2	0	1	1	$\lambda_q^{(+,-)}=2-4a$
3	±1	1	1	$\lambda_q^{(+,-)}=6-4a$
For $(r,p)=(-,+)$				
1	0	0	1	$\lambda_q^{(-,+)}=-a$
2	±1	0	1	$\lambda_q^{(-,+)}=2-a$
3	0	0	$\sqrt{\dfrac{\lambda_q^{(-,+)}+9a-3}{2(\lambda_q^{(-,+)}+5a-3)}}$	$\left[3-9a-\lambda_q^{(-,+)}\right]\left[3-a-\lambda_q^{(-,+)}\right]=9$
3	0	1	$-\sqrt{\dfrac{\lambda_q^{(-,+)}+a-3}{2(\lambda_q^{(-,+)}+5a-3)}}$	$\left[3-9a-\lambda_q^{(-,+)}\right]\left[3-a-\lambda_q^{(-,+)}\right]=9$
3	±2	0	1	$\lambda_q^{(-,+)}=6-a$
For $(r,p)=(-,-)$				
1	0	0	1	$\lambda_q^{(-,-)}=2-a$
2	±1	0	1	$\lambda_q^{(-,-)}=6-a$
3	0	0	$\sqrt{\dfrac{\lambda_q^{(-,-)}+9a-3}{2(\lambda_q^{(-,-)}+5a-7)}}$	$\left[3-9a-\lambda_q^{(-,-)}\right]\left[11-a-\lambda_q^{(-,-)}\right]=9$
3	0	1	$-\sqrt{\dfrac{\lambda_q^{(-,-)}+a-11}{2(\lambda_q^{(-,-)}+5a-7)}}$	$\left[3-9a-\lambda_q^{(-,-)}\right]\left[11-a-\lambda_q^{(-,-)}\right]=9$
3	±2	0	1	$\lambda_q^{(-,-)}=12-a$

with the coefficients C_l and E_l being given by

$$C_l = \sqrt{\frac{(J-l-1)(J-l)(J+l+2)(J+l+3)}{(2l+1)(2l+3)^2(2l+5)}} \times \sqrt{(l-|m|+1)(l-|m|+2)(l+|m|+1)(l+|m|+2)} \tag{3.33}$$

$$E_l = \frac{(J-l+1)(J+l+1)(l-|m|)(l+|m|)}{(2l-1)(2l+1)} + \frac{(J-l)(J+l+2)(l-|m|+1)(l+|m|+1)}{(2l+1)(2l+3)}. \tag{3.34}$$

Inserting equation (3.32) in (3.30) we obtain a three-term recurrence relation for the interbasis coefficients $N_{ql} \equiv N_{Jql}(a)$:

$$C_l N_{q,l+2} + \left\{\frac{1}{a}\left[l(l+1) - \lambda_q\right] - E_l\right\} N_{ql} + C_{l-2} N_{q,l-2} = 0. \tag{3.35}$$

From the three-term recurrence relation (3.35) we see that the index l increases by $\Delta l = 2$. This means that we get four classes of coefficients which can again be classified according to their parity. The first class emerges if the index l starts with $l = |m|$, the second for $l = |m| + 1$. Their respective parity is given by $p = (-1)^{l-|m|}$. The third (fourth) class is obtained if l is between $|m|$ and J (between $|m| + 1$ and $J - 1$). Their respective parity is given by $r = (-1)^{J-|m|}$. Therefore the interbasis expansion (3.29) splits into four contributions with parity classes $p = (-1)^{l-|m|}$ and $r = (-1)^{J-|m|}$, and they have the form

$$\Psi^{(+,+)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|,|m|+2}^{J} N^{(+,+)}_{ql} \Psi_{Jlm}(\chi,\vartheta,\varphi) \tag{3.36}$$

$$\Psi^{(+,-)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|+1,|m|+3}^{J-1} N^{(+,-)}_{ql} \Psi_{Jlm}(\chi,\vartheta,\varphi) \tag{3.37}$$

$$\Psi^{(-,+)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|,|m|+2}^{J-1} N^{(-,+)}_{ql} \Psi_{Jlm}(\chi,\vartheta,\varphi) \tag{3.38}$$

$$\Psi^{(-,-)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|+1,|m|+3}^{J} N^{(-,-)}_{ql} \Psi_{Jlm}(\chi,\vartheta,\varphi). \tag{3.39}$$

The interbasis coefficients N_{ql} satisfy the following four sets of homogeneous equations, and we have listed in table 3 some simple examples for small values of J, l, m.

1. $(r, p) = (+,+)$. $J - |m|$ even, $l - |m|$ even:

$$\begin{aligned}
0 &= \left[\frac{|m|(|m|+1) - \lambda_q}{a} - E_{|m|}\right] N_{q,|m|} + C_{|m|} N_{q,|m|+2} \\
0 &= C_{|m|} N_{q,|m|} + \left[\frac{(|m|+2)(|m|+3) - \lambda_q}{a} - E_{|m|+2}\right] N_{q,|m|+2} + C_{|m|+2} N_{q,|m|+4} \\
&\dots \\
0 &= \left[\frac{J(J+1) - \lambda_q}{a} - E_J\right] N_{qJ} + C_{J-2} N_{q,J-2}.
\end{aligned} \tag{3.40}$$

Table 3. Coefficients for the interbasis expansion (3.29).

J	m	l	$N_{ql}^{(+,+)}$	$\lambda_q^{(+,+)}$
				For $(r, p) = (+, +)$
0	0	0	1	$\lambda_q^{(+,+)} = 0$
1	±1	1	1	$\lambda_q^{(+,+)} = 2$
2	0	0	$\sqrt{\frac{\lambda_q^{(+,+)} + 4a/3 - 6}{2(\lambda_q^{(+,+)} + 2a - 3)}}$	$\left[6 - 4a/3 - \lambda_q^{(+,+)}\right]\left[8a/3 + \lambda_q^{(+,+)}\right] = -32a^2/9$
2	0	2	$\sqrt{\frac{\lambda_q^{(+,+)} + 8a/3}{2(\lambda_q^{(+,+)} + 2a - 3)}}$	$\left[6 - 4a/3 - \lambda_q^{(+,+)}\right]\left[8a/3 + \lambda_q^{(+,+)}\right] = -32a^2/9$
2	±2	2	1	$\lambda_q^{(+,+)} = 6$
3	±1	1	$\sqrt{\frac{\lambda_q^{(+,+)} + 8a/5 - 12}{2(\lambda_q^{(+,+)} + 2a - 7)}}$	$\left[2 - 12a/5 - \lambda_q^{(+,+)}\right]\left[12 - 8a/5 - \lambda_q^{(+,+)}\right] = 96a^2/25$
3	±1	3	$\sqrt{\frac{\lambda_q^{(+,+)} + 12a/5 - 2}{2(\lambda_q^{(+,+)} + 2a - 7)}}$	$\left[2 - 12a/5 - \lambda_q^{(+,+)}\right]\left[12 - 8a/5 - \lambda_q^{(+,+)}\right] = 96a^2/25$
3	±3	3	1	$\lambda_q^{(+,+)} = 12$
				For $(r, p) = (+, -)$
2	0	1	1	$\lambda_q^{(+,-)} = 2 - 4a$
3	±1	2	1	$\lambda_q^{(+,-)} = 6 - 4a$
				For $(r, p) = (-, +)$
1	0	0	1	$\lambda_q^{(-,+)} = -a$
2	±1	1	1	$\lambda_q^{(-,+)} = 2 - a$
3	0	0	$\sqrt{\frac{\lambda_q^{(-,+)} + 5a}{2(\lambda_q^{(-,+)} + 5a - 3)}}$	$\left[\lambda_q^{(-,+)} + 5a\right]\left[-6 + 5a + \lambda_q^{(-,+)}\right] = 16a^2$
3	0	2	$\sqrt{\frac{\lambda_q^{(-,+)} + 5a - 6}{2(\lambda_q^{(-,+)} + 5a - 3)}}$	$\left[\lambda_q^{(-,+)} + 5a\right]\left[-6 + 5a + \lambda_q^{(-,+)}\right] = 16a^2$
3	±2	2	1	$\lambda_q^{(-,+)} = 6 - a$
				For $(r, p) = (-, -)$
1	0	1	1	$\lambda_q^{(-,-)} = 2 - a$
2	±1	2	1	$\lambda_q^{(-,-)} = 6 - a$
3	0	1	$\sqrt{\frac{\lambda_q^{(-,-)} + 41a/5 - 2}{2(\lambda_q^{(-,-)} + 5a - 7)}}$	$\left[\lambda_q^{(-,-)} - 2 + 41a/5\right]\left[\lambda_q^{(-,-)} - 12 + 9a/5\right] = 144a^2/25$
3	0	3	$\sqrt{\frac{\lambda_q^{(-,-)} + 9a/5 - 12}{2(\lambda_q^{(-,-)} + 5a - 7)}}$	$\left[\lambda_q^{(-,-)} + 41a/5 - 2\right]\left[\lambda_q^{(-,-)} + 9a/5 - 12\right] = 144a^2/25$
3	±2	3	1	$\lambda_q^{(-,-)} = 12 - a$

