DESY 96–242
November 1996

ISSN 0418 - 9833

PATH INTEGRAL DISCUSSION FOR SUPER-INTEGRABLE POTENTIALS: IV. THE THREE-DIMENSIONAL HYPERBOLOID

C. Grosche

Institut für Theoretische Physik
Technische Universität Clausthal, Sommerfeldstraße 6
38678 Clausthal-Zellerfeld, Germany
and
II. Institut für Theoretische Physik
Universität Hamburg, Luruper Chaussee 149
22761 Hamburg, Germany

G. S. Pogosyan and A. N. Sissakian

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research (Dubna)
141980 Dubna, Moscow Region, Russia

ABSTRACT

In our fourth paper on super-integrable potentials on spaces of constant curvature we discuss the case of the three-dimensional hyperboloid. Whereas in many coordinate systems an explicit path integral solution for the corresponding potential is not possible, we list in the soluble cases the path integral solutions explicitly in terms of the propagators, Green functions, and the spectral expansions into the wave-functions.

We discuss in some detail the formulation and construction of coordinate systems on the three-dimensional hyperboloid, which includes the statement of the Schrödinger operator, the general form a potential must have to be separable in the coordinate system, and the relevant observables.

Some special care is taken for the proper generalization of the harmonic oscillator and the Kepler problem. We find the analogues of the maximally and minimally super-integrable potentials of $\mathbb{R}^3$ on the hyperboloid, and many minimally super-integrable potentials which emerge from the subgroup chains corresponding to $SO(3,1)$. In an appendix we give a list of super-integrable potentials in the spaces of constant curvature in two and three dimensions, including the corresponding observables.

Contents

List of Tables

1 Introduction

1.1 Motivation and Symmetry Methods in Physics.

The present paper is the fourth in a sequel concerning super-integrable potentials in spaces of constant curvature. It continues our studies which started from the investigation in two- and three-dimensional Euclidean space, i.e., in $\mathbb{R}^2$ and $\mathbb{R}^3$, on the two- and three-dimensional sphere $S^{(2)}$ and $S^{(3)}$, and on the two-dimensional hyperboloid $\Lambda^{(2)}$. Our goal is devoted to the study of physical systems in spaces of constant curvature which have *accidental degeneracies*, i.e., systems which have due to their peculiar features a so-called *hidden* or *dynamical* group structure, thus giving rise to degeneracies in the energy spectrum, and additional integrals of motion, respectively observables.

The most well-known of these kinds of potential systems in three-dimensional flat space are the harmonic oscillator with quantum energy spectrum

$$E_N = \hbar\omega(N + \tfrac{3}{2}) \ , \qquad N \in \mathbb{N}_0 \ , \tag{1.1}$$

and the Kepler-Coulomb problem with the quantum energy spectrum

$$E_N = -\frac{Me^4}{2\hbar^2(N+1)^2} \ , \qquad N \in \mathbb{N}_0 \ . \tag{1.2}$$

Here, N denotes the principal quantum number, and for fixed N each level E_N for the oscillator is $(N+1)(N+2)/2$-fold degenerate, and in the Coulomb problem $(N+1)^2$-fold degenerate.

The particular symmetry features have the consequence that there are additional constants of motion in classical mechanics, respectively observables in quantum mechanics. In comparison, the orbits of a simple integrable system, e.g., a three-dimensional anharmonic oscillator, are generally only periodic with respect to each coordinate, but not globally[1]. For a physical system in D dimensions just to be integrable, a number of D constants of motion is required, with one of them the energy E. In classical mechanics these constants of motion have vanishing Poisson brackets with the Hamiltonian, and with each other; in quantum mechanics they are operators which commute with the quantum Hamiltonian and with each other. For instance, for a spherical symmetric system, the constants of motion are the energy E, the square of the total angular momentum $\mathbf{L}^2$, and the square of the (usually chosen) z-component of the angular momentum L_z^2, in classical mechanics as well as in quantum mechanics.

In systems like the isotropic harmonic oscillator or the Kepler-Coulomb problem in three dimensions, there are two more functionally independent constants of motion. In the case of the harmonic oscillator the additional constants of motion correspond to the conservation of the quadrupol moment, the so-called Demkov tensor $T_{ik} = p_i p_k + \omega^2 x_i x_k$ [37], and in the case of the Kepler-Coulomb problem the conservation of the square of another component of the angular momentum and the third component of the Pauli-Lenz-Runge vector $\mathbf{A} = \frac{1}{2M}(\mathbf{L}\times\mathbf{P} - \mathbf{P}\times\mathbf{L}) - e^2\mathbf{x}/|\mathbf{x}|$, and both systems have five constants of motion, respectively observables.

A more careful investigation shows that the highly spherical symmetric systems of the isotropic harmonic oscillator and the Kepler-Coulomb problem can be perturbed in various ways by the incorporation of additional potential terms: First, this does not spoil the degeneracy of the energy levels at all, i.e., there are still five observables[2]; second, one of the observables is removed, i.e., they are four left, and third, only the minimum number of three observables

[1]However, they are periodic globally if the frequencies $\omega_{1,2,3}$ are commensurable, i.e., if their respective quotients are rational numbers.

[2]In the sequel we use the notions "energy levels" and "periodicity of closed orbits", "observables" and "constants of motion", "Coulomb-" or "Kepler-problem", referring to quantum mechanical or classical mechanical properties, respectively, as synonymous.

for integrability remains. The first possibility is described by the notion of a *maximally super-integrable* system, the second possibility by the notion of a *minimally super-integrable* systems, and the last possibility just describes an *integrable* system. If the system in three dimensions has less than three observables it is not integrable, and can be referred as a *chaotic* system, for instance the anisotropic Kepler problem, c.f. [17, 94, 97, 177], or the Coulomb potential with a magnetic field, e.g. [94].

In this respect, the physical significance of the consideration of separation of variables in several coordinate systems is as follows. The free motion in some space is, of course, the most symmetric one, and the search for the number of coordinate systems which allow the separation of the Hamiltonian is equivalent to the investigation how many inequivalent sets of observables can be found. In particular, the free motion in various coordinate systems on the hyperboloid has been studied in Refs. [63, 70, 75, 87, 89, 134]. The incorporation of potentials usually removes at least some of the symmetry properties of the space. Well-known examples are spherically symmetric systems, and they are most conveniently studied in spherical coordinates.

All the super-integrable systems have the particular property that all the energy-levels of the system are organized in representations of the non-invariance group which contain representations of the dynamical subgroup realized in terms of the wave-functions of these energy-levels [51]. In the case of the hydrogen it enabled Pauli [168], Fock [49] and Bargmann [5] to solve the quantum mechanical Kepler problem without explicitly solving the Schrödinger equation. The additional integrals of motion also have the consequence that in the case of the super-integrable systems in two dimensions and maximally super-integrable systems in three dimensions all finite trajectories are found to be periodic; in the case of minimally super-integrable systems in three dimensions all finite trajectories are found to be quasi-periodic[3] [122]. Of course, in the case of the pure Kepler or the isotropic harmonic oscillator all finite trajectories are periodic.

Generally, a physical system in D dimensions is called *minimally super-integrable* if it has $2D-2$ integrals of motion, and it is called *maximally super-integrable* if it has $2D-1$ integrals of motion, respectively observables. Therefore we are led to the search of more (potential-) systems which have similar features concerning degeneracy and number of observables as the isotropic harmonic oscillator and the Coulomb problem.

Attempts to classify separable potentials along these lines in general go back to Eisenhart [43], Aly and Spector [2], Bose [15], and Luming and Predazzi [142], however with the emphasize on "solvable potentials" in quantum mechanics. A systematic study was undertaken by Smorodinsky, Winternitz and co-workers [51, 148, 193], i.e., they looked for potentials which are separable in more than one coordinate system. The separation of a quantum mechanical potential problem in more than one coordinate systems has the consequence that there are additional integrals of motion and that the spectrum is degenerate. The Noether theorem [161] connects the particular symmetries of a Lagrangian, i.e., the invariances with respect to the dynamical symmetries, with conservation laws in classical mechanics and with observables in quantum mechanics, respectively. The choice of a coordinate system then emphasizes which observables are considered to be the most appropriate for a particular investigation.

1.2 Super-Integrable Systems.

The harmonic oscillator in various coordinate systems has been studied by Evans [44], Kallies et al. [111], and Pogosyan et al. [171]; in spaces of constant curvature by e.g. by Bonatsos et al. [14], Higgs [99], Lemon [137], Granovsky et al. [59] and [81, 82]. The Coulomb-Kepler problem in flat space in various coordinate systems has been studied by many authors including, e.g., Cisneros and McIntosh [28], Coulson et al. [30, 31] concerning the Coulomb problem in spheroidal

[3] The notion quasi-periodic means that they are periodic in each coordinate, but not necessarily periodic in a global way. They are periodic globally if the respective periods are commensurable.

coordinates, Davtyan et al. [33]–[36], Fock [49], Guha and Mukherjee [93], Hartmann [95], Higgs [99] and Lemon [137], Hodge [100], Kibler et al. [122]-[128], Lutsenko et al. [144], Pauli [168] with his famous solution based on the algebra without solving the Schrödinger equation explicitly, Teller [184], who first dealt with the two-center Coulomb problem, Vaidya and Boschi-Filho [187], and Zhedanov [197], and in spaces of constant curvature by Barut et al. [9] in the general context of the $SO(4,2)$ dynamical algebra, Granovsky et al. [60], Hietarinta [98], Katayama [120], Mardoyan et al. [149, 151], Otchik and Red'kov [165], Pogosyan et al. [171], Schrödinger [173], Stevenson [182] and Vinitsky et al. [190].

Corresponding path integral discussions are due to, e.g., Carpio-Bernido et al. [18]–[21] for the generalized Kepler-Coulomb problem and the Hartmann potential, Chetouani at al. [24]–[27] concerning parabolic coordinates and the D-dimensional Coulomb problem, Duru and Kleinert [42] with the first evaluation of the Coulomb problem within the path integral formalism by means of the Kustaanheimo-Stiefel transformation, Gerry [53], Refs. [62, 72] and in particular Refs. [68, 69, 71, 80] concerning parabolic coordinates, modified Coulomb-potentials in two and three-dimensions, and the general aspect of separation of variables and super-integrable potentials, Inomata [105] with a careful lattice investigation of the Kustaanheimo-Stiefel transformation, and by Sökmen [175] and Steiner [180] in the context of space-time transformations in polar coordinates.

The notion of "super-integrability" [44, 118, 194] can now be introduced in spaces of constant curvature [81, 82]. Whereas the general form of potentials which are "super-integrable" in some kind is not clear until now, one knows that the corresponding Higgs-oscillator (c.f. Bonatsos et al. [14], Granovsky et al. [59], Higgs [99], Ikeda and Katayama [102], Katayama [120]), Leemon [137], Nishino [160], and Pogosyan et al. [171]) and Kepler problems (c.f. Granovsky et al. [60], Infeld [103], Infeld and Schild [104], Kalnins et al. [119], Kibler et al. [123], Kurochkin and Otchik [135], Nishino [160], Otchik and Red'kov [165], Vinitsky et al. [190, 191], Zhedanov [196]) in spaces of constant curvature do have additional constants of motion: the analogues of the flat space. For the Higgs-oscillator this is the Demkov-tensor [37, 58, 160] and in the Kepler problem a Pauli-Runge-Lenz vector on spaces of constant curvature can be defined, c.f. [60, 99, 135, 137, 160]. Corresponding path integral considerations are due to Barut et al. [7, 8], Otchik and Red'kov [165], and [65] (D-dimensional case), and [81] (super-integrable aspects).