2. $(r, p) = (+, -)$. $J - |m|$ even, $l - |m|$ odd:

$$0 = \left[\frac{(|m|+1)(|m|+2) - \lambda_q}{a} - E_{|m|+1}\right] N_{q,|m|+1} + C_{|m|+1} N_{q,|m|+3}$$

$$0 = C_{|m|+1} N_{q,|m|+1} + \left[\frac{(|m|+3)(|m|+4) - \lambda_q}{a} - E_{|m|+5}\right] N_{q,|m|+3} + C_{|m|+3} N_{q,|m|+5} \qquad (3.41)$$

...

$$0 = C_{J-3} N_{q,-3} + \left[\frac{(J-1)J - \lambda_q}{a} - E_{J-1}\right] N_{q,-1}.$$

3. $(r, p) = (-, +)$. $J - |m|$ odd, $l - |m|$ even:

$$0 = \left[\frac{|m|(|m|+1) - \lambda_q}{a} - E_{|m|}\right] N_{q,|m|} + C_{|m|} N_{q,|m|+2}$$

$$0 = C_{|m|} N_{q,|m|} + \left[\frac{(|m|+2)(|m|+3) - \lambda_q}{a} - E_{|m|+2}\right] N_{q,|m|+2} + C_{|m|+2} N_{q,|m|+4} \qquad (3.42)$$

...

$$0 = C_{J-3} N_{q,-3} + \left[\frac{J(J-1) - \lambda_q}{a} - E_{J-1}\right] N_{q,-1}.$$

4. $(r, p) = (-, -)$. $J - |m|$ odd, $l - |m|$ odd:

$$0 = \left[\frac{(|m|+1)(|m|+2) - \lambda_q}{a} - E_{|m|+1}\right] N_{q,|m|+1} + C_{|m|+1} N_{q,|m|+3}$$

$$0 = C_{|m|+1} N_{q,|m|+1} + \left[\frac{(|m|+3)(|m|+4) - \lambda_q}{a} - E_{|m|+3}\right] N_{q,|m|+3} + C_{|m|+3} N_{q,|m|+5} \qquad (3.43)$$

...

$$0 = C_{J-2} N_{q,-2} + \left[\frac{J(J+1) - \lambda_q}{a} - E_J\right] N_{qJ}.$$

The following normalization conditions hold:

$$1 = \sum_{l=|m|,|m|+2}^{J} \left|N_{ql}^{(+,+)}\right|^2 = \sum_{l=|m|,|m|+2}^{J} \left|N_{ql}^{(-,+)}\right|^2$$

$$= \sum_{l=|m|+1}^{J} \left|N_{ql}^{(+,-)}\right|^2 = \sum_{l=|m|+1}^{J} \left|N_{ql}^{(-,-)}\right|^2 \qquad (3.44)$$

and the ellipso-cylindrical wavefunctions are again subject to the orthogonality relation (3.24). As in (3.28) we have a unitary condition

$$\left(N_{ql}^{(r,p)}\right)^{-1} = N_{ql}^{(r,p)\,*}. \qquad (3.45)$$

If we take into account the parity of the hyper-spherical wavefunctions according to $\Psi_{Jlm} \equiv \Psi_{Jlm}^{(r,p)}$, we can obtain the inverse interbasis expansion of the hyper-spherical basis in terms of the elliptic basis by means of

$$\Psi_{Jlm}^{(r,p)}(\chi, \vartheta, \varphi) = \sum_q N_{ql}^{(r,p)\,*} \Psi_{Jqm}^{(r,p)}(\mu, \nu, \varphi; a) \qquad (3.46)$$

with q as in (3.16), (3.17).

3.3. Symmetry properties of the wavefunctions

The interbasis expansions (3.18)–(3.21) or (3.36)–(3.39), together with their corresponding three-term recurrence relations, completely determine the oblate and prolate elliptic bases. Now we consider the symmetry properties of the elliptic bases under reflection. For the cylindrical and hyper-spherical bases we have:

Reflection Π_3: $s_3 \to -s_3$:

$$\begin{aligned}\Pi_3\Psi^{(r,p)}_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2) &= p\Psi^{(r,p)}_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2)\\ \Pi_3\Psi^{(r,p)}_{Jlm}(\chi,\vartheta,\varphi) &= p\Psi^{(r,p)}_{Jlm}(\chi,\vartheta,\varphi).\end{aligned} \qquad (3.47)$$

Reflection Π_4: $s_4 \to -s_4$:

$$\begin{aligned}\Pi_4\Psi^{(r,p)}_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2) &= r\Psi^{(r,p)}_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2)\\ \Pi_4\Psi^{(r,p)}_{Jlm}(\chi,\vartheta,\varphi) &= r\Psi^{(r,p)}_{Jlm}(\chi,\vartheta,\varphi).\end{aligned} \qquad (3.48)$$

Applying the reflection operators to the interbasis expansions (3.18)–(3.21) or (3.36)–(3.39) we obtain the complete set of operators defining the elliptic bases which are given by

$$\begin{aligned}\mathcal{L}_1\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a) &= \hbar^2 J(J+2)\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a)\\ \mathcal{L}_2\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a) &= \hbar^2\lambda_q(a)\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a)\\ \Pi_3\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a) &= p\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a)\\ \Pi_4\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a) &= r\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a).\end{aligned} \qquad (3.49)$$

3.4. Limiting cases

As we have seen in the previous section, the two ellipso-cylindrical systems are the most general one-parametric coordinate systems on $S^{(3)}$ and contain as limiting cases the cylindrical and the hyper-spherical coordinate systems.

According to this, the elliptic bases in the limiting cases $a \to \infty$ and $a \to 0$ degenerate to the hyperspherical and cylindrical bases. Therefore we must determine the quantum numbers l_0 and $\widetilde{m}_2$ which characterize the limiting hyperspherical and cylindrical wavefunctions, for q fixed. Using the fact that the number q is independent of the parameter a and the expression for the wavefunctions (B.1) and (3.22), (3.23), we obtain

$$l_0 = q + |m| \qquad \widetilde{m}_2 \equiv \widetilde{m}_2^{(r,p)} \qquad (3.50)$$

where

$$\begin{aligned}\widetilde{m}_2^{(+,+)} &= J - |m| - q = 2k - q\\ \widetilde{m}_2^{(+,-)} &= J - |m| - q + 1 = 2k - q + 1\\ \widetilde{m}_2^{(-,+)} &= J - |m| - q - 1 = 2k - q\\ \widetilde{m}_2^{(-,-)} &= J - |m| - q = 2k - q + 1.\end{aligned} \qquad (3.51)$$

We analyse the two limits separately.

3.4.1. The hyper-spherical limit. The limit $a \to 0$ can be analysed by means of the recurrence relation (3.35). Ignoring all terms which depend explicitly on a we get

$$\lim_{a\to 0} \lambda_q(a) = l_0(l_0+1) = (q+|m|)(q+|m|+1) \tag{3.52}$$

$$\lim_{a\to 0} N_{ql}(a) = \delta_{ll_0} \tag{3.53}$$

and in particular for the wavefunctions

$$\lim_{a\to 0} \Psi_{Jqm}(\mu,\nu,\varphi;a) = \Psi_{Jl_0m}(\chi,\vartheta,\varphi). \tag{3.54}$$

Let us consider the expansion of the elliptic bases in terms of cylindrical basis, as $a \to 0$. By means of the quantum numbers $\{J, m, l, m_2\}$ we define the new quantum numbers

$$\mu_1 = \frac{|m|+m_2}{2} \qquad \mu_2 = \frac{|m|-m_2}{2} \qquad J = 2j_1 \qquad l = j \tag{3.55}$$

and the three-term recurrence relation (3.10) for the $T_{qm} \equiv T_{\mu_1\mu_2}$ takes the form

$$\begin{aligned}
&\sqrt{(j_1-\mu_1)(j_1+\mu_2)(j_1+\mu_1+1)(j_1-\mu_2+1)}\, T_{\mu_1+1,\mu_2-1} \\
&\quad + \big[j(j+1) - 2j_1(j_1+1) - 2\mu_1\mu_2\big] T_{\mu_1\mu_2} \\
&\quad + \sqrt{(j_1+\mu_1)(j_1-\mu_2)(j_1+\mu_2+1)(j_1-\mu_1+1)}\, T_{\mu_1-1,\mu_2+1} = 0
\end{aligned} \tag{3.56}$$

which coincides up to a phase factor with the three-term recurrence relation for the Clebsch–Gordan coefficients $\langle j_1 j \mu_1 \mu_2 | j, \mu_1+\mu_2\rangle$ [30], and we find

$$\lim_{a\to 0} T_{Jqm}(a) = (-1)^{(\mu_1-\mu_2)/2} \langle j_1 j \mu_1 \mu_2 | j, \mu_1+\mu_2\rangle. \tag{3.57}$$

Therefore the expansion (3.1) yields

$$\Psi_{j_1,j,\mu_1+\mu_2}(\chi,\vartheta,\varphi) = \sum_{\mu_1,\mu_2} (-1)^{(\mu_1-\mu_2)/2} \langle j_1 j \mu_1 \mu_2 | j, \mu_1+\mu_2\rangle \Psi_{j_1,\mu_1,\mu_2}(\gamma,\varphi_1,\varphi_2) \tag{3.58}$$

which is the well known expansion of the hyper-spherical basis in terms of the cylindrical basis [31].

3.4.2. The cylindrical limit. For the limit $a \to \infty$ we find from the recurrence relations (3.14), (3.15)

$$\begin{aligned}
&\lim_{a\to\infty} \frac{\lambda_q^{(r,p)}(a)}{a} = -\big[\widetilde{m}_2^{(r,p)}\big]^2 \\
&\lim_{a\to\infty} T_{qm_2}^{(r,p)}(a) = \delta_{\widetilde{m}_2^{(r,p)},m_2}
\end{aligned} \tag{3.59}$$

and therefore we get

$$\lim_{a\to\infty} \Psi_{Jqm}^{(r,p)}(\mu,\nu,\varphi;a) = \Psi^{(r,p)}_{J,\widetilde{m}_2^{(r,p)},m_2}(\gamma,\varphi_1,\varphi_2). \tag{3.60}$$

We consider the interbasis expansion (3.29). Using the quantum numbers (3.55) we can rewrite the three-term recurrence relation (3.35) as follows ($N_j \equiv N_{ql}$):

$$\begin{aligned}
&\sqrt{\frac{(2j_1-j-1)(2j_1-j)(2j_1+j+2)(2j_1+j+3)}{(2j+1)(2j+3)^2(2j+5)}} \\
&\quad \times \sqrt{(j-\mu_1-\mu_2+1)(j-\mu_1-\mu_2+2)(j+\mu_1+\mu_2+1)}
\end{aligned}$$

$$\times\sqrt{(j+\mu_1+\mu_2+2)}\,N_{j+2}$$

$$+\left\{(\mu_1-\mu_2)^2-\left[\frac{\big((2j+1)^2-j_1^2\big)\big(j^2-(\mu_1+\mu_2)^2\big)}{4j^2-1}\right.\right.$$

$$\left.\left.+\frac{\big((2j-1)^2-(j_1+1)^2\big)\big((j+1)^2-(\mu_1+\mu_2)^2\big)}{(2j+1)(2j+3)}\right]\right\}N_j$$

$$+\sqrt{\frac{(2j_1-j+1)(2j_1-j+2)(2j_1+j)(2j_1+j+1)}{(2j-1)(2j+1)^2(2j+3)}}$$

$$\times\sqrt{(j-\mu_1-\mu_2-1)(j-\mu_1-\mu_2)(j+\mu_1+\mu_2-1)}$$

$$\times\sqrt{(j+\mu_1+\mu_2)}\,N_{j-2}=0. \qquad (3.61)$$

This three-term recurrence relation coincides with the corresponding three-term recurrence relation for the Clebsch–Gordan coefficients which can be obtained from [32, equation 8.6.5(27)]. Therefore, in the limit $a\to\infty$ we obtain

$$\lim_{a\to\infty}N_{Jql}(a)=(-1)^{j/2}\langle j_1,j,\mu_1,\mu_2|j,\mu_1+\mu_2\rangle \qquad (3.62)$$

and the expansion (3.29) yields

$$\Psi_{j_1\mu_1\mu_2}(\gamma,\varphi_1,\varphi_2)=\sum_j(-1)^{j/2}\langle j_1,j,\mu_1,\mu_2|j,\mu_1+\mu_2\rangle\Psi_{j_1,j,\mu_2+\mu_2}(\chi,\vartheta,\varphi) \qquad (3.63)$$

which is the inverse expansion of (3.58).

4. Perturbation theory

The determination of the ellipso-cylindrical wavefunctions $\Psi_{Jqm}^{(r,p)}$ by using (3.14), (3.15) or (3.35), respectively, and solving them recursively is related to the solution of higher-order algebraic equations. In general, such algebraic equations cannot be solved analytically. Nevertheless, the spherical and the cylindrical bases can be considered as a zero-approximation for a perturbation expansion. We do this in what follows, first for $0<|a|\ll 1$, and then for $a\gg 1$.

4.1. The $0<|a|\ll 1$ case

According to (3.50) we can label the coefficients N_{ql} and λ_q with the index l_0 instead of q. Let us set $N_{ql}=\tilde{N}_{ll_0}$, $\lambda_q=\tilde{\lambda}_{l_0}$.

For the calculation of the higher-order corrections we consider the perturbation expansion

$$\tilde{N}_{ll_0}(a)=\delta_{ll_0}+\sum_{s=1}^{\infty}N_{ll_0}^{(s)}a^s \qquad (4.1)$$

$$\tilde{\lambda}_{l_0}(a)=l_0(l_0+1)+\sum_{s=1}^{\infty}\lambda_{ll_0}^{(s)}a^s \qquad (4.2)$$

which fixes l_0 as the angular momentum number corresponding to the $\boldsymbol{L}^2$ operator in the limit $a=0$. Inserting these expansions in the three-term recurrence relations (3.35) and comparing coefficients with the same power of a, after some algebraic manipulation we arrive at the following equations for $N_{ll_0}^{(s)}$ and $\lambda_{ll_0}^{(s)}$:

$$(l-l_0)(l+l_0+1)N_{ll_0}^{(s)} + C_l N_{l+2,l_0}^{(s-1)} + C_{l-2}N_{l-2,l_0}^{(s-1)} - E_l N_{ll_0}^{(s-1)} - \sum_{t=0}^{s-1}\lambda_{l_0}^{(s-1)}N_{ll_0}^{(t)} = 0. \tag{4.3}$$

According to standard perturbation theory, e.g. Landau and Lifshitz [33], we use the initial conditions

$$N_{ll_0}^{(0)} = \delta_{ll_0} \qquad N_{ll}^{(s)} = \delta_{s0} \tag{4.4}$$

and we can express $\lambda_{l_0}^{(s)}$ for $s > 1$ in terms of the coefficients $N_{l_0l_0}^{(s-1)}$, $N_{l_0,l_0\pm2}^{(s-1)}$ by means of

$$\lambda_{l_0}^{(s)} = C_{l_0}N_{l_0+2,l_0}^{(s-1)} - E_{l_0}N_{l_0l_0}^{(s-1)} + C_{l_0-2}N_{l_2-2,l_0}^{(s-1)}. \tag{4.5}$$

Explicitly for $s = 1$ and $s = 2$ we have

$$\begin{aligned}\lambda_{l_0}^{(1)} &= -E_{l_0}\\ \lambda_{l_0}^{(2)} &= C_{l_0}N_{l_0+2,l_0}^{(1)} + C_{l_0-2}N_{l_0-2,l_0}^{(1)}\end{aligned} \tag{4.6}$$

and from (4.3) we obtain

$$N_{ll_0}^{(1)} = \frac{C_{l_0-2}}{2(2l_0-1)}\delta_{l,l_0-2} - \frac{C_{l_0}}{2(2l_0+3)}\delta_{l,l_0+2} \tag{4.7}$$

for $s = 1$, which determines the second approximation for $\tilde{N}_{ll_0}(a)$ and $\tilde{\lambda}_{l_0}(a)$. Explicitly, for $\lambda_q(a)$ we obtain in second-order perturbation theory

$$\lambda_q(a) = (q+|m|)(q+|m|+1) - E_{q+|m|}\cdot a - \frac{1}{2}\left[\frac{C_{q+|m|}^2}{2q+2|m|+3} - \frac{C_{q+|m|+2}^2}{2q+2|m|-1}\right]a^2 + \mathrm{O}(a^3). \tag{4.8}$$

Substituting equation (4.7) in (4.1) and expanding (3.29), for the elliptic basis in first-order perturbation theory with respect to the hyper-spherical basis we obtain

$$\Psi_{Jqm}(\mu,\nu,\varphi;a) = \Psi_{J,q+|m|,m}(\chi,\vartheta,\varphi) - \frac{a}{2}\left[\frac{C_{q+|m|}}{2q+2|m|+3}\Psi_{J,q+|m|+2,m}(\chi,\vartheta,\varphi) - \frac{C_{q+|m|-2}}{2q+2|m|-1}\Psi_{J,q+|m|-2,m}(\chi,\vartheta,\varphi)\right] + \mathrm{O}(a^2). \tag{4.9}$$

Finally, the case of the prolate elliptic basis is obtained if we replace a with $-a$ in all relevant formulae.