It is interesting to remark that the Kaluza-Klein monopole system, e.g. [55], is a super-integrable system as well. Its observables can be constructed in complete analogy to the observables in the Kepler-Coulomb system [29, 121], i.e., a Pauli-Lenz-Runge vector exists. Path integral discussions are due to Bernido [10] and Inomata and Junker [106] in spherical coordinates, and due to Ref. [67] in parabolic coordinates. However, a system which is separable in spherical and parabolic coordinates is also separable in prolate spheroidal coordinates, similarly as the Coulomb or the Hartmann potential. In [83] we have analysed the corresponding interbasis expansions of the parabolic basis with respect to the spherical basis, and in [84] we will analyse the spheroidal basis with respect to the spherical basis, such that all three coordinate bases of the Kaluza-Klein monopole system are explicitly known, and can be transformed into each other.

Disturbing the spherical symmetry usually spoils it. The first step consists of deforming the ring-shaped feature of the (maximally super-integrable) modified oscillator and Coulomb potential. One gets in the former a ring-shaped oscillator and in the latter the Hartmann potential, two minimally super-integrable systems. The number of coordinate systems which allow a separation of variables drop from eight to four (namely spherical, circular polar, oblate spheroidal and prolate spheroidal coordinates Kibler et al. [123, 128]), and from four to three, namely spherical, parabolic, and prolate spheroidal II coordinates. The ring-shaped oscillator has been discussed by, e.g., Carpio-Bernido et al. [20, 21], Kibler et al. [122, 125, 127], Lutsenko et al. [143], and Quesne [172]. The Hartmann system has been discussed by, e.g., Carpio-Bernido et al. [18]–[22], Chetouani at al. [24], Gal'bert et al. [52], Gerry [53], Granovsky et al. [58], [68], Guha and Mukherjee [93], Hartmann [95], Kibler et al. [122]-[125], [127], Lutsenko et al. [144], Vaidya and Boschi-Filho [187], and Zhedanov [197]; compare also the connection with a Coulomb plus Aharonov-Bohm solenoid [1], e.g., Chetouani et al. [25], Kibler and Negadi [126], and Sökmen [175].

Disturbing the system further, and one is left with, say, one coordinate systems which still allows separation of variables. A constant electric field (Stark effect) allows only the separation in parabolic coordinates [68]. Here it is interesting to remark that in the momentum representation of the hydrogen atom the bound state spectrum is described by the free motion on the sphere $S^{(3)}$. To be more precise, the dynamical group O(4) describes the discrete spectrum, and the Lorentz group O(3,1) the continuous spectrum [4]. Now, there are six coordinate systems on $S^{(3)}$ which separate the corresponding Laplacian. The solution in spherical and cylindrical coordinates corresponds to the spherical and parabolic solution in the coordinate space representation. The elliptic cylindrical system is of special interest because it enables one to set up a complete classification for the energy-levels of the quadratic Zeeman effect (c.f. Solov'ev [177], Brown and Solov'ev [17], Herrick [97], Lakshmann and Hasegawa [136]).

The separation in parabolic coordinates is also possible in the case of a perturbation of the pure Coulomb field with a potential force $\propto z/r$ which allows an exact solution [69, 71]. The two-center Coulomb problem turns out to separable only in spheroidal coordinates (Coulson and Josephson [30], Coulson and Robinson [31], Morse [157]) as has been studied first in the connection with the hydrogen-molecule ion by Teller [184].

Another possibility to disturb the spherical symmetry is to remove the invariance to rotations with respect to some axis, e.g., about a uniform magnetic field. Usually this invariance is used to illustrate the azimuthal quantum number m of the L_z operator. The physical meaning of this quantum number then is that there exists a preferred axis in space. This symmetry can be broken if one considers a Hamiltonian of a nucleus with an electric quadrupole moment Q and spin J in a spatially varying electric field [139, 178]. Here sphero-conical coordinates are most convenient, and the projection of the terminus of the angular momentum vector traces out a cone of elliptic cross section about the z-axis [178]. Also the problem of the asymmetric top (Kramers and Ittmann [133], Lukač [138], Smorodinsky et al. [140, 176, 192]), the symmetric oblate top [138], or the case of tensor-like potentials (Lukač and Smorodinsky [141]) can be treated best in sphero-conical coordinates. Therefore sphero-conical coordinates are most suitable for problems which have spherical symmetry but not a sphero-axial symmetry.

In order that a potential problem can be separated in ellipsoidal coordinates is that the shape of the potential resembles the shape of a ellipsoid. Of course, the anisotropic harmonic oscillator belongs to this class. Introducing quartic and sextic [186] interaction terms then eventually allows only a separation of variables in ellipsoidal coordinates. Another example is the Neumann model [159], which describes a particle moving on a sphere subject to anisotropic harmonic forces (Babelon and Talon [3], and MacFarlane [145]).

The case of magnetic fields on the two-dimensional hyperboloid has been considered by means of path integrals in [64], and it has been found that in spherical, horicyclic and equidistant coordinates a separation of variables is possible, i.e., in coordinate systems which have one ignorable coordinate [117], and the corresponding solutions are circular, respectively plane waves in this (ignorable) coordinate. Depending on the strength of the magnetic field a finite number of bound states can exist, with energy levels given by ($b = eMB/2c\hbar > 0$, e the electric charge, B the magnetic field strength, and c the velocity of light), e.g. [96]

$$E_n = \frac{\hbar^2}{2M}\Big[b^2 + \tfrac{1}{4} - (b - n - \tfrac{1}{2})^2\Big] \ , \qquad n = 0, \dots, N_M < b - \tfrac{1}{2} \ . \tag{1.3}$$

Such investigations play an important rôle in the theory of tensor-weighted Laplacians, automorphic forms, determinants of Laplacians and zeta-function regularization, and quantum field theory on (super-) Riemann surfaces, e.g. [75] and references therein.

Our first paper [80] dealt with super-integrable potentials in two- and three-dimensional flat space, where we distinguished minimally and maximally super-integrable systems. In two dimensional Euclidean space they are four (maximally) super-integrable systems [44], i.e., the (generalized) harmonic oscillator $V_1(\mathbf{x})$, the Holt-potential $V_2(\mathbf{x})$, the (generalized) Coulomb potential $V_3(\mathbf{x})$, and a modified Coulomb potential $V_4(\mathbf{x})$.[4]

In three-dimensional Euclidean space there has been found five maximally and nine minimally super-integrable systems. Among the maximally super-integrable are the (generalized) harmonic oscillator $V_1(\mathbf{x})$, the Holt-potential in $\mathrm{I\!R}^3$, $V_2(\mathbf{x})$, and the (generalized) Coulomb potential $V_3(\mathbf{x})$; among the minimally super-integrable systems were a double-ring shaped oscillator $V_6(\mathbf{x})$, the Hartmann potential $V_7(\mathbf{x})$, a three-dimensional analogue of the Holt-potential $V_6(\mathbf{x})$, four potentials $V_2(\mathbf{x}), V_3(\mathbf{x}), V_4(\mathbf{x}), V_8(\mathbf{x})$ which emerged from the group chain $E(3) \supset E(2)$, i.e., they are super-integrable in $\mathrm{I\!R}^2$, and the two potentials $V_1(\mathbf{x}), V_9(\mathbf{x})$ which emerged from the group chain $E(3) \supset \mathrm{SO}(3)$, i.e., they are super-integrable on the two-dimensional sphere $S^{(2)}$. In appendix D we give in tables D.1–D.3 a list of the super-integrable potentials in $\mathrm{I\!R}^2$ and $\mathrm{I\!R}^3$, where we list in each case the separating coordinate systems, and the corresponding integrals of motion. In the <u>underlined</u> coordinate systems an explicit path integral solution has been found.

In our second paper [81] we continued our study on the two- and three-dimensional sphere. On $S^{(2)}$ we found only two potentials with the required properties, i.e., the (generalized) Higgs oscillator $V_1(\mathbf{s})$ and the (generalized) Coulomb potential $V_2(\mathbf{s})$. We have not been able to find the super-integrable analogues of the Holt-potential and the modified Coulomb potential. On the three-dimensional sphere $S^{(3)}$ we have found three maximally super-integrable and four minimally super-integrable potentials, respectively. Among the maximally super-integrable potentials were the (generalized) Higgs oscillator $V_1(\mathbf{s})$, the Coulomb potential $V_2(\mathbf{s})$, and as a third potential $V_3(\mathbf{s})$ a pure scattering potential which corresponds to $V_4(\mathbf{x})$ in $\mathrm{I\!R}^3$. Among the minimally super-integrable systems have been the analogues of the double ring-shaped oscillator $V_4(\mathbf{s})$ and the Hartmann potential $V_5(\mathbf{s})$ on $S^{(3)}$, and the two remaining potentials $V_6(\mathbf{s}), V_7(\mathbf{s})$ emerged from the group chain $\mathrm{SO}(4) \supset \mathrm{SO}(3)$. However, we have not been able to find a super-integrable potential on $S^{(3)}$ incorporating a constant electric field, c.f. $V_5(\mathbf{x})$ in $\mathrm{I\!R}^3$. In appendix D our results are listed in tables D.4–D.5, similarly as in the flat space cases.

In [82] we considered the super-integrable potentials on the two-dimensional hyperboloid $\Lambda^{(2)}$. We have found the analogues of the (generalized) harmonic oscillator $V_1(\mathrm{u})$, i.e., the Higgs oscillator in a space of constant negative curvature, the (generalized) Coulomb potential $V_2(\mathrm{u})$, and of the Holt-potential $V_3(\mathrm{u})$ on $\Lambda^{(2)}$. We also found two more systems $V_3(\mathrm{u}), V_4(\mathrm{u})$, which are due to the peculiarity of the hyperboloid that in spaces of constant negative curvature there are generally more orthogonal coordinate systems which separate the Schrödinger, respectively Helmholtz equation, in comparison to flat or constant positive curvature spaces. However, we have not been able to find a super-integrable version of the modified Coulomb potential, c.f. $V_4(\mathbf{x})$ in $\mathrm{I\!R}^2$. The results are compiled in appendix D in table D.7.

1.3 Interbasis Expansions.

An important aspect of group path integration in quantum mechanics is the so-called interbasis expansion technique for problems which allow the representation of the wave-functions in various coordinate space representations. The basic formula is quite simple being

$$|\mathbf{k}>= \int dE_{\mathbf{l}} C_{\mathbf{p},\mathbf{k}} |\mathbf{p}> + \sum_{\mathbf{n}} C_{\mathbf{n},\mathbf{k}} |\mathbf{n}> \quad , \tag{1.4}$$

[4]The notion of minimally super-integrable systems in two dimensions does not make sense, because the number of integrals of motions equals two, and is thus equal to the number of integrals of motion which are required that the system is separable at all.

where $|\mathbf{k}>$ stands for a basis of eigenfunctions of the Hamiltonian in the coordinate space representation $\mathbf{k}$, and $\int dE_{\mathbf{l}}$ is the spectral-expansion with respect to the coordinate space representation $\mathbf{l}$ with coefficients $C_{\mathbf{p},\mathbf{k}}, C_{\mathbf{n},\mathbf{k}}$ which can be discrete, continuous or both. The main difficulty is, in case one has two coordinate space representations in the quantum numbers $\mathbf{k}$ and $\mathbf{p}, \mathbf{n}$, respectively, to find the expansion coefficients $C_{\mathbf{p},\mathbf{k}}$ and $C_{\mathbf{n},\mathbf{k}}$. Well-know n are the expansions which involve cartesian coordinates and polar coordinates. In the simple case of free quantum motion in Euclidean space, this means that exponentials representing plane waves are expanded in terms of Bessel functions and spherical waves, a discrete interbasis expansion, i.e., $e^{z\cos\psi} = \sum_{\nu\in\mathbb{Z}} e^{i\nu\psi} I_\nu(z)$.

This general method of changing a coordinate basis in quantum mechanics can now be used in the path integral. We assume that we can expand the short-time kernel, respectively the exponential $e^{z\mathbf{x}_{j-1}\cdot\mathbf{x}_j}$ in terms of matrix elements of a group [75] by choosing a specific coordinate basis. We then can change the coordinate basis by means of (1.4). Due to the unitarity of the expansion coefficients $C_{\mathbf{l},\mathbf{k}}$ the short-time kernel is expanded in the new coordinate basis, and the orthonormality of the basis allows to perform explicitly the path integral, exactly in the same way as in the original coordinate basis. However, to find the dynamical group and its corresponding coordinate space representation in a super-integrable system, one of the principal problems, is not always very easy.

From the two (or more) different equivalent coordinate space representations, formulæ and path integral identities can be derived. These identities actually correspond to integral and summation identities between special functions. The case of the expansion from cartesian coordinates to polar coordinates has been studied by Peak and Inomata [169], and they obtained the solution of the isotropic harmonic oscillator as well. The path integral solution of the isotropic harmonic oscillator in turn enables one to calculate numerous path integral problems related to the radial harmonic oscillator, actually problems which are of the so-called Besselian type, including the Coulomb problem.

Summarizing, the following small list of interbasis expansion have been investigated until now in the literature (we do not claim completeness):

1. Homogeneous Spaces of Constant Curvature:
   (a) The Two-Dimensional Euclidean Space:
      i. The expansion of plane waves with respect to the polar basis is a well-known expansion theorem involving Bessel-functions and circular waves [57, p.971].
      ii. The expansion of plane wave in terms of the elliptic basis can be found in [153, p.185], and
      iii. the expansion of the polar basis with respect to the elliptic bases in [153, p.183].
   (b) The Three-Dimensional Euclidean Space:
      i. A formal discussion of the interbasis expansion of ten of the eleven bases with respect to the cartesian basis is due to [16], and references therein.
      ii. The expansion of the sphero-conical basis in terms of the in terms of the spherical basis corresponds to the subgroup chain $E(3) \supset \mathrm{SO}(3)$, and is therefore essentially the same as in [167].
      iii. The expansion of plane waves into the spherical basis is a expansion theorem involving Bessel-functions and Gegenbauer polynomials [57, p.980].
      iv. The expansion of plane waves in terms of (oblate and prolate) spheroidal wave-functions is due to [153, p.315].
   (c) Two- and three-dimensional Minkowski space:

i. Interbasis expansions relating the cartesian basis with the polar and elliptic bases in two dimensions, and the cartesian basis with the spherical and spheroidal bases in three dimensions have been discussed in [75].

(d) The Two-Dimensional Sphere:

i. On the two-dimensional sphere, there is only the interbasis expansion of the elliptic basis with respect to the spherical basis which can be found in [167].

(e) The Three-Dimensional Sphere:

i. The expansion of the spherical basis with respect to the cylindrical basis is determined by the Clebsch-Gordan coefficients, e.g. [4, 188].

ii. The expansion of the sphero-elliptic basis in terms of the in terms of the spherical basis corresponds to the subgroup chain $SO(4) \supset SO(3)$, and is therefore essentially the same as in [167].

iii. The expansion of the two ellipso-cylindrical bases (the oblate elliptic and prolate elliptic) with respect to the spherical, respectively the cylindrical, is due to [78].

iv. The expansion of the ellipsoidal basis with respect to another basis is not known yet, but will be the subject of a future investigation [79].

(f) The Two-Dimensional Hyperboloid:

i. The only known interbasis expansions on $\Lambda^{(2)}$ deal with the expansions of eight out of nine bases with respect to the spherical basis [114].

(g) The Three-Dimensional Hyperboloid:

i. The interbasis expansions of the four bases corresponding to the subgroup chain $SO(3,1) \supset E(2)$ are the same as in $\mathbb{R}^2$,

ii. the interbasis expansions of the two bases corresponding to the subgroup chain $SO(3,1) \supset SO(3)$ are the same as on $S^{(2)}$,

iii. the interbasis expansions of the nine bases corresponding to the subgroup chain $SO(3,1) \supset SO(2,1)$ are the same as on $\Lambda^{(2)}$.

iv. Interbasis expansions corresponding to the non-subgroup bases on the three-dimensional hyperboloid are not known.

2. Potential Problems:

(a) The Oscillator in $\mathbb{R}^2$:

i. The expansion of the elliptic and cartesian basis with respect to the polar basis is due to [150].

(b) The Coulomb potential in $\mathbb{R}^2$:

i. The expansion of the elliptic basis with respect to the polar and parabolic bases is due to [151].

(c) The Coulomb potential in $\mathbb{R}^3$:

i. The expansion of the parabolic basis with respect to the spherical bases is determined by the Clebsch-Gordan coefficients, e.g. [4, 188].

ii. The expansion of the spheroidal basis with respect to the spherical bases is due to [30], and with respect to the parabolic bases due to [149].

(d) Generalized Oscillator and Kepler-Coulomb Problem in $\mathbb{R}^3$:

i. In the Coulomb problem the expansion of the spheroidal basis with respect to the spherical and parabolic bases is due to [123].

ii. In the oscillator problem the expansion of the spheroidal basis with respect to the spherical and cylindrical bases is due to [124].

(e) Oscillator and Kepler-Coulomb Problem on $S^{(3)}$:

i. In the Coulomb problem the expansion of the ellipso-cylindrical basis with respect to the spherical basis is due to [170].

ii. In the oscillator problem the expansion of the cylindrical basis with respect to the spherical basis and vice versa is due to [58, 171].

(f) The Kaluza-Klein Monopole system:

i. The expansion of the parabolic basis with respect to the spherical basis is determined by the Clebsch-Gordan coefficients [83].

ii. The expansion of the spheroidal basis with respect to the spherical and parabolic bases will be discussed in [84].

1.4 Path Integral Approach.

In our investigations the path integral turns out to be a very convenient tool to formulate and solve the super-integrable potentials on spaces of constant curvature, in particular on the hyperboloid. The subsequent separation of variables in each problem can be performed in a straightforward and easy way.

This line of reasoning must be seen in the endevaour to solve quantum mechanical problems "from the point of view of fluctuating paths" [42]. In the contributions [90, 91] we have already presented our approach "How To Solve Path Integrals in Quantum Mechanics" by listing the relevant *Basic Path Integrals* necessary for the calculations, and a comprehensive "Table of Path Integrals" will appear soon [92]. This includes path integral problems which are based on the general *Gaussian Path Integral* [46, 174] based on the path integral of the general quadratic Lagrangian, the *Besselian Path Integral* [40, 47, 56, 108, 169] based on the path integral of the radial harmonic oscillator, and the *Legendrian Path Integral* [12, 13, 41, 48, 108, 109, 132] based on the path integral of the (modified) Pöschl-Teller potential, respectively, as well as path integral formulæ which deal with point interactions and boundary conditions [91].

In tables 1.1 and 1.2 we have summarized the wide range of quantum mechanical problems which are accessible by path integration. For details we refer to [75] and [90]–[92].

Higher dimensional path integral problems can usually be put into a hierarchy corresponding which one-dimensional path integral is most important for their evaluation. For instance, in the Coulomb problem, the Besselian part is the most important because after separating off the angular path integrations, and a space-time transformation, a path integral for a radial harmonic oscillator remains.

The class of Natanzon potentials, where the confluent and hypergeometric case are both subject to path integration [77], are the most general kinds of Besselian and Legendrian path integral potential problems, with six parameters which may be chosen freely.

The classes of "conditionally solvable" potentials are somewhat more restricted in comparison to the Natanzon potentials by having only four free parameters. However, fractional power dependences can be incorporated, e.g., a long-range $1/\sqrt{r}$ behaviour [76] and references therein. For the conditionally solvable potentials confluent (Besselian) and hypergeometric (Legendrian) versions can be found as well.

Table 1.1: Application of Basic Path Integrals

Quadratic Lagrangian	Radial Harmonic Oscillator	Pöschl-Teller Potential	Modified Pöschl-Teller Potential
Infinite square well	Liouville potential	Scarf potentials	Reflectionless potential
Linear potential	Morse potential	Symmetric top	Rosen-Morse potential
Repelling oscillator	Uniform magnetic field	Magnetic top	Wood-Saxon potential
Forced oscillator	Motion in a section	Higgs oscillator on spheres	Hultén potential
Saddle point potential	Calogero model		Manning-Rosen potential
Uniform magnetic field	Aharonov-Bohm problems		Hyperbolic Scarf potential
Driven coupled oscillators	Coulomb- and confinement potentials		Hyperbolic barrier and modified Rosen-Morse potentials
Two-time action (Polaron)	Smorodinsky-Winternitz potentials		Hyperbolic spaces of rank one
Second derivative Lagrangians	Coulomb-like potentials in polar and parabolic coordinates		Kepler problem on (pseudo-) spheres
Semi-classical expansion	Nonrelativistic monopoles		Hyperbolic strip
Generating functional	Kaluza-Klein monopole		Higgs oscillator on pseudospheres
Moments formula	Poincaré plane + magnetic field + potentials		Hermitean spaces
Effective potentials	Klein-Gordon- and Dirac Coulomb problem		
Anharmonic oscillator	Anyons		
Conditionally solvable pots.	Natanzon potentials		Natanzon potentials Conditionally solvable pots.

1.5 Presentation of Results.

The contents of this paper is as follows. In the next section we give an introduction into the formulation and construction of coordinate systems on the three-dimensional hyperboloid. This includes an enumeration of the coordinate systems according to [74, 75, 114, 116, 162]. The enumeration includes the explicit statement of the quantity $\mathrm{u} = (u_0, u_1, u_2, u_3)$ in terms of the coordinate variables $\boldsymbol{\varrho} = (\varrho_1, \varrho_2, \varrho_3)$, the line element $ds^2 = ds^2(\boldsymbol{\varrho})$, the momentum operators P_{ϱ_i}, the Hamiltonian H_0, the form a potential $V(\mathrm{u})$ must have in order that the Schrödinger equation $H\Psi = (H_0 + V)\Psi = E\Psi$ is separable, and the corresponding integrals of motion, respectively observables.