4.2. The $a \gg 1$ case

For the case of $a \gg 1$ we consider the power expansions

$$T_{qm_2}^{(r,p)}(a) = \delta_{\widetilde{m}_2^{(r,p)},m_2} + \sum_{s=1}^{\infty} T_{qm_2}^{(r,p),(s)}a^{-s} \tag{4.10}$$

$$\frac{\lambda_q^{(r,p)}(a)}{a} = -\left[\widetilde{m}_2^{(r,p)}\right]^2 + \sum_{s=1}^{\infty}\lambda_q^{(r,p),(s)}a^{-s} \tag{4.11}$$

where $\widetilde{m}_2^{(r,p)}$ as in (3.50). Substituting equations (4.10) and (4.11) in the recurrence relation (3.14), (3.15), by the same reasoning as for small $|a|$, in second-order perturbation theory we obtain the following expressions for the quantities $\lambda_q^{(r,p)}(a)$:

$$\lambda_q^{(+,+)}(a) \simeq -a(2k-q)^2 + \tfrac{1}{2}\big[J(J+2) + (J-q)(J-4k+q)\big] + \frac{1}{a}\left\{ \frac{q(2k+\frac{1}{2}q+1)(J-2k+\frac{1}{2}q)(J-\frac{1}{2}q+1)}{8(2k-q+1)}\big(1+\delta_{k,\frac{1}{2}q}\big) - \frac{(q+2)(2k-\frac{1}{2}q)(J-\frac{1}{2}q)(J-2k+\frac{1}{2}q+1)}{8(2k-q-1)}\big(1+\delta_{k,\frac{1}{2}(q+1)}\big) \right\} \tag{4.12}$$

$$\lambda_q^{(+,-)}(a) \simeq -a(2k-q+1)^2 + \tfrac{1}{2}\big[J(J+2) + (J-q+1)(J-4k+q-1)\big] + \frac{1}{a}\left\{ \frac{(q-2)(2k-\frac{1}{2}q+\frac{3}{2})(J-2k+\frac{1}{2}q-\frac{1}{2})(J-\frac{1}{2}q+\frac{3}{2})}{8(2k-q+2)} \times \big(1-\delta_{k,\frac{1}{2}(q-1)}\big) - \frac{(q+1)(2k-\frac{1}{2}q+\frac{1}{2})(J-2k+\frac{1}{2}q+\frac{1}{2})(J-\frac{1}{2}q+\frac{1}{2})}{8(2k-q)} \times \big(1-\delta_{k,\frac{1}{2}(q+1)}\big) \right\} \tag{4.13}$$

$$\lambda_q^{(-,+)}(a) \simeq -a(2k-q+1)^2 + J(J+2) + (2k+1)J + \tfrac{1}{2}q(q-2) + (J-k)(k+1)\delta_{k,\frac{1}{2}q} + \frac{1}{a}\left\{ \frac{q(2k-\frac{1}{2}q+2)(J-2k+\frac{1}{2}q-1)(J-\frac{1}{2}q)}{8(2k-q+2)} - \frac{(q+1)(2k-\frac{1}{2}q+\frac{1}{2})(J-2k+\frac{1}{2}q)(J-\frac{1}{2}q)}{8(2k-q)}\big(1-\delta_{k,\frac{1}{2}q}\big) \right\} \tag{4.14}$$

$$\lambda_q^{(-,-)}(a) \simeq -a(2k-q+2)^2 + J(J+2) + (2k+1)J + \tfrac{1}{2}(q^2-1) - (J-k)(k+1)\delta_{k,\frac{1}{2}(q-1)} + \frac{1}{a}\left\{ \frac{(q-1)(2k-\frac{1}{2}q+\frac{5}{2})(J-2k+\frac{1}{2}q-\frac{3}{2})(J-\frac{1}{2}q+\frac{3}{2})}{8(2k-q+3)} - \frac{(q+1)(2k-\frac{1}{2}q+\frac{3}{2})(J-2k+\frac{1}{2}q-\frac{1}{2})(J-\frac{1}{2}q+\frac{1}{2})}{8(2k-q+1)}\big(1-\delta_{k,\frac{1}{2}(q-1)}\big) \right\}. \tag{4.15}$$

Therefore, to first order, we obtain the following expressions for the elliptic wavefunction $\Psi_{Jqm}^{(r,p)}(\mu,\nu,\varphi;a)$:

$$\Psi^{(+,+)}_{Jqm}(\mu,\nu,\varphi;a) \simeq \Psi^{(+,+)}_{J,2k-q,m}(\gamma,\varphi_1,\varphi_2) - \frac{1}{4a}\left\{ \frac{\sqrt{q(2k+\frac{1}{2}q+1)(J-2k+\frac{1}{2}q)(J-\frac{1}{2}q+1)}}{\sqrt{2}\,(2k-q+1)} \times \left(1+\delta_{k,\frac{1}{2}q+1}\right)\Psi^{(+,+)}_{J,2k-q+2,m}(\gamma,\varphi_1,\varphi_2) - \frac{\sqrt{(q+2)(2k-\frac{1}{2}q)(J-\frac{1}{2}q)(J-2k+\frac{1}{2}q+1)}}{\sqrt{2}\,(2k-q-1)} \times \Psi^{(+,+)}_{J,2k-q-2,m}(\gamma,\varphi_1,\varphi_2)\right\} \tag{4.16}$$

$$\Psi^{(+,-)}_{Jqm}(\mu,\nu,\varphi;a) \simeq \Psi^{(+,-)}_{J,2k-q+1,m}(\gamma,\varphi_1,\varphi_2) - \frac{1}{4a}\left\{ \frac{\sqrt{(q-1)(2k-\frac{1}{2}q+\frac{3}{2})(J-2k+\frac{1}{2}q-\frac{1}{2})(J-\frac{1}{2}q+\frac{3}{2})}}{\sqrt{2}\,(2k-q+2)} \times \left(1+\delta_{k,\frac{1}{2}(q+1)}\right)\Psi^{(+,-)}_{J,2k-q+3,m}(\gamma,\varphi_1,\varphi_2) - \frac{\sqrt{(q+1)(2k-\frac{1}{2}q+\frac{1}{2})(J-\frac{1}{2}q+\frac{1}{2})(J-2k+\frac{1}{2}q+\frac{1}{2})}}{\sqrt{2}\,(2k-q)} \times \Psi^{(+,-)}_{J,2k-q-1,m}(\gamma,\varphi_1,\varphi_2)\right\} \tag{4.17}$$

$$\Psi^{(-,+)}_{Jqm}(\mu,\nu,\varphi;a) \simeq \Psi^{(-,+)}_{J,2k-q,m}(\gamma,\varphi_1,\varphi_2) - \frac{1}{4a}\left\{ \frac{\sqrt{q(2k-\frac{1}{2}q+2)(J-2k+\frac{1}{2}q-1)(J-\frac{1}{2}q)}}{\sqrt{2}\,(2k-q+2)}\,\Psi^{(-,+)}_{J,2k-q+2,m}(\gamma,\varphi_1,\varphi_2) - \frac{\sqrt{(q+1)(2k-\frac{1}{2}q+1)(J-\frac{1}{2}q)(J-2k+\frac{1}{2}q)}}{\sqrt{2}\,(2k-q)} \times \left(1-\delta_{k,\frac{1}{2}q}\right)\Psi^{(-,+)}_{J,2k-q-2,m}(\gamma,\varphi_1,\varphi_2)\right\} \tag{4.18}$$

$$\Psi^{(-,-)}_{Jqm}(\mu,\nu,\varphi;a) \simeq \Psi^{(-,-)}_{J,2k-q+1,m}(\gamma,\varphi_1,\varphi_2) - \frac{1}{4a}\left\{ \frac{\sqrt{(q-1)(2k-\frac{1}{2}q+\frac{5}{2})(J-2k+\frac{1}{2}q-\frac{3}{2})(J-\frac{1}{2}q+\frac{3}{2})}}{\sqrt{2}\,(2k-q+3)}\right.$$

$$\times\,\Psi^{(-,-)}_{J,2k-q+3,m}(\gamma,\varphi_1,\varphi_2)$$

$$-\frac{\sqrt{(q+1)(2k-\frac{1}{2}q+\frac{3}{2})(J-\frac{1}{2}q+\frac{1}{2})(J-2k+\frac{1}{2}q-\frac{1}{2})}}{\sqrt{2}\,(2k-q+1)}$$

$$\left.\times\left(1-\delta_{k,\frac{1}{2}(q-1)}\right)\Psi^{(-,-)}_{J,2k-q-1,m}(\gamma,\varphi_1,\varphi_2)\right\}. \tag{4.19}$$

Higher-order perturbation corrections can be calculated analogously.

5. Two path integral representations in ellipso-cylindrical coordinates on $S^{(3)}$

Equation (3.1) describes the expansion of the elliptic basis in terms of the cylindrical basis, where in addition the parity classes according to (3.18)–(3.21) must be taken into account properly. Conversely, for the expansion of the cylindrical basis in terms of the elliptic basis we have the expansion (3.27), which must be interpreted according to the parity classes of the expansion coefficients T_{qm_2}. However, we use the shorthand notation of (3.27) in order not to make the corresponding formulae too lengthy.

The path integral representation of the propagator in cylindrical coordinates is given by [29, 34] (note that $\gamma\in(0,\pi)$, $\vartheta\in(0,\pi)$)

$$\begin{aligned}
K(\gamma'',\gamma',\varphi_1'',\varphi_1',\varphi_2'',\varphi_2';T) &= \int\limits_{\gamma(t')=\gamma'}^{\gamma(t'')=\gamma''}\mathcal{D}\gamma(t)\,\sin\gamma\cos\gamma\int\limits_{\varphi_1(t')=\varphi_1'}^{\varphi_1(t'')=\varphi_1''}\mathcal{D}\varphi_1(t)\int\limits_{\varphi_2(t')=\varphi_2'}^{\varphi_2(t'')=\varphi_2''}\mathcal{D}\varphi_2(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\left(\dot\gamma^2+\cos^2\gamma\,\dot\varphi_1^2+\sin^2\gamma\,\dot\varphi_2^2\right)\right.\right.\\
&\quad\left.\left.+\frac{\hbar^2}{8M}\left(4+\frac{1}{\cos^2\gamma}+\frac{1}{\sin^2\gamma}\right)\right]\mathrm{d}t\right\}\\
&:=\lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\epsilon\hbar}\right)^{3N/2}\prod_{j=1}^{N-1}\int_0^{\pi/2}\sin\gamma_j\cos\gamma_j d\gamma_j\int_0^{2\pi}\mathrm{d}\varphi_j^{(1)}\int_0^{2\pi}\mathrm{d}\varphi_j^{(2)}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\epsilon}\left((\Delta\gamma_j)^2+\widehat{\cos^2}\gamma_j(\Delta\varphi_j^{(1)})^2+\widehat{\sin^2}\gamma_j(\Delta\varphi_j^{(2)})^2\right)\right.\right.\\
&\quad\left.\left.+\epsilon\frac{\hbar^2}{8M}\left(4+\frac{1}{\cos^2\gamma_j\sin^2\gamma_j}\right)\right]\right\}\\
&=\sum_{Jmm_2}\exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\Psi^*_{Jmm_2}(\gamma',\varphi_1',\varphi_2')\Psi_{Jmm_2}(\gamma'',\varphi_1'',\varphi_2'').
\end{aligned}\tag{5.1}$$

Here we have formulated the path integral in the canonical way according to, e.g. [35–40], by means of a 'product lattice' [29] which is used in what follows. We have used the abbreviations $\varphi_0^{(1)}=\varphi_1',\varphi_N^{(1)}=\varphi_1''$, etc, $\epsilon=(t''-t')/N\equiv T/N$, $\Delta\boldsymbol{q}_j=\boldsymbol{q}_j-\boldsymbol{q}_{j-1}$, $\boldsymbol{q}_j=\boldsymbol{q}(t'+j\epsilon)$ $(t_j=t'+\epsilon j,\ j=0,\ldots,N)$, $\widehat{f^2(\boldsymbol{q}_j)}=f(\boldsymbol{q}_{j-1})f(\boldsymbol{q}_j)$ for some function

$f(\boldsymbol{q})$ of coordinates $\boldsymbol{q}$, and we interpret the limit $N \to \infty$ as equivalent to $\epsilon \to 0$, for fixed T.

The path integral representation of the propagator in the hyper-spherical basis is given by [29, 34] (note that $\chi \in (0, \pi)$)

$$K(\chi'', \chi', \vartheta'', \vartheta', \varphi'', \varphi'; T) = \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t)\ \sin^2\chi \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\ \sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$
$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{M}{2}\left(\dot\chi^2 + \sin^2\chi\,\dot\vartheta^2 + \sin^2\chi\,\sin^2\vartheta\,\dot\varphi^2\right) \right.\right.$$
$$\left.\left. + \frac{\hbar^2}{8M}\left(4 + \frac{1}{\sin^2\chi}\left(1 + \frac{1}{\sin^2\vartheta}\right)\right)\right] \mathrm{d}t \right\}$$
$$= \sum_{Jlm} \exp\left[-\frac{\mathrm{i}\hbar T}{2M} J(J+2)\right] \Psi^*_{Jlm}(\chi', \vartheta', \varphi')\Psi_{Jlm}(\chi'', \vartheta'', \varphi''). \tag{5.2}$$

In [36] we have derived the path integral representation on the sphere $S^{(D-1)}$ which has the form

$$K(\boldsymbol{\Omega}'', \boldsymbol{\Omega}'; T) = \int\limits_{\boldsymbol{\Omega}(t')=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(t'')=\boldsymbol{\Omega}''} \mathcal{D}\boldsymbol{\Omega}(t) \exp\left\{ \frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{M}{2}\dot{\boldsymbol{\Omega}}^2 + \hbar^2 \frac{(D-1)(D-3)}{8M}\right] \mathrm{d}t \right\}. \tag{5.3}$$

Here $\boldsymbol{\Omega}$ is a unit vector on the D-dimensional sphere. The quantity $\dot{\boldsymbol{\Omega}}^2$ is defined in its lattice representation by means of $\dot{\boldsymbol{\Omega}}^2 \to (\boldsymbol{\Omega}_j - \boldsymbol{\Omega}_{j-1})^2/\epsilon$, which in turn can be restated, yielding

$$\frac{1}{\epsilon}(\boldsymbol{\Omega}_j - \boldsymbol{\Omega}_{j-1})^2 = \frac{2}{\epsilon}(1 - \cos\psi_{j,j-1}) \tag{5.4}$$

where $\psi_{j,j-1}$ is the angle between two vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{\Omega}_{j-1}$ on $S^{(D-1)}$. The addition theorem for $\cos\psi_{1,2}$ has the form

$$\cos\psi_{1,2} = \cos\vartheta_1^{(1)}\cos\vartheta_2^{(1)} + \sum_{m=1}^{D-2}\cos\vartheta_1^{(m+1)}\cos\vartheta_2^{(m+1)}\prod_{n=1}^{m}\sin\vartheta_1^{(n)}\sin\vartheta_2^{(n)}$$
$$+ \prod_{n=1}^{D-1}\sin\vartheta_1^{(n)}\sin\vartheta_2^{(n)}$$
$$= \cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2\left[\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\varphi_2 - \varphi_1)\right] \tag{5.5}$$

where in the last line we have inserted the spherical coordinates on $S^{(3)}$. In order to evaluate the path integral, we need the expansion theorem [41, p 980]

$$\mathrm{e}^{z\cos\psi} = \left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)\sum_{J=0}^{\infty}(J+\nu)I_{J+\nu}(z)C_J^{\nu}(\cos\psi) \tag{5.6}$$

for $\nu = (D-2)/2$. The $C_n^\nu(x)$ are Gegenbauer polynomials [41, p 1029] and the $I_\mu(z)$ are modified Bessel functions [41, p 958]. By means of the addition theorem for the hyper-spherical harmonics on the $S^{(D-1)}$-sphere [42]

$$\sum_{\mu=1}^{M} S_l^\mu(\mathbf{\Omega}') S_l^\mu(\mathbf{\Omega}'') = \frac{1}{\Omega(D)} \frac{2J+D-2}{D-2} C_J^{\frac{1}{2}(D-2)}(\cos\psi^{\prime,\prime\prime}) \tag{5.7}$$

where $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$, $M = (2J+D-2)(J+D-3)!/J!(D-2)!$, we obtain the following expansion:

$$\mathrm{e}^{z(\mathbf{\Omega}'\cdot\mathbf{\Omega}')} = \mathrm{e}^{z\cos\psi^{\prime,\prime\prime}} = 2\pi\left(\frac{2\pi}{z}\right)^{\frac{1}{2}(d-2)} \sum_{J=0}^{\infty}\sum_{\mu=1}^{M} S_J^\mu(\mathbf{\Omega}') S_J^\mu(\mathbf{\Omega}'') I_{J+\frac{1}{2}(D-2)}(z). \tag{5.8}$$