Table 1.2: Group Path Integration and Perturbation Expansions

Group Path Integration	Perturbation Expansions
Euclidean space	δ-functions
pseudo-Euclidean space	δ'-functions
Spheres	Point interaction for Dirac particle
Single-sheeted pseudospheres	Dirichlet boundary conditions
Double-sheeted pseudospheres	Neumann boundary conditions
Bispherical coordinates	Boxes and radial rings
Pseudo-bispherical coordinates	Absolute value potentials
Klein-Gordon propagator	Point interactions in $\mathbb{R}^2, \mathbb{R}^3$
	Discontinuous potentials

In Section III we present the three found maximally super-integrable potentials on the three-dimensional hyperboloid, including an analogue of a Stark effect potential which is, however, in comparison to $\mathbb{R}^3$ only minimally super-integrable. The maximally super-integrable systems have five integrals of motion. For instance, in the pure Coulomb problem in $\mathbb{R}^3$ they are the energy E, the square of the absolute value of the angular momentum $\mathbf{L}^2$, L_z^2, an observable corresponding to the semi-hyperbolic system, and the third component of the Pauli-Lenz-Runge vector $\mathbf{A}$ (the whole set of $E, \mathbf{L}^2, L_z^2, \mathbf{A}$ is not functionally independent). Actually, the first three of these constants of motion are typical for each radial problem, and the minimum number of three observables is required that a three-dimensional system is separable at all (in [44] a systematic listing of these constants of motion has been presented). We treat the first two potentials, i.e., the Higgs oscillator and the Coulomb potential on $\Lambda^{(3)}$ in some detail.

In Section IV we discuss the minimally super-integrable potentials on $\Lambda^{(3)}$. We find four potentials which have their counterparts in three-dimensional Euclidean space. The remaining potentials emerge from the subgroup structure of $\mathrm{SO}(3,1)$, i.e., we find four potentials corresponding to the chain $\mathrm{SO}(3,1) \supset E(2)$, two potentials corresponding to the chain $\mathrm{SO}(3,1) \supset \mathrm{SO}(3)$, where one of them is however equivalent to a previous one, and five potentials corresponding to the chain $\mathrm{SO}(3,1) \supset \mathrm{SO}(2,1)$, respectively. This yields 15 minimally super-integrable potentials on $\Lambda^{(3)}$. We do not explicitly list each solution again, because this would blow up our paper to much, and refer instead to our previous work concerning the super-integrable potentials in flat space [80], on the sphere [81], and on the two-dimensional hyperboloid [82]. In Sections III and IV we make frequently use of the path integral formulations of the Pöschl-Teller, the modified Pöschl-Teller, and the Rosen-Morse potential, whose solutions are cited in Appendix A.

In the fifth Section we summarize and discuss our results. Here we also establish a correspondence of maximally and minimally super-integrable potentials in two and three dimensions in the three spaces of constant curvature, i.e., Euclidean space, the sphere, and the hyperboloid. In addition we suggest analogues of the Holt-potential on the two- and three-dimensional sphere and on the two- and three-dimensional hyperboloid, respectively. However, these potentials turn out to be only integrable. On the sphere the corresponding separating coordinate systems are the $k = k' = 1/\sqrt{2}$ particular case of the rotated elliptic, respectively rotated prolate spheroidal systems. On the hyperboloid the separating coordinate system are the semi-hyperbolic systems. The flat space limits of these systems are parabolic coordinates in two and three dimensions.

In four appendices we present some additional material. This includes on the one hand side the presentation of elementary path integral solutions which we need in our evaluations. On the other we complete the consideration of super-integrable potentials in $\mathbb{R}^3$ (Appendix B) and on $S^{(3)}$ (Appendix C) corresponding to the $S^{(2)}$ subgroup chain reduction by adding a minimally super-integrable potential due to the subgroup reduction corresponding to SO(3). Appendix D summarizes all our former findings of minimally and maximally super-integrable potentials in two- and three-dimensional Euclidean space, on the two- and three-dimensional sphere, and on the two-dimensional hyperboloid, respectively.

2 Coordinate Systems on Hyperboloids

In this Section we construct the coordinate systems on the three-dimensional hyperboloid. However, first we cite some useful information concerning the construction of coordinate systems on the most important spaces of constant curvature. These are Euclidean spaces, spheres and hyperboloids.

Table 2.1: Operators and Coordinate Systems in Flat Space

Observable I in $\mathbb{R}^2$	Coordinate System
$I = P_1^2 - P_2^2$	Cartesian
$I = L_3^2$	Polar
$I = L_3^2 + d^2(P_1^2 - P_2^2)$	Elliptic
$I = \{L_3, P_2\}$	Parabolic
Observables I_1, I_2 in $\mathbb{R}^3$	**Coordinate System**
$I_1 = P_2^2, I_2 = P_3^2$	Cartesian
$I_1 = L_3^2, I_2 = P_3^2$	Circular Polar
$I_1 = L_3^2 + d^2 P_1^2, I_2 = P_3^2$	Circular Elliptic
$I_1 = \{L_3, P_2\}, I_2 = P_3^2$	Circular Parabolic
$I_1 = \mathbf{L}^2, I_2 = L_1^2 + {k'}^2 L_2^2$	Sphero-Elliptic
$I_1 = \mathbf{L}^2, I_2 = L_3^2$	Spherical
$I_1 = \mathbf{L}^2 - d^2(P_1^2 + P_2^2), I_2 = L_3^2$	Prolate Spheroidal
$I_1 = \mathbf{L}^2 + d^2(P_1^2 + P_2^2), I_2 = L_3^2$	Oblate Spheroidal
$I_1 = \{L_1, P_2\} - \{L_2, P_1\}, I_2 = L_3^2$	Parabolic
$I_1 = \mathbf{L}^2 + (a_2 + a_3)P_3^2 + (a_1 + a_3)P_2^2 + (a_1 + a_2)P_1^2$, $I_2 = a_1^2 L_1^2 + a_2 L_2^2 + a_3 L_3^2 + a_2 a_3 P_3^2 + a_1 a_3 P_2^2 + a_1 a_2 P_1^2$	Ellipsoidal
$I_1 = L_3^2 - d^2 P_3^2 + d(\{L_2, P_1\} + \{L_1, P_2\})$, $I_2 = d(P_2^2 - P_1^2) + \{L_2, P_1\} - \{L_1, P_2\}$	Paraboloidal

2.1 Coordinate Systems and Observables.

For the classification of coordinate systems in an homogeneous space, and hence for sets of inequivalent observables, we need second-order differential operators I_i ($i \in J$, J an index set) which are at most quadratic in the derivatives. In order that they can characterize a coordinate system which separates the Hamiltonian we must require that they commute with the Hamiltonian and with each other, i.e., $[H, I_i] = [I_i, I_j] = 0$. This property characterizes them as observables (in classical mechanics as constants of motion). In two-dimensional spaces we have *one characteristic operator* I which corresponds to the additional observable, and in three-dimensional spaces there are *two characteristic operators* I_1, I_2 which correspond to the two separation constants which appear for each coordinate system. Finding all inequivalent sets of $\{I\}$, respectively $\{I_1, I_2\}$ is equivalent in finding all inequivalent sets of observables for the

Hamiltonian of the free motion. Because the operators $I_{1,2}$ commute with the Hamiltonian and with each other one can find simultaneously eigenfunctions of H, I, respectively H, I_1, I_2.

Table 2.1 illustrates this for the four coordinate systems in $\mathbb{R}^2$ and the eleven coordinate systems in $\mathbb{R}^3$ ($a, d > 0, 0 < r \leq 1$ parameters, $\{\cdot,\cdot\}$ is the anticommutator, and $0 < a_1 < \varrho_1 < a_2 < \varrho_2 < a_3 < \varrho_3$ with $\varrho_{1,2,3}$ the coordinates in the ellipsoidal system), according to Refs. [16, 111]. Here $P_{1,2,3} = -\mathrm{i}\hbar\partial_{x_i}$ and $L_{1,2,3}$ are the usual momentum and angular momentum operators taken in Cartesian coordinates (2.1), see below, and c.f. [16, 75] for more details.

Table 2.2 illustrates this for the coordinate systems on $S^{(2)}$ and $S^{(3)}$. Again $L_{1,2,3}$ are the usual angular momentum operators, where we have set [119]

$$L_1 = \frac{\hbar}{\mathrm{i}} D_{23} \ , \qquad L_2 = \frac{\hbar}{\mathrm{i}} D_{31} \ , \qquad L_3 = \frac{\hbar}{\mathrm{i}} D_{12} \ , \tag{2.1}$$

$$N_1 = \frac{\hbar}{\mathrm{i}} D_{14} \ , \qquad N_2 = \frac{\hbar}{\mathrm{i}} D_{24} \ , \qquad N_3 = \frac{\hbar}{\mathrm{i}} D_{34} \ . \tag{2.2}$$

with $D_{ij} = s_j\partial_{s_i} - s_i\partial_{s_j}$, and for the parameters we have $0 < k^2 < 1, k'^2 = 1 - k^2$, and in the ellipsoidal system $a_1 < \rho_1 < a_2 < \rho_2 < a_3 < \rho_3 < a_4$. We have also included the elliptic II and the prolate elliptic II systems as examples for a rotated elliptic system, c.f. for the proper definition of the parameters [81]. We have the commutation relations (ϵ_{ijk} is the Levi-Civita tensor)

$$[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_k \ , \qquad [L_i, N_j] = \mathrm{i}\hbar\epsilon_{ijk}N_k \ , \qquad [N_i, N_j] = \mathrm{i}\hbar\epsilon_{ijk}L_k \ . \tag{2.3}$$

Table 2.2: Operators and Coordinate Systems on Spheres

Observable I on $S^{(2)}$	Coordinate System
$I = L_3^2$	Polar
$I = L_1^2 + k'^2 L_2^2$	Elliptic
$I = \cos 2f L_3^2 - \frac{1}{2}\sin 2f\{L_1, L_3\}$	Elliptic II*
Observables I_1, I_2 on $S^{(3)}$	**Coordinate System**
$I_1 = N_3^2, I_2 = L_3^2$	Cylindrical
$I_1 = \mathbf{L}^2, I_2 = L_1^2 + k'^2 L_2^2$	Sphero-Elliptic
$I_1 = \mathbf{L}^2, I_2 = L_3^2$	Spherical
$I_1 = (1-k^2)\mathbf{L}^2 + k^2(L_3^2 - N_3^2), I_2 = L_3^2$	Prolate Elliptic
$I_1 = \cos 2f\mathbf{L}^2 + \frac{1}{2}\sin 2f(\{L_1, N_2\} - \{L_2, N_1\}), I_2 = L_3^2$	Prolate Elliptic II*
$I_1 = (1-k^2)\mathbf{L}^2 - k^2 N_3^2, I_2 = L_3^2$	Oblate Elliptic
$I_1 = (a_1+a_4)L_1^2 + (a_2+a_4)L_2^2 + (a_3+a_4)L_3^2$ $+(a_2+a_3)N_1^2 + (a_1+a_3)N_2^2 + (a_1+a_2)N_3^2$ $I_2 = a_1a_4L_1^2 + a_2a_4L_2^2 + a_3a_4L_3^2 + a_2a_3N_1^2 + a_1a_3N_2^2 + a_1a_2N_3^2$	Ellipsoidal

* $\sin^2 f = k^2$

2.2 Coordinate Systems on the Three-Dimensional Hyperboloid.

Before we are going to discuss the coordinate systems on the three-dimensional hyperboloid in some detail, let us start with some remarks concerning harmonic analysis on $\Lambda^{(3)}$, and we cite some results from [116].