In the short-time kernel of the path integral (5.3) we consider the quantity $\mathrm{e}^{z_j\cos\psi_{j-1,j}}$ with $z_j = M/\mathrm{i}\epsilon\hbar$. Using the expansion (5.8) and the interbasis expansion (3.46) we obtain

$$\begin{aligned}
\mathrm{e}^{z_j\cos\psi_{j-1,j}} &= 2\pi\left(\frac{2\pi}{z_j}\right)^2 \sum_{Jlm} \Psi^*_{Jlm}(\chi_{j-1},\vartheta_{j-1},\varphi_{j-1})\Psi_{Jlm}(\chi_j,\vartheta_j,\varphi_j) I_{J+2}(z_j)\\
&= 2\pi\left(\frac{2\pi}{z_j}\right)^2 \sum_{Jlm}\sum_{qq'} N_{ql}^{-1*} N_{q'l}^{-1}\\
&\quad\times \Psi^*_{Jqm}(\mu_{j-1},\nu_{j-1},\varphi_{j-1};a)\Psi_{Jq'm}(\mu_j,\nu_j,\varphi_j;a) I_{J+2}(z_j)\\
&\simeq \left(\frac{2\pi\mathrm{i}\epsilon}{M}\right)^{3/2} \sum_{Jqm} \Psi^*_{Jqm}(\mu_{j-1},\nu_{j-1},\varphi_{j-1};a)\Psi_{Jq'm}(\mu_j,\nu_j,\varphi_j;a)\\
&\quad\times \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{M}{\epsilon} - \mathrm{i}\epsilon\hbar\frac{(J+2)^2-1/4}{2M}\right].
\end{aligned} \tag{5.9}$$

In the last line we have applied the asymptotic expansion $I_\nu(z) \simeq (2\pi z)^{-1/2}\,\mathrm{e}^{z-(\nu^2-1/4)/2z}$, $|z| \to \infty$ according to [43], where a Wick rotation must be performed in order that $|\arg(z)| < \pi/2$ [36]. The path integration of the propagator (5.3) starting from the hyper-spherical basis is thus reduced to a multiple integration over the orthogonal basis $\Psi_{Jqm}(\mu,\nu,\varphi;a)$, yielding

$$\begin{aligned}
K(\chi'',\chi',\vartheta'',\vartheta',\varphi'',\varphi';T) &= \sum_{Jlm} \exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\Psi^*_{Jlm}(\chi',\vartheta',\varphi')\Psi_{Jlm}(\chi'',\vartheta'',\varphi'')\\
&= \sum_{J} \exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\sum_{qm}\Psi^*_{Jqm}(\mu',\nu',\varphi';a)\Psi_{Jqm}(\mu'',\nu'',\varphi'';a)
\end{aligned} \tag{5.10}$$

for oblate elliptic coordinates, $a \geqslant 0$:

$$\begin{aligned}
&\equiv \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)\ (k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)\,\mathrm{sn}\mu\,\mathrm{dn}\nu \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\\
&\quad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\big((k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2) + \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu\dot\varphi^2\big)\right.\right.
\end{aligned}$$

$$+\frac{\hbar^2}{8M}\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{cn}^2\nu\,\mathrm{dn}^2\nu}{\mathrm{sn}^2\nu}+k^4\frac{\mathrm{sn}^2\mu\,\mathrm{cn}^2\mu}{\mathrm{dn}^2\mu}\right)\Bigg]\mathrm{d}t+\frac{\mathrm{i}\hbar T}{2M}\Bigg\} \qquad (5.11)$$

for prolate elliptic coordinates, $a \in [-1, 0]$:

$$\equiv \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\ (k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\,\mathrm{cn}\mu\,\mathrm{cn}\nu\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}\big((k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)+\,\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu\dot\varphi^2\big)$$

$$+\frac{\hbar^2}{8M}\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu}{\mathrm{cn}^2\nu}+k^4\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu}\right)\Bigg]\mathrm{d}t+\frac{\mathrm{i}\hbar T}{2M}\Bigg\}\,. \qquad (5.12)$$

In addition, we have included a direct implementation of the interbasis expansion (3.46) in the propagator. Of course, a similar relation holds for the case where the interbasis expansion with the coefficients T_{qm_2} is used.

6. Discussion and summary

In this paper we have discussed the solution of the Schrödinger equation on the three-dimensional sphere $S^{(3)}$ in terms of the two one-parametric ellipso-cylindrical coordinate systems, i.e. the oblate elliptic and prolate elliptic coordinate systems. The elliptic wavefunctions $\Psi_{Jqm}(\mu, \nu, \varphi; a)$ have been constructed in terms of the cylindrical and hyperspherical wavefunctions, respectively. The coefficients of the interbasis expansions have been found to satisfy three-term recurrence relations.

The two interbasis expansions (3.1) and (3.29) have allowed us to determine the elliptic wavefunctions to any desired order. In analogy with the corresponding wavefunctions on the two-dimensional sphere $S^{(2)}$, where *Lamé polynomials* are involved, following [17] we may call the elliptic wavefunctions *associated Lamé polynomials*. The improvement of our solution compared with [17] lies in the fact that we have not just defined a power-series expansion in an unambiguous way. Our approach at the same time provides the necessary coefficients of the interbasis expansions which allows us on the one hand to determine the elliptic wavefunctions and the corresponding observables to any desired accuracy, and on the other enables us to switch from one basis to another, as appropriate for our purposes.

We have also developed a perturbation theory for the elliptic wavefunctions Ψ_{Jqm} for the two limiting cases of the elliptic parameter a, i.e, for the cases $0 < |a| \ll 1$ and $a \gg 1$, respectively. Equations (4.9), (4.16) represent first-order perturbation theory for the elliptic wavefunctions. The perturbation theory for the elliptic parameter and the wavefunctions are considered as asymptotic expansions. For higher-order contributions, the recurrence relations (3.14), (3.15) or (3.40)–(3.43) can be used in perturbation theory by implementing them in a symbolic computer program, like REDUCE, MAPLE, or MATHEMATICA.

The elliptic observable is contained in the approximate 'third observable' which describes the diamagnetic Kepler problem (the quadratic Zeeman effect) [13–16], which is not separable. The 'third observable' can be introduced having the form [13, 14] $\hat\Lambda = 4\boldsymbol{A}^2 - 5A_z$ ($\boldsymbol{A} = \boldsymbol{M}/(J+1)^2$ is the Pauli-Runge-Lenz vector). Then, '$\hat\Lambda$ commutes with the operator $\hat{\rho}^2$ ($= (J+1)^2\{5J(J+2) - 4 + L_z^2 - 4\mathcal{L}_2\}/2$, $\rho^2 = x^2 + y^2$, $d = -\frac{5}{4}$) in an n-layer, since the operator $\hat{\Lambda}$ is an exact integral of motion for the hydrogen atom in the absence of a magnetic field. ... The eigenfunctions Ψ_{nqm} are the correct wave functions in the zeroth approximation for the hydrogen atom in a weak magnetic field' [14], and the elliptic observable can be used in the weak-field limit in an algebraic perturbation investigation according to (4.11).

Using the results for the interbasis coefficients T_{qm_2} and N_{ql} we have derived path integral identities for the propagator on $S^{(3)}$ in oblate elliptic and prolate elliptic coordinates by a simple change of basis. Furthermore, in the two identities (5.11) and (5.12) we could factorize the wavefunctions Ψ_{Jqm} according to (2.17), and separate off the φ-path integral. This would yield path integral identities for potentials according to

$$V_1(\mu,\nu) = \frac{\hbar^2}{2M}\frac{m^2 - 1/4}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu} \qquad V_2(\mu,\nu) = \frac{\hbar^2}{2M}\frac{m^2 - 1/4}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu} \qquad m \in \mathbb{Z} \tag{6.1}$$

and m^2 can be continued analytically to any real number [34, 43]. However, this is obvious, and we omit the explicit formulae, cf [29], for the two spheroidal cases in $\mathbb{R}^3$.

It is legitimate to ask whether the two path integral identities (5.11) and (5.12) are of use from a practical point of view. Here one must keep in mind that all known path integral relations and identities require approximate knowledge of the expansion formulae, where some kind of higher transcendental functions are involved. Calculation of the radial harmonic oscillator requires the expansion of plane waves into a product of circular waves, i.e. spherical harmonics, and Bessel functions [43], while calculation of the path integral for the (modified) Pöschl–Teller potential requires knowledge of expansion theorems on the SU(2) and SU(1, 2) group manifolds [34]. In the cases of path integration on homogeneous spaces in terms of (one-) parametric coordinate systems, i.e. elliptic and spheroidal coordinates in Euclidean space [29], where Matthieu and spheroidal wavefunctions are involved, or elliptic (conical) [29] and ellipso-cylindrical coordinates on spheres, where Lamé and associated Lamé polynomials are involved, one has to accept the fact that these unusual wavefunctions are simply another kind of higher transcendental special function. Unfamiliarity does not imply unreasonability.

We have not explicitly included in our discussion the dependence on the radius R of the three-dimensional sphere. In the flat-space limit the following identities must emerge (up to normalization, $\delta = pd/2$, $R \to \infty$, $k \to 0$, where $Rk \equiv d$ is fixed):

$$\begin{aligned} \psi_1(\mu;k) &\to \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,\mathrm{ps}_l^m(\vartheta;\delta^2) \\ \psi_2(\nu;k) &\to \sqrt{\frac{2}{\pi}}\,pS_l^{m\,(1)}(\cosh\xi;\delta). \end{aligned} \tag{6.2}$$

Here ps_l^n and $S_l^{n\,(1)}$ are prolate spheroidal wavefunctions in terms of spheroidal coordinates ξ, ϑ in $\mathbb{R}^3$ [44]. This would require an expansion of the quantity ($\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^4$, where for $\boldsymbol{x}$ appropriate prolate elliptic and for $\boldsymbol{p}$ spherical coordinates are used)

$$\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = \exp\left[\mathrm{i}pR\big(\sin\chi\sin\vartheta\,\mathrm{sn}\mu\,\mathrm{dn}\nu\cos(\varphi-\psi) + \sin\chi\cos\vartheta\,\mathrm{cn}\mu\,\mathrm{cn}\nu + \cos\chi\,\mathrm{dn}\mu\,\mathrm{sn}\nu\big)\right] \tag{6.3}$$

in terms of the ellipso-cylindrical bases Ψ_{Jqm}, as $R \to \infty$. However, such considerations go beyond the scope of the present paper and will be discussed elsewhere.

It is obvious that our methods lead to further investigations: (1) determination of the interbasis expansion for ellipsoidal coordinates on the three-dimensional sphere;

(2) determination of the interbasis expansion for the analogous one-parametric coordinate systems on the three-dimensional hyperboloid; (3) calculation of matrix elements for the diamagnetic Kepler problem. This would also include a detailed investigation of the limiting cases, as $R \to \infty$. We hope to return to these topics in the future.

Acknowledgments

The authors gratefully acknowledge financial support from the Heisenberg–Landau programme. GSP would like to thank the members of the II. Institut für Theoretische Physik, Hamburg University, for their kind hospitality. GSP and KhHK are grateful to V M Ter-Antonyan and L G Mardoyan for useful discussions.

Appendix A. Matrix elements in the cylindrical basis

The wavefunctions in the cylindrical coordinates have the following form:

$$\Psi_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2) = \frac{e^{im_1\varphi_1}}{\sqrt{2\pi}} C^J_{m_1m_2}(\gamma) \frac{e^{im_2\varphi_2}}{\sqrt{2\pi}} \tag{A.1}$$

and are normalized according to

$$\int_0^{\pi/2} \sin\gamma\cos\gamma\,d\gamma \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2\ \Psi^*_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2)\Psi_{J'm_1'm_2'}(\gamma,\varphi_1,\varphi_2)$$
$$= \delta_{JJ'}\delta_{m_1m_1'}\delta_{m_2m_2'}. \tag{A.2}$$

The cylindrical wavefunction $C^J_{m_1m_2}(\gamma)$ is given by

$$C^J_{m_1m_2}(\gamma) = \sqrt{\frac{2(J+1)\left(\frac{1}{2}(J-|m_1|-|m_2|)\right)!\left(\frac{1}{2}(J+|m_1|+|m_2|)\right)!}{\left(\frac{1}{2}(J-|m_1|+|m_2|)\right)!\left(\frac{1}{2}(J+|m_1|-|m_2|)\right)!}}$$
$$\times (\sin\gamma)^{|m_1|}(\cos\gamma)^{|m_2|} P^{(|m_1|,|m_2|)}_{(J-|m_1|-|m_2|)/2}(\cos 2\gamma) \tag{A.3}$$
$$= \sqrt{2(J+2)}\, d^{J/2}_{\frac{1}{2}(m_2+m_1),\ \frac{1}{2}(m_1-m_2)}(\cos 2\gamma). \tag{A.4}$$

The $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials [41, p 1035], and the $d^n_{\kappa,\lambda}(x)$ are Wigner functions [32]. We have for the quantum numbers $m_{1,2} \in \mathbb{Z}$, where for fixed J they vary according to $-J \leqslant m_1 + m_2 \leqslant J$, and $(J - |m_1| - |m_2|)$ is even. The operators L_1, L_2, L_3 are given explicitly by

$$L_1 = -\frac{\hbar}{i}\left(\sin\varphi_1\sin\varphi_2\frac{\partial}{\partial\gamma} - \tan\gamma\sin\varphi_1\cos\varphi_2\frac{\partial}{\partial\varphi_2} + \cot\gamma\cos\varphi_1\sin\varphi_2\frac{\partial}{\partial\varphi_1}\right)$$
$$L_2 = \frac{\hbar}{i}\left(\cos\varphi_1\sin\varphi_2\frac{\partial}{\partial\gamma} - \tan\gamma\cos\varphi_1\cos\varphi_2\frac{\partial}{\partial\varphi_2} - \cot\gamma\sin\varphi_1\sin\varphi_2\frac{\partial}{\partial\varphi_1}\right) \tag{A.5}$$
$$L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\varphi_1}.$$

The functions (A.1) are eigenfunctions of the set of the three operators $\{\mathcal{L}_2, M_3^2, L_3^2\}$:

$$\begin{aligned}\mathcal{L}_1\Psi_{Jm_1m_2} &= \hbar^2 J(J+2)\Psi_{Jm_1m_2}\\ L_3^2\Psi_{Jm_1m_2} &= \hbar^2 m_1^2\Psi_{Jm_1m_2}\\ M_3^2\Psi_{Jm_1m_2} &= \hbar^2 m_2^2\Psi_{Jm_1m_2}.\end{aligned} \tag{A.6}$$

The matrix elements of the operators $L_\pm$ have the following form ($L_\pm = L_1 \pm \mathrm{i}L_2$):

$$\langle Jm_1'm_2'|L_+|Jm_1m_2\rangle = -\frac{\hbar}{2\mathrm{i}}\sqrt{(J-m_1-m_2)(J+m_1+m_2+2)}\,\delta_{m_1',m_1+1}\delta_{m_2',m_2+1} + \frac{\hbar}{2\mathrm{i}}\sqrt{(J-m_1+m_2)(J+m_1-m_2+2)}\,\delta_{m_1',m_1+1}\delta_{m_2',m_2-1} \tag{A.7}$$

$$\langle Jm_1'm_2'|L_-|Jm_1m_2\rangle = -\frac{\hbar}{2\mathrm{i}}\sqrt{(J+m_1-m_2)(J-m_1+m_2+2)}\,\delta_{m_1',m_1-1}\delta_{m_2',m_2+1} + \frac{\hbar}{2\mathrm{i}}\sqrt{(J+m_1+m_2)(J-m_1-m_2+2)}\,\delta_{m_1',m_1-1}\delta_{m_2',m_2-1}. \tag{A.8}$$

Appendix B. Matrix elements in the hyper-spherical basis

The wavefunctions in spherical coordinates have the following form:

$$\Psi_{Jlm}(\chi,\vartheta,\varphi) = S_{Jl}(\chi)Y_l^m(\vartheta,\varphi) \tag{B.1}$$

where the $Y_l^m(\vartheta,\varphi)$ are the usual spherical harmonics, and the $S_{Jl}(\chi)$ are given by Gegenbauer polynomials, i.e.

$$S_{Jl}(\chi) = 2^l l!\sqrt{\frac{2(J+1)(J-l)!}{\pi(J+l+1)!}}\,(\sin\chi)^l C_{J-l}^{l+1}(\cos\chi). \tag{B.2}$$

They are normalized according to

$$\int_0^\pi \sin^2\chi\ \mathrm{d}\chi \int_0^\pi \sin\vartheta\ \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\ \Psi_{Jlm}^*(\chi,\vartheta,\varphi)\Psi_{J'l'm'}(\chi,\vartheta,\varphi) = \delta_{JJ'}\delta_{ll'}\delta_{mm'}. \tag{B.3}$$

The functions (A.1) are eigenfunctions of the set of the three operators $\{\mathcal{L}_2, \boldsymbol{L}^2, L_3^2\}$:

$$\begin{aligned}\mathcal{L}_1\Psi_{Jlm} &= \hbar^2 J(J+2)\Psi_{Jlm}\\ \boldsymbol{L}^2\Psi_{Jlm} &= \hbar^2 l(l+1)\Psi_{Jlm}\\ L_3^2\Psi_{Jlm} &= \hbar^2 m^2\Psi_{Jlm}.\end{aligned} \tag{B.4}$$