The homogeneous Lorentz group SO(3,1) consists of those proper real linear transformations which leave the hyperboloid ($u_0 > 0$)

$$\mathrm{u}\cdot\mathrm{u} = \mathrm{u}^2 = u_0^2 - (u_1^2 + u_2^2 + u_3^2) = u_0^2 - \mathbf{u}^2 = R^2 \tag{2.4}$$

invariant. The Lie algebra is six-dimensional, and is generated by the spatial rotation generators

$$L_1=\frac{\hbar}{\mathrm{i}}\bigg(u_2\frac{\partial}{\partial u_3}-u_3\frac{\partial}{\partial u_2}\bigg)\,,\quad L_2=\frac{\hbar}{\mathrm{i}}\bigg(u_1\frac{\partial}{\partial u_3}-u_3\frac{\partial}{\partial u_1}\bigg)\,,\quad L_3=\frac{\hbar}{\mathrm{i}}\bigg(u_1\frac{\partial}{\partial u_2}-u_2\frac{\partial}{\partial u_1}\bigg)\,,\tag{2.5}$$

(note the sign convention in comparison to the sphere) and the Lorentz transformation generators

$$K_1=\frac{\hbar}{\mathrm{i}}\bigg(u_0\frac{\partial}{\partial u_1}+u_1\frac{\partial}{\partial u_0}\bigg)\,,\quad K_2=\frac{\hbar}{\mathrm{i}}\bigg(u_0\frac{\partial}{\partial u_2}+u_2\frac{\partial}{\partial u_0}\bigg)\,,\quad K_3=\frac{\hbar}{\mathrm{i}}\bigg(u_0\frac{\partial}{\partial u_3}+u_3\frac{\partial}{\partial u_0}\bigg)\,.\tag{2.6}$$

The commutation relations are

$$[L_i,L_j]=-\mathrm{i}\hbar\epsilon_{ijk}L_k\ ,\qquad [L_i,K_j]=-\mathrm{i}\hbar\epsilon_{ijk}K_k\ ,\qquad [K_i,K_j]=\mathrm{i}\hbar\epsilon_{ijk}K_k\ .\tag{2.7}$$

The Hamiltonian on $\Lambda^{(3)}$ can then be written as ($V(\mathrm{u})$ a potential on $\Lambda^{(3)}$)

$$H=H_0+V(\mathrm{u})\ ,\qquad H_0=-\frac{\hbar^2}{2MR^2}\Delta_{LB}=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)\ .\tag{2.8}$$

The irreducible representations of the identity component of $\mathrm{SO}(3,1)$ are labeled by two numbers (j_0,σ), where j_0 is an integer or half-integer, and σ is complex. The Eigenvalues of the Schrödinger operator H_0 are found to have the following form

$$E_{\sigma,j_0}=-\frac{\hbar^2}{2MR^2}[j_0^2+\sigma(\sigma+2)]\ ,\qquad \begin{array}{ll}\text{continuous spectrum:} & j_0=0,\sigma=-1+\mathrm{i}p\ ,\\ \text{discrete spectrum:} & j_0=2n\ (n\in\mathbb{N}),\sigma=-1\ .\end{array}\tag{2.9}$$

Actually, the discrete spectrum is not present in our case; for instance, it must be taken into account for the quantum motion on the single sheeted hyperboloid [73], on the $\mathrm{SU}(1,1)$ [12, 13] and on the $\mathrm{O}(2,2)$ group manifold [75, 115]. We have for the energy-spectrum of the free quantum motion on $\Lambda^{(3)}$

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)\ ,\qquad p>0\ .\tag{2.10}$$

In the following enumeration we list in each case the definition of the coordinate systems, the metric, the momentum operators, the Hamiltonian and the observables I_1,I_2, respectively.

In the sequel we only consider *orthogonal* coordinate systems on the three-dimensional hyperboloid. $\mathrm{u}\in\Lambda^{(3)}$ is expressed as $\mathrm{u}=\mathrm{u}(\boldsymbol{\varrho})$, where $\boldsymbol{\varrho}=(\varrho_1,\varrho_2,\varrho_3)$ are three-dimensional coordinates on $\Lambda^{(3)}$. Following Olevskiĭ [162] the line element is found to have the form

$$\begin{aligned}ds^2&=\epsilon_a g_{aa}d\varrho_a^2\\&=\frac14\bigg[\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{P(\varrho_1)}\mathrm{d}\varrho_1^2+\frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{P(\varrho_2)}\mathrm{d}\varrho_2^2+\frac{(\varrho_3-\varrho_1)(\varrho_2-\varrho_1)}{P(\varrho_3)}\mathrm{d}\varrho_3^2\bigg]\ ,\end{aligned}\tag{2.11}$$

which must be a positive-definite quantity, hence $\epsilon_a=-1$, $a=1,2,3$, and where $P(\varrho)$ is the *characteristic polynomial* corresponding to the coordinate system. In algebraic form a coordinate system on $\Lambda^{(3)}$ is described in the following way

$$\left.\begin{aligned}u_0^2&=R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)(\varrho_3-a_1)}{(a_2-a_1)(a_3-a_1)(a_4-a_1)}\ ,\\u_1^2&=-R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)(\varrho_3-a_2)}{(a_1-a_2)(a_3-a_2)(a_4-a_2)}\ ,\\u_2^2&=-R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)(\varrho_3-a_3)}{(a_1-a_3)(a_2-a_3)(a_4-a_3)}\ ,\\u_3^2&=-R^2\frac{(\varrho_1-a_4)(\varrho_2-a_4)(\varrho_3-a_4)}{(a_1-a_4)(a_2-a_4)(a_3-a_4)}\ ,\end{aligned}\right\}\tag{2.12}$$

and we have for the characteristic polynomial

$$P(\varrho) = (\varrho - a_1)(\varrho - a_2)(\varrho - a_3)(\varrho - a_4) \ . \tag{2.13}$$

Fixing the numbers a_i, $i = 1, 2, 3, 4$, and the range of the $\boldsymbol{\varrho}$ specifies a coordinate system. For the metric tensor then follows

$$g_{ab} = G_{ik} \frac{\partial u_i}{\partial \varrho_a} \frac{\partial u_k}{\partial \varrho_b} \ , \tag{2.14}$$

where G_{ik} is the metric tensor of the ambient space, which in the present case has the form $G_{ik} = \mathrm{diag}(1, -1, -1, -1)$, and in order that the line element $ds^2 = \sum_{ab} \epsilon_{ab} g_{ab} dq^a dq^b$ is positive definite appropriate $\epsilon_{aa} = \pm 1$ must be taken into account. Actually $\epsilon_{ab} = \epsilon_{aa} = -1, \forall_{a,b}$. In the following we state for convenience only the explicit form of ds^2. In table 2.3 we summarize the results on the coordinate systems on $\Lambda^{(3)}$ according to [112, 116, 162, 181], including, if known, the limiting cases for $\mathbb{R}^3$, as $R \to \infty$. The coordinate systems on $\Lambda^{(3)}$ now are the following:

2.3 Enumeration of the Coordinate Systems.

1. *Cylindrical* coordinates are defined by $(\tau_1, \tau_2 \in \mathbb{R}, 0 \le \varphi < 2\pi)$

$$\left.\begin{aligned} u_0 &= R \cosh \tau_1 \cosh \tau_2 \ , \\ u_1 &= R \cosh \tau_1 \sinh \tau_2 \ , \\ u_2 &= R \sinh \tau_1 \sin \varphi \ , \\ u_3 &= R \sinh \tau_1 \cos \varphi \ . \end{aligned}\right\} \tag{2.15}$$

The characteristic operators are

$$I_1 = K_3^2 \ , \qquad I_2 = L_3^2 \ . \tag{2.16}$$

The line element is $ds^2 = R^2(d\tau_1^2 + \cosh^2 \tau_1 d\tau_2^2 + \sinh^2 \tau_1 d\varphi^2)$, and for the momentum operators we have

$$P_{\tau_1} = \frac{\hbar}{\mathrm{i}} \left(\frac{\partial}{\partial \tau_1} + \frac{1}{2} \coth \tau_1 + \frac{1}{2} \tanh \tau_1 \right) \ , \tag{2.17}$$

$$P_{\tau_2} = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \tau_2} \ , \qquad P_\varphi = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \varphi} \ . \tag{2.18}$$

For the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2} \left[\frac{\partial^2}{\partial \tau_1^2} + \Big(\tanh \tau_1 + \coth \tau_1 \Big) \frac{\partial}{\partial \tau_1} + \frac{1}{\cosh^2 \tau_1} \frac{\partial^2}{\partial \tau_2^2} + \frac{1}{\sinh^2 \tau_1} \frac{\partial^2}{\partial \varphi^2} \right] \\ &= \frac{1}{2MR^2} \left(P_{\tau_1}^2 + \frac{1}{\cosh^2 \tau_1} P_{\tau_2}^2 + \frac{1}{\sinh^2 \tau_1} P_\varphi^2 \right) + \frac{\hbar^2}{8MR^2} \left(4 + \frac{1}{\cosh^2 \tau_1} - \frac{1}{\sinh^2 \tau_1} \right) \ . \end{aligned} \tag{2.19}$$

A potential separable in cylindrical coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau_1) + \frac{V_2(\tau_2)}{\cosh^2 \tau_1} + \frac{V_3(\varphi)}{\sinh^2 \tau_1} \ , \tag{2.20}$$

and the corresponding observables are

$$I_1^V = \frac{1}{2M} K_3^2 + V_2(\tau_2) \ , \qquad I_2^V = \frac{1}{2M} L_3^2 + V_3(\varphi) \ . \tag{2.21}$$

2. *Horicyclic* coordinates are defined by ($(x_1, x_2) = \mathbf{x} \in \mathbb{R}^2, y > 0$)

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\left(y + \frac{x_1^2 + x_2^2}{y} + \frac{1}{y}\right) \quad , \quad u_1 = R\frac{x_1}{y} \ , \\ u_3 &= \frac{R}{2}\left(y + \frac{x_1^2 + x_2^2}{y} - \frac{1}{y}\right) \quad , \quad u_2 = R\frac{x_2}{y} \ . \end{aligned}\right\} \tag{2.22}$$

The characteristic operators are

$$I_1 = (K_1 + L_2)^2 \ , \qquad I_2 = (K_2 - L_1)^2 \ . \tag{2.23}$$

Note the relations $I_1 = P_{x_1}^2, I_2 = P_{x_2}^2$, with the $P_{x_{1,2}} = -\mathrm{i}\hbar\partial_{x_{1,2}}$ the usual Cartesian momentum operators. The line elements is $ds^2 = R^2(dx_1^2 + dx_2^2 + dy^2)/y^2$. For the momentum operators we have $P_{x_{1,2}}$ and

$$P_y = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial y} - \frac{3}{2y}\right) \ . \tag{2.24}$$

For the Hamiltonian we have

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}y^2\left(\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \\ &= \frac{1}{2MR^2}y\Big(P_y^2 + P_{x_1}^2 + P_{x_2}^2\Big)y + \frac{3\hbar^2}{8MR^2} \ . \end{aligned} \tag{2.25}$$

A potential separable in horicyclic coordinates must have the form

$$V(\mathrm{u}) = V_1(y) + y^2\Big[V_2(x_1) + V_3(x_2)\Big] \ , \tag{2.26}$$

and the constants of motion are

$$I_1^V = \frac{1}{2M}(K_1 + L_2)^2 + V_2(x_1) \ , \qquad I_2^V = \frac{1}{2M}(K_2 - L_1)^2 + V_3(x_2) \ . \tag{2.27}$$