The operators $M_{1,2,3}$ in spherical coordinates have the form

$$\begin{aligned}M_1 &= \frac{\hbar}{\mathrm{i}}\left(\sin\vartheta\cos\varphi\frac{\partial}{\partial\chi} + \cot\chi\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - \cot\chi\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\\ M_2 &= \frac{\hbar}{\mathrm{i}}\left(\sin\vartheta\sin\varphi\frac{\partial}{\partial\chi} + \cot\chi\cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} + \cot\chi\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\\ M_3 &= \frac{\hbar}{\mathrm{i}}\left(\cos\vartheta\frac{\partial}{\partial\chi} - \cot\chi\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\\ M_\pm &= M_1 \pm \mathrm{i}M_2 = \frac{\hbar}{\mathrm{i}}\mathrm{e}^{\pm\mathrm{i}\varphi}\left(\sin\vartheta\frac{\partial}{\partial\chi} + \cot\chi\cos\vartheta\frac{\partial}{\partial\vartheta} \pm \mathrm{i}\frac{\cot\chi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right).\end{aligned} \tag{B.5}$$

The corresponding matrix elements are given by

$$\langle Jl'm'|M_+|Jlm\rangle = \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l+m+1)(l+m+2)(J+l+2)(J-l)}{(2l+1)(2l+3)}}\,\delta_{l',l+1}\delta_{m',m+1} + \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m-1)(l-m)(J+l+1)(J-l+1)}{(2l+1)(2l-1)}}\,\delta_{l',l-1}\delta_{m',m+1} \tag{B.6}$$

$$\langle Jl'm'|M_-|Jlm\rangle = \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m+1)(l-m+2)(J+l+2)(J-l)}{(2l+1)(2l+3)}}\,\delta_{l',l+1}\delta_{m',m-1} - \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l+m+1)(l+m)(J+l+1)(J-l+1)}{(2l+1)(2l-1)}}\,\delta_{l',l-1}\delta_{m',m-1} \tag{B.7}$$

$$\langle Jl'm|M_3|Jlm\rangle = -\frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m+1)(l+m+1)(J+l+2)(J-l)}{(2l+1)(2l+3)}}\,\delta_{l',l+1} + \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m)(l+m)(J+l+1)(J-l+1)}{(2l+1)(2l-1)}}\,\delta_{l',l-1}. \tag{B.8}$$

References

[1] Olevskiĭ M N 1950 Triorthogonal systems in spaces of constant curvature in which the equation $\Delta_2 u+\lambda u = 0$ allows the complete separation of variables *Math. Sb.* **27** 379–426 (in Russian)

[2] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw Hill)

[3] Evans N W 1990 Superintegrability in classical mechanics *Phys. Rev.* A **41** 5666–76

[4] Friš J, Mandrosov V, Smorodinsky Ya A, Uhlir M and Winternitz P 1965 On higher symmetries in quantum mechanics *Phys. Lett.* **16** 354–6
Friš J, Smorodinskiĭ Ya A, Uhlíř M and Winternitz P 1967 Symmetry groups in classical and quantum mechanics *Sov. J. Nucl. Phys.* **4** 444–50

[5] Makarov A A, Smorodinsky J A, Valiev Kh and Winternitz P 1967 A systematic search for nonrelativistic systems with dynamical symmetries *Nuovo Cimento* A **52** 1061–84

[6] Winternitz P, Smorodinskiĭ Ya A, Uhlir M and Fris I 1967 Symmetry groups in classical and quantum mechanics *Sov. J. Nucl. Phys.* **4** 444–50

[7] Grosche C, Pogosyan G S and Sissakian A N 1995 Path integral discussion for Smorodinsky–Winternitz potentials: I. Two- and three-dimensional Euclidean space *Fortschr. Phys.* **43** 453–521

[8] Grosche C, Pogosyan G S and Sissakian A N 1995 Path integral discussion for Smorodinsky–Winternitz potentials: II. The two- and three-dimensional sphere *Fortschr. Phys.* **43** 523–63

[9] Grosche C, Pogosyan G S and Sissakian A N 1996 Path integral discussion for Smorodinsky–Winternitz potentials: III. The two-dimensional hyperboloid *Phys. Part. Nucl.* **27** 244–78

[10] Grosche C, Pogosyan G S and Sissakian A N 1995 Path integral discussion for Smorodinsky–Winternitz potentials: IV. The three-dimensional hyperboloid *DESY Report* 130 pp (*Phys. Part. Nucl.* to appear)

[11] Bander M and Itzykson C 1968 Group theory and the hydrogen atom (I) *Rev. Mod. Phys.* **38** 330–45
Bander M and Itzykson C 1968 Group theory and the hydrogen atom (II) *Rev. Mod. Phys.* **38** 346–58

[12] Lakshmann M and Hasegawa H 1984 On the canonical equivalence of the Kepler problem in coordinate and momentum space *J. Phys. A: Math. Gen.* **17** L889–93

[13] Herrick D R 1982 Symmetry of the quadratic Zeemann effect for hydrogen *Phys. Rev.* A **26** 323–329

[14] Solov'ev E A 1982 The hydrogen atom in a weak magnetic field *Sov. Phys.–JETP* **55** 1017–22

[15] Brown P A and Solov'ev E A 1984 The stark effect for the hydrogen atom in a magnetic field *Sov. Phys.–JETP* **59** 38–46

[16] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (Berlin: Springer)

[17] Kalnins E G, Miller M Jr and Winternitz P 1976 The group O(4), separation of variables and the hydrogen atom *SIAM J. Appl. Math.* **30** 630–64

[18] Bessis G and Bessis N 1979 Electronic wavefunctions in a space of constant curvature *J. Phys. A: Math. Gen.* **12** 1991–7

[19] Otchik V S 1991 Symmetry and separation of the Coulomb two-center problem in three-dimensional space with constant curvature *Dokl. AN BSSR* **35** 420–4 (in Russian)
Otchik V S 1994 On the two Coulomb centres problem in a spherical geometry *Proc. Int. Workshop on Symmetry Methods in Physics in Memory of Professor Ya A Smorodinsky (Dubna, 1993)* ed G S Pogosyan, A N Sissakian and S I Vinitsky (Dubna: JINR Publications) pp 384–8

[20] Neumann C 1859 De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur *J. Reine Angew. Math.* **56** 46–63

[21] Babelon O and Talon M 1992 Separation of variables for the classical and Neumann model *Nucl. Phys.* B **379** 321–39

[22] Lukàčs I 1977 Complete sets of observables on the sphere in four-dimensional Euclidean space *Theor. Math. Phys.* **31** 457–61

[23] Lukàčs I 1973 A complete set of the quantum-mechanical observables on a two-dimensional sphere *Theor. Math. Phys.* **14** 271–81

[24] Patera J and Winternitz P 1973 A new basis for the representations of the rotation group. Lamé and Heun polynomials *J. Math. Phys.* **14** 1130–9

[25] Coulson A C and Joseph A 1967 Spheroidal wave functions for the hydrogen atom *Proc. Phys. Soc.* **90** 887–93

[26] Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1985 Spheroidal corrections to the spherical and parabolic bases of the hydrogen atom *Theor. Math. Phys.* **64** 762–4

[27] Kibler M, Mardoyan L G and Pogosyan G S 1994 On a generalized Kepler–Coulomb system: interbasis expansions *Int. J. Quantum Chem.* **52** 1301–16

[28] Davtyan L S, Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1987 Spheroidal basis of the four-dimensional isotropic oscillator *JINR Communications* P2-87-453, 15pp, unpublished (in Russian)

[29] Grosche C 1996 *Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae* (Singapore: World Scientific)

[30] Kibler M and Grenet G 1980 On the SU_2 unit tensor *J. Math. Phys.* **21** 422–39

[31] Stone A P 1956 Some properties of Wigner coefficients and hyperspherical harmonics *Proc. Camb. Phil. Soc.* **52** 424–30

[32] Varshalovich D A, Moskalev A N and Khersonskii V K 1988 *Quantum Theory of Angular Momentum* (Singapore: World Scientific)

[33] Landau L D and Lifshitz 1958 *Quantum Mechanics* (Reading, MA: Addison-Wesley)

[34] Böhm B and Junker G 1987 Path integration over compact and noncompact rotation groups *J. Math. Phys.* **28** 1978–94

[35] Feynman R P and Hibbs A 1965 *Quantum Mechanics and Path Integrals* (Singapore: World Scientific)

[36] Grosche C and Steiner F 1987 Path integrals on curved manifolds *Z. Phys.* C **36** 699–714

[37] Grosche C and Steiner F 1995 How to solve path integrals in quantum mechanics *J. Math. Phys.* **36** 2354–85

[38] Grosche C and Steiner F 1997 Table of Feynman path integrals *Springer Tracts in Modern Physics* (Berlin: Springer) to appear

[39] Kleinert K 1990 *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (Singapore: World Scientific)

[40] Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)

[41] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)

[42] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G (ed) 1995 *Higher Transcendental Functions* vol II (New York: McGraw-Hill)

[43] Peak D and Inomata A 1969 Summation over Feynman histories in polar coordinates *J. Math. Phys.* **10** 1422–28

[44] Meixner J and Schäfke F W 1954 *Mathieusche Funktionen und Sphäroidfunktionen* (Berlin: Springer)