3. *Sphero-Elliptic* coordinates in algebraic form are given by ($\tau > 0, a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3$)

$$\left.\begin{aligned} u_0^2 &= R^2\cosh^2\tau \ , \\ u_1^2 &= R\sinh^2\tau\frac{(\rho_1 - a_1)(\rho_2 - a_1)}{(a_2 - a_1)(a_3 - a_1)} \ , \\ u_1^2 &= R\sinh^2\tau\frac{(\rho_1 - a_2)(\rho_2 - a_2)}{(a_3 - a_2)(a_1 - a_2)} \ , \\ u_1^2 &= R\sinh^2\tau\frac{(\rho_1 - a_3)(\rho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)} \ . \end{aligned}\right\} \tag{2.28}$$

If we put $\rho_1 = a_1 + (a_2 - a_1)\,\mathrm{sn}^2(\tilde\alpha, k)$ and $\rho_2 = a_2 + (a_3 - a_2)\,\mathrm{cn}^2(\tilde\beta, k')$, where the functions $\mathrm{sn}(\tilde\alpha, k)$, $\mathrm{cn}(\tilde\alpha, k)$ and $\mathrm{dn}(\tilde\alpha, k)$ are the Jacobi elliptic functions with modulus k [57, p.923], we obtain for the coordinates u on the hyperboloid ($\tilde\alpha \in [-K, K], \tilde\beta \in [-2K', 2K'], k^2 + {k'}^2 = 1$)

$$\left.\begin{aligned} u_0 &= R\cosh\tau \ , \\ u_1 &= R\sinh\tau\,\mathrm{sn}(\tilde\alpha, k)\,\mathrm{dn}(\tilde\beta, k') \ , \\ u_2 &= R\sinh\tau\,\mathrm{cn}(\tilde\alpha, k)\,\mathrm{cn}(\tilde\beta, k') \ , \\ u_3 &= R\sinh\tau\,\mathrm{dn}(\tilde\alpha, k)\,\mathrm{sn}(\tilde\beta, k') \ . \end{aligned}\right\} \tag{2.29}$$

$K = K(k)$ and $K' = K(k')$ are the complete elliptic integrals with k and k', respectively. The characteristic operators are

$$I_1 = \mathbf{L}^2 \ , \qquad I_2 = L_1^2 + {k'}^2 L_2^2 \ . \tag{2.30}$$

A rotated sphero-elliptic system can be introduced according to [81], i.e. (for shorthand notation we omit the moduli),

$$\left.\begin{aligned} u_0 &= R\cosh\tau \ , \\ u_1 &= R\sinh\tau(k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\beta + k\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta) \ , \\ u_2 &= R\sinh\tau\,\mathrm{cn}\tilde\alpha\,\mathrm{cn}\tilde\beta \ , \\ u_3 &= R\sinh\tau(k'\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta - k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\beta) \ . \end{aligned}\right\} \tag{2.31}$$

In the following the moduli k, k' are omitted if no confusion can arise. The line element is given by

$$ds^2 = R^2[d\tau^2 + \sinh^2\tau(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)(d\tilde\alpha^2 + d\tilde\beta^2)] \ , \tag{2.32}$$

and the momentum operators are

$$P_\tau = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\tau} + \coth\tau\bigg) \ , \tag{2.33}$$

$$P_{\tilde\alpha} = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\tilde\alpha} - \frac{k^2\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}\bigg) \ , \tag{2.34}$$

$$P_{\tilde\beta} = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\tilde\beta} - \frac{k'^2\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}\bigg) \ . \tag{2.35}$$

The Hamiltonian has the form

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2} + 2\coth\tau\frac{\partial}{\partial\tau} + \frac{1}{\sinh^2\tau}\cdot\frac{1}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}\bigg(\frac{\partial^2}{\partial\tilde\alpha^2} + \frac{\partial^2}{\partial\tilde\beta^2}\bigg)\bigg] \\ &= \frac{1}{2MR^2}\Bigg[P_\tau^2 + \frac{1}{\sinh^2\tau}\cdot\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}}(P_{\tilde\alpha}^2 + P_{\tilde\beta}^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}}\Bigg] + \frac{\hbar^2}{2MR^2} \ . \end{aligned} \tag{2.36}$$

A potential separable in sphero-elliptic coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\sinh^2\tau}\cdot\frac{V_2(\tilde\alpha) + V_3(\tilde\beta)}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta} \ , \tag{2.37}$$

and the second observable is changed into

$$I_2^V = \frac{1}{2M}(L_1^2 + k'^2 L_2^2) + \frac{V_2(\tilde\alpha) + V_3(\tilde\beta)}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta} \ . \tag{2.38}$$

4. The *Equidistant-Elliptic* coordinate system is defined by

$$\left.\begin{aligned} u_0^2 &= R^2\frac{(\varrho_1 - a_3)(\varrho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)}\cosh^2\tau \ , \\ u_1^2 &= R^2\frac{(\varrho_1 - a_2)(\varrho_2 - a_2)}{(a_1 - a_2)(a_2 - a_3)}\cosh^2\tau \ , \\ u_2^2 &= R^2\frac{(\varrho_1 - a_1)(a_1 - \varrho_2)}{(a_1 - a_2)(a_1 - a_3)}\cosh^2\tau \ , \\ u_3^2 &= R^2\sinh^2\tau \end{aligned}\right\} \tag{2.39}$$

($a_3 < a_2 < \varrho_2 < a_1 < \varrho_1$). Here $\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K), \beta \in [0, 4K')$, and $\tau \in \mathbb{R}$. The characteristic operators are

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = L_3^2 + \sinh^2 f K_1^2 \ . \tag{2.40}$$

with $\sinh^2 f$ as defined below in (2.42), and $2f$ is the distance between the foci. We set $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$, and after putting $\varrho_1 = a_1 - (a_1 - a_3)\,\mathrm{dn}^2(\alpha, k)$, $\varrho_2 = a_1 - (a_1 - a_2)\,\mathrm{sn}^2(\beta, k')$, and $k^2 = (a_2 - a_3)/(a_1 - a_3)$, $k'^2 = (a_1 - a_2)/(a_1 - a_3)$ with the property $k^2 + k'^2 = 1$, we get

$$\left.\begin{aligned} u_0 &= \phantom{\mathrm{i}}R\,\mathrm{sn}(\alpha, k)\,\mathrm{dn}(\beta, k')\cosh\tau \ , \\ u_1 &= \mathrm{i}R\,\mathrm{cn}(\alpha, k)\,\mathrm{cn}(\beta, k')\cosh\tau \ , \\ u_2 &= \mathrm{i}R\,\mathrm{dn}(\alpha, k)\,\mathrm{sn}(\beta, k')\cosh\tau \ , \\ u_3 &= \phantom{\mathrm{i}}R\sinh\tau \ . \end{aligned}\right\} \tag{2.41}$$

Analogously as for the elliptic system on the two-dimensional hyperboloid we can introduce a *rotated equidistant-elliptic* system, also called *equidistant-elliptic II* system. Instead of a trigonometric rotation as for the case on the sphere [81] we must consider in the present case a hyperbolic rotation [82]. We define

$$\sinh^2 f = \frac{a_1 - a_2}{a_2 - a_3} = \frac{k'^2}{k^2} \ , \qquad \cosh^2 f = \frac{a_1 - a_3}{a_2 - a_3} = \frac{1}{k^2} \ . \tag{2.42}$$

The rotated elliptic system is then obtained by

$$\begin{pmatrix} u_0' \\ u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} \cosh f & \sinh f & 0 & 0 \\ \sinh f & \cosh f & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_0\cosh f + u_1\sinh f \\ u_0\sinh f + u_1\cosh f \\ u_2 \\ u_3 \end{pmatrix} \ . \tag{2.43}$$

Explicitly this yields

$$\left.\begin{aligned} u_0' &= \frac{R}{a_2 - a_3}\Big(\sqrt{(\varrho_1 - a_3)(\varrho_2 - a_3)} + \sqrt{(\varrho_1 - a_2)(\varrho_2 - a_2)}\Big)\cosh\tau \\ &= R\left[\frac{1}{k}\,\mathrm{sn}(\alpha, k)\,\mathrm{dn}(\beta, k') \ + \mathrm{i}\frac{k'}{k}\,\mathrm{cn}(\alpha, k)\,\mathrm{cn}(\beta, k')\right]\cosh\tau \ , \\ u_1' &= \frac{R}{a_2 - a_3}\left(\sqrt{\frac{a_1 - a_2}{a_1 - a_3}(\varrho_1 - a_3)(\varrho_2 - a_3)} + \sqrt{\frac{a_1 - a_3}{a_1 - a_2}(\varrho_1 - a_2)(\varrho_2 - a_2)}\right)\cosh\tau \\ &= R\left[\frac{k'}{k}\,\mathrm{sn}(\alpha, k)\,\mathrm{dn}(\beta, k') + \frac{\mathrm{i}}{k}\,\mathrm{cn}(\alpha, k)\,\mathrm{cn}(\beta, k')\right]\cosh\tau \ , \\ u_2 &= R\sqrt{\frac{(\varrho_1 - a_1)(a_1 - \varrho_2)}{(a_1 - a_2)(a_1 - a_3)}}\,\cosh\tau = \mathrm{i}R\,\mathrm{dn}(\alpha, k)\,\mathrm{sn}(\beta, k')\cosh\tau \ , \\ u_3 &= R\sinh\tau \ . \end{aligned}\right\} \tag{2.44}$$

In the rotated elliptic system the second observable is changed into

$$I_2' = \cosh 2f L_3^2 - \tfrac{1}{2}\sinh 2f\{K_2, L_3\} \ , \tag{2.45}$$

where $\{X, Y\} = XY + YX$ is the anti-commutator of two operators X and Y. The line element in each case is given by $ds^2 = R^2[d\tau^2 + \cosh^2\tau(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)(d\alpha^2 + d\beta^2)]$. In the limit $R \to \infty$ the equidistant-elliptic system yields *circular-elliptic*, and the rotated system *circular-elliptic II* coordinates in $\mathbb{R}^3$. For the momentum operators in either case we obtain

$$P_\tau = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau} + \tanh\tau\right) \ , \tag{2.46}$$

$$P_\alpha = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\alpha} - \frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\right) \ , \tag{2.47}$$

$$P_\beta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\beta} - \frac{k^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\right) \ , \tag{2.48}$$

and for the Hamiltonian

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2}+2\tanh\tau\frac{\partial}{\partial\tau}+\frac{1}{\cosh^2\tau(k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta)}\bigg(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\bigg)\bigg]\\ &= \frac{1}{2MR^2}\left[P_\tau^2+\frac{1}{\cosh^2\tau}\cdot\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}}(P_\alpha^2+P_\beta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}}\right]\\ &\qquad+\frac{\hbar^2}{8MR^2}\bigg(4+\frac{1}{\cosh^2\tau}\bigg)\ . \end{aligned}\tag{2.49}$$

A potential separable in equidistant-elliptic coordinates must have the form

$$V(\mathrm{u})=V_1(\tau)+\frac{1}{\cosh^2\tau}\cdot\frac{V_2(\alpha)+V_3(\beta)}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\ .\tag{2.50}$$

The second observable then takes on the form

$$I_2^V=\frac{1}{2M}(L_3^2+\sinh^2 fK_2^2)+\frac{V_2(\alpha)+V_3(\beta)}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\ .\tag{2.51}$$

5. The *Equidistant-Hyperbolic* system is given by

$$\left.\begin{aligned} u_0^2 &= R^2\frac{(\varrho_1-a_2)(a_2-\varrho_2)}{(a_1-a_2)(a_2-a_3)}\cosh^2\tau\ ,\\ u_1^2 &= R^2\frac{(\varrho_1-a_3)(a_3-\varrho_2)}{(a_1-a_3)(a_2-a_3)}\cosh^2\tau\ ,\\ u_2^2 &= R^2\frac{(\varrho_1-a_1)(a_1-\varrho_2)}{(a_1-a_2)(a_1-a_3)}\cosh^2\tau\ ,\\ u_3^2 &= R^2\sinh^2\tau \end{aligned}\right\}\tag{2.52}$$

$(\varrho_2<a_3<a_2<a_1<\varrho_1)$. The characteristic operators are

$$I_1=K_1^2+K_2^2-L_3^2\ ,\qquad I_2=K_1^2-\sin^2\alpha L_3^2\ ,\tag{2.53}$$

where $\sin^2\alpha=(a_2-a_3)/(a_1-a_3)$ and 2α is the angle between the two focal lines. We set $\boldsymbol{\varrho}=(\varrho_1,\varrho_2)$, and after putting $\varrho_1=a_2-(a_2-a_3)\,\mathrm{cn}^2(\mu,k)$, $\varrho_2=a_2+(a_1-a_2)\,\mathrm{cn}^2(\eta,k')$, and $k^2=(a_2-a_3)/(a_1-a_3)$, $k'^2=(a_1-a_2)/(a_1-a_3)$, where $\mu\in(\mathrm{i}K',\mathrm{i}K'+2K)$, $\eta\in[0,4K')$, we get

$$\left.\begin{aligned} u_0 &= -R\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\eta,k')\cosh\tau\ ,\\ u_1 &= \mathrm{i}R\,\mathrm{sn}(\mu,k)\,\mathrm{dn}(\eta,k')\cosh\tau\ ,\\ u_2 &= \mathrm{i}R\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\eta,k')\cosh\tau\ ,\\ u_3 &= R\sinh\tau\ . \end{aligned}\right\}\tag{2.54}$$

The line element has the form $ds^2=R^2[d\tau^2+\cosh^2\tau(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta)(d\mu^2+d\eta^2)]$, and the momentum operators are given by

$$P_\mu=\frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\mu}-\frac{k^2\,\mathrm{sn}\mu\,\mathrm{cn}\mu\,\mathrm{dn}\mu}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}\bigg)\ ,\qquad P_\eta=\frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\eta}-\frac{k'^2\,\mathrm{sn}\eta\,\mathrm{cn}\eta\,\mathrm{dn}\eta}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}\bigg)\ ,\tag{2.55}$$

with P_τ as in (2.46), and for the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2}+2\tanh\tau\frac{\partial}{\partial\tau}+\frac{1}{\cosh^2\tau(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta)}\bigg(\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\eta^2}\bigg)\bigg]\\ &= \frac{1}{2MR^2}\left[P_\tau^2+\frac{1}{\cosh^2\tau}\cdot\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}}(P_\mu^2+P_\eta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}}\right]\\ &\qquad+\frac{\hbar^2}{8MR^2}\bigg(4+\frac{1}{\cosh^2\tau}\bigg)\ . \end{aligned}\tag{2.56}$$

A potential separable in equidistant-hyperbolic coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\cosh^2\tau}\cdot\frac{V_2(\mu)+V_3(\nu)}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\eta} \quad . \tag{2.57}$$

For the second observable we then have

$$I_2^V = \frac{1}{2M}(K_1^2 - \sin^2\alpha L_3^2) + \frac{V_2(\mu)+V_3(\nu)}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\eta} \quad . \tag{2.58}$$

6. The *Equidistant-Semi-Hyperbolic* system is given by

$$\left.\begin{aligned}
u_0^2 &= \frac{R^2}{2}\left(\frac{1}{\delta}\sqrt{\frac{[(\varrho_1-\gamma)^2+\delta^2][(\varrho_2-\gamma)^2+\delta^2]}{(a-\gamma)^2+\delta^2}} + \frac{(\varrho_1-a)(a-\varrho_2)}{[(a-\gamma)^2+\delta^2]} + 1\right)\cosh^2\tau \ , \\
u_1^2 &= \frac{R^2}{2}\left(\frac{1}{\delta}\sqrt{\frac{[(\varrho_1-\gamma)^2+\delta^2][(\varrho_2-\gamma)^2+\delta^2]}{(a-\gamma)^2+\delta^2}} - \frac{(\varrho_1-a)(a-\varrho_2)}{[(a-\gamma)^2+\delta^2]} - 1\right)\cosh^2\tau \ , \\
u_2^2 &= R^2\frac{(\varrho_1-a)(a-\varrho_2)}{(a-\gamma)^2+\delta^2}\cosh^2\tau \ , \\
u_3^2 &= R^2\sinh^2\tau
\end{aligned}\right\} \tag{2.59}$$

($\varrho_2 < a < \varrho_1, \gamma, \delta \in \mathbb{R}$). The characteristic operators are

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = \{K_2, L_3\} - \tfrac{1}{2}\sinh 2f K_1^2 \ , \tag{2.60}$$

where $\sinh 2f = (a-\gamma)/\delta$ and $2f$ is the distance between the focus of the semi-hyperbolas and the basis of the equidistants. In the flat space limit the case of $\sinh 2f \to 0$ gives *circular-parabolic* coordinates, and the case $\sinh 2f \to \infty$ *Cartesian* coordinates. The special choice of the parameters $a = \gamma = 0, \delta = 1$ together with $\varrho_1 = \mu_1 > 0, -\varrho_2 = \mu_2 > 0$ yields

$$\left.\begin{aligned}
u_0^2 &= \frac{R^2}{2}\left(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} + \mu_1\mu_2 + 1\right)\cosh^2\tau \ , \\
u_1^2 &= \frac{R^2}{2}\left(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} - \mu_1\mu_2 - 1\right)\cosh^2\tau \ , \\
u_2^2 &= R^2\mu_1\mu_2\cosh^2\tau \ , \\
u_3^2 &= R^2\sinh^2\tau \ .
\end{aligned}\right\} \tag{2.61}$$

The characteristic operators then have the form

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = \{K_2, L_3\} \ , \tag{2.62}$$

which shows that the coordinate system (2.61) yields in the flat space limit *circular-parabolic* coordinates. The line element is ($P(\mu) = \mu(1+\mu^2)$)

$$ds^2 = R^2\left[d\tau^2 + \cosh^2\tau\cdot\frac{\mu_1+\mu_2}{4}\left(\frac{d\mu_1^2}{P(\mu_1)} - \frac{d\mu_2^2}{P(\mu_2)}\right)\right] \ , \tag{2.63}$$

the momentum operators are ($i = 1, 2$)

$$P_{\mu_i} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu_i} + \frac{1}{2(\mu_1+\mu_2)} - \frac{1}{4}\frac{P'(\mu_i)}{P(\mu_i)}\right) \ , \tag{2.64}$$

with P_τ as in (2.46), and for the Hamiltonian we obtain

$$\begin{aligned}
H_0 &= -\frac{\hbar^2}{2MR^2}\Bigg[\frac{\partial^2}{\partial\tau^2}+2\tanh\tau\frac{\partial}{\partial\tau}\\
&\quad+\frac{1}{\cosh^2\tau}\cdot\frac{4}{\mu_1+\mu_2}\bigg(P(\mu_1)\Big(\frac{\partial^2}{\partial\mu_1^2}+\frac{P'(\mu_1)}{2P(\mu_1)}\frac{\partial}{\partial\mu_1}\Big)-P(\mu_2)\Big(\frac{\partial^2}{\partial\mu_2^2}+\frac{P'(\mu_2)}{2P(\mu_2)}\frac{\partial}{\partial\mu_2}\Big)\bigg)\Bigg]\\
&= \frac{1}{2MR^2}\Bigg[P_\tau^2+\frac{1}{\cosh^2\tau}\bigg(\sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}}P_{\mu_1}^2\sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}}-\sqrt{\frac{4P(\mu_2)}{\mu_1+\mu_2}}P_{\mu_2}^2\sqrt{\frac{4P(\mu_2)}{\mu_1+\mu_2}}\bigg)\\
&\quad+\frac{\hbar^2}{8MR^2}\Bigg[4+\frac{1}{\cosh^2\tau}\bigg(1+\frac{1}{\mu_1+\mu_2}\Big(P''(\mu_1)-P''(\mu_2)-\frac{3P'^2(\mu_1)}{4P(\mu_1)}+\frac{3P'^2(\mu_2)}{4P(\mu_2)}\Big)\bigg)\Bigg]\quad.
\end{aligned}\tag{2.65}$$

A potential separable in equidistant-semi-hyperbolic coordinates must have the form

$$V(\mathrm{u})=V_1(\tau)+\frac{1}{\cosh^2\tau}\cdot\frac{V_2(\mu_1)+V_3(\mu_2)}{\mu_1+\mu_2}\quad.\tag{2.66}$$

For the second observable we then have

$$I_2^V=\frac{1}{4M}\{K_2,L_3\}+\frac{V_2(\mu_1)+V_3(\mu_2)}{\mu_1+\mu_2}\quad.\tag{2.67}$$

7. *Equidistant-Elliptic-Parabolic* coordinates have the form

$$\left.\begin{aligned}
u_0 &= \frac{R}{2}\bigg(\frac{(\varrho_1-a_1)(a_1-\varrho_2)}{(a_1-a_2)^{3/2}\sqrt{(\varrho_1-a_2)(\varrho_2-a_2)}}\\
&\qquad+\sqrt{\frac{a_1-a_2}{(\varrho_1-a_2)(\varrho_2-a_2)}}+\sqrt{\frac{(\varrho_1-a_2)(\varrho_2-a_2)}{a_1-a_2}}\bigg)\cosh\tau\ ,\\
u_1 &= \frac{R}{2}\bigg(\frac{(\varrho_1-a_1)(a_1-\varrho_2)}{(a_1-a_2)^{3/2}\sqrt{(\varrho_1-a_2)(\varrho_2-a_2)}}\\
&\qquad+\sqrt{\frac{a_1-a_2}{(\varrho_1-a_2)(\varrho_2-a_2)}}-\sqrt{\frac{(\varrho_1-a_2)(\varrho_2-a_2)}{a_1-a_2}}\bigg)\cosh\tau\ ,\\
u_2 &= R\frac{\sqrt{(\varrho_1-a_1)(a_1-\varrho_2)}}{a_1-a_2}\cosh\tau\ ,\\
u_3 &= R\sinh\tau
\end{aligned}\right\}\tag{2.68}$$

$(a_2<\varrho_2<a_1<\varrho_1)$. The characteristic operators are

$$I_1=K_1^2+K_2^2-L_3^2\ ,\qquad I_2=K_2^2+(a_1-a_2)K_1^2+L_3^2-\{K_2,L_3\}\ .\tag{2.69}$$

Making the special choice $a_1=0, a_2=-1$ together with $\varrho_1=\tan^2\vartheta, \varrho_2=-\tanh^2 a$ $(\vartheta\in(-\pi/2,\pi/2), a\in\mathbb{R})$, we obtain

$$\left.\begin{aligned}
u_0 &= R\frac{\cosh^2 a+\cos^2\vartheta}{2\cosh a\cos\vartheta}\cosh\tau\ ,\\
u_1 &= R\frac{\sinh^2 a-\sin^2\vartheta}{2\cosh a\cos\vartheta}\cosh\tau\ ,\\
u_2 &= R\tan\vartheta\tanh a\cosh\tau\ ,\\
u_3 &= R\sinh\tau\ .
\end{aligned}\right\}\tag{2.70}$$

In this case the second characteristic operator has the form

$$I_2 = K_1^2 + K_2^2 + L_3^2 - \{K_2, L_3\} \ . \tag{2.71}$$

The line element is given by

$$ds^2 = R^2\bigg[d\tau^2 + \cosh^2\tau \cdot \frac{\cosh^2 a - \cos^2\vartheta}{\cosh^2 a \cos^2\vartheta}(da^2 + d\vartheta^2)\bigg] \ . \tag{2.72}$$

For the momentum operators we have

$$P_a = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial a} + \frac{\sinh a \cosh a}{\cosh^2 a - \cos^2\vartheta} - \tanh a\bigg) \ , \ P_\vartheta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\vartheta} + \frac{\sin\vartheta\cos\vartheta}{\cosh^2 a - \cos^2\vartheta} + \tan\vartheta\bigg) \ , \tag{2.73}$$

with P_τ as in (2.46), and the Hamiltonian has the form

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{1}{\cosh^2\tau}\cdot\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\bigg(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial\vartheta^2}\bigg)\bigg] \\ &= \frac{1}{2MR^2}\bigg[P_\tau^2 + \frac{1}{\cosh^2\tau}\frac{\cosh a\cos\vartheta}{\sqrt{\cosh^2 a - \cos^2\vartheta}}(P_a^2 + P_\vartheta^2)\frac{\cosh a\cos\vartheta}{\sqrt{\cosh^2 a - \cos^2\vartheta}}\bigg] \\ &\qquad + \frac{\hbar^2}{8MR^2}\bigg(4 + \frac{1}{\cosh^2\tau}\bigg) \ . \end{aligned} \tag{2.74}$$

A potential separable in equidistant-elliptic-parabolic coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\cosh^2\tau}\cdot\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\Big[V_2(a) + V_3(\vartheta)\Big] \ . \tag{2.75}$$

The second observable then is given by

$$I_2^V = \frac{1}{2M}(K_1^2 + K_2^2 + L_3^2 - \{K_2, L_3\}) + \frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\Big[V_2(a) + V_3(\vartheta)\Big] \ . \tag{2.76}$$

8. The *Equidistant-Hyperbolic-Parabolic* system has the form

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\Bigg(\frac{(\varrho_1 - a_1)(a_1 - \varrho_2)}{(a_1 - a_2)^{3/2}\sqrt{(\varrho_1 - a_2)(a_2 - \varrho_2)}} \\ &\qquad + \sqrt{\frac{a_1 - a_2}{(\varrho_1 - a_2)(a_2 - \varrho_2)}} + \sqrt{\frac{(\varrho_1 - a_2)(a_2 - \varrho_2)}{a_1 - a_2}}\Bigg)\cosh\tau \ , \\ u_1 &= \frac{R}{2}\Bigg(\frac{(\varrho_1 - a_1)(a_1 - \varrho_2)}{(a_1 - a_2)^{3/2}\sqrt{(\varrho_1 - a_2)(a_2 - \varrho_2)}} \\ &\qquad + \sqrt{\frac{a_1 - a_2}{(\varrho_1 - a_2)(a_2 - \varrho_2)}} - \sqrt{\frac{(\varrho_1 - a_2)(a_2 - \varrho_2)}{a_1 - a_2}}\Bigg)\cosh\tau \ , \\ u_2 &= R\frac{\sqrt{(\varrho_1 - a_1)(a_1 - \varrho_2)}}{a_1 - a_2}\cosh\tau \\ u_3 &= R\sinh\tau \end{aligned}\right\} \tag{2.77}$$

($\varrho_2 < a_2 < a_1 < \varrho_1$). The characteristic operators are

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = K_2^2 - (a_1 - a_2)K_1^2 + L_3^2 - \{K_2, L_3\} \ . \tag{2.78}$$

Making the special choice $a_1 = 0, a_2 = -1$ together with $\varrho_1 = \cot^2\vartheta, \varrho_2 = -\coth^2 b$ ($\vartheta \in (0,\pi), b > 0$), we obtain

$$\left.\begin{aligned} u_0 &= R\frac{\cosh^2 b + \cos^2\vartheta}{2\sinh b\sin\vartheta}\cosh\tau \ , \\ u_1 &= R\frac{\sinh^2 b - \sin^2\vartheta}{2\sinh b\sin\vartheta}\cosh\tau \ , \\ u_2 &= R\cot\vartheta\coth b\cosh\tau \ , \\ u_3 &= R\sinh\tau \ . \end{aligned}\right\} \tag{2.79}$$

In this case the second characteristic operator has the form

$$I_2 = -K_1^2 + K_2^2 + L_3^2 - \{K_2, L_3\} \ . \tag{2.80}$$

The line element is given by

$$ds^2 = R^2\bigg[d\tau^2 + \cosh^2\tau\cdot\frac{\sinh^2 b + \sin^2\vartheta}{\sinh^2 b\sin^2\vartheta}(db^2 + d\vartheta^2)\bigg] \ . \tag{2.81}$$

For the momentum operators we have

$$P_b = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial b} + \frac{\sinh b\cosh b}{\sinh^2 b + \sin^2\vartheta} - \coth b\bigg) \ , \ P_\vartheta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\vartheta} + \frac{\sin\vartheta\cos\vartheta}{\sinh^2 b + \sin^2\vartheta} - \cot\vartheta\bigg) \ , \tag{2.82}$$

with P_τ as in (2.46), and the Hamiltonian has the form

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{1}{\cosh^2\tau}\cdot\frac{\sinh^2 b\sin^2\vartheta}{\sinh^2 b + \sin^2\vartheta}\bigg(\frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial\vartheta^2}\bigg)\bigg] \\ &= \frac{1}{2MR^2}\bigg[P_\tau^2 + \frac{1}{\cosh^2\tau}\cdot\frac{\sinh b\sin\vartheta}{\sqrt{\sinh^2 b + \sin^2\vartheta}}(P_b^2 + P_\vartheta^2)\frac{\sinh b\sin\vartheta}{\sqrt{\sinh^2 b + \sin^2\vartheta}}\bigg] \\ &\qquad + \frac{\hbar^2}{8MR^2}\bigg(4 + \frac{1}{\cosh^2\tau}\bigg) \ . \end{aligned} \tag{2.83}$$

A potential separable in equidistant-hyperbolic-parabolic coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\cosh^2\tau}\cdot\frac{\sinh^2 b\sin^2\vartheta}{\sinh^2 b + \sin^2\vartheta}\Big[V_2(b) + V_3(\vartheta)\Big] \ , \tag{2.84}$$

and the second constant of motion then is given by

$$I_2^V = \frac{1}{2M}(-K_1^2 + K_2^2 + L_3^2 - \{K_2, L_3\}) + \frac{\sinh^2 b\sin^2\vartheta}{\sinh^2 b + \sin^2\vartheta}\Big[V_2(b) + V_3(\vartheta)\Big] \ . \tag{2.85}$$

9. The *Equidistant-Semi-Circular-Parabolic* coordinate system is given by ($\tau \in \mathbb{R}$)

$$\left.\begin{aligned} u_0 &= R\bigg(\frac{(\varrho_1 - \varrho_2)^2}{8[(\varrho_1 - a)(a - \varrho_2)]^{3/2}} + \frac{1}{2}\sqrt{(\varrho_1 - a)(a - \varrho_2)}\bigg)\cosh\tau \\ &= R\frac{(\xi^2 + \eta^2)^2 + 4}{8\xi\eta}\cosh\tau \ , \\ u_1 &= R\bigg(\frac{(\varrho_1 - \varrho_2)^2}{8[(\varrho_1 - a)(a - \varrho_2)]^{3/2}} - \frac{1}{2}\sqrt{(\varrho_1 - a)(a - \varrho_2)}\bigg)\cosh\tau \\ &= R\frac{(\xi^2 + \eta^2)^2 - 4}{8\xi\eta}\cosh\tau \ , \\ u_2 &= \frac{R}{2}\Bigg(\sqrt{\frac{\varrho_1 - a}{a - \varrho_2}} - \sqrt{\frac{a - \varrho_2}{\varrho_1 - a}}\Bigg)\cosh\tau = R\frac{\eta^2 - \xi^2}{2\xi\eta}\cosh\tau \ , \\ u_3 &= R\sinh\tau \end{aligned}\right\} \tag{2.86}$$

($\varrho_2 < a < \varrho_1$), and we have made the choice $a = 0$, $\varrho_1 = -1/\eta^2, \varrho_2 = 1/\xi^2$, $\xi, \eta > 0$. The characteristic operators have the form

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = \{K_1, K_2\} - \{K_1, L_3\} \ . \tag{2.87}$$

The line element is given by

$$ds^2 = R^2\bigg[d\tau^2 + \cosh^2\tau \cdot \frac{\xi^2+\eta^2}{\xi^2\eta^2}(d\xi^2 + d\eta^2)\bigg] \ . \tag{2.88}$$

The momentum operators are

$$P_\xi = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2+\eta^2} - \frac{1}{\xi}\bigg) \ , \qquad P_\eta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\eta} + \frac{\eta}{\xi^2+\eta^2} - \frac{1}{\eta}\bigg) \ , \tag{2.89}$$

with P_τ as in (2.46), and for the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{1}{\cosh^2\tau}\cdot\frac{\xi^2\eta^2}{\xi^2+\eta^2}\bigg(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\bigg)\bigg] \\ &= \frac{1}{2MR^2}\bigg[P_\tau^2 + \frac{1}{\cosh^2\tau}\cdot\frac{\xi\eta}{\sqrt{\xi^2+\eta^2}}(P_\xi^2 + P_\eta^2)\frac{\xi\eta}{\sqrt{\xi^2+\eta^2}}\bigg] + \frac{\hbar^2}{8MR^2}\bigg(4 + \frac{1}{\cosh^2\tau}\bigg) \ . \end{aligned} \tag{2.90}$$

A potential separable in equidistant-semi-circular-parabolic coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\cosh^2\tau}\cdot\frac{\xi^2\eta^2}{\xi^2+\eta^2}\Big[V_2(\xi) + V_3(\eta)\Big] \ . \tag{2.91}$$

The second observable is given by

$$I_2^V = \frac{1}{4M}(\{K_1, K_2\} - \{K_1, L_3\}) + \frac{\xi^2\eta^2}{\xi^2+\eta^2}\Big[V_2(\xi) + V_3(\eta)\Big] \ . \tag{2.92}$$

10. The *Spherical* coordinate system has the form ($\tau > 0, \vartheta \in (0,\pi), \varphi \in [0, 2\pi)$)

$$\left.\begin{aligned} u_0 &= R\cosh\tau \ , \\ u_1 &= R\sinh\tau\sin\vartheta\cos\varphi \ , \\ u_2 &= R\sinh\tau\sin\vartheta\sin\varphi \ , \\ u_3 &= R\sinh\tau\cos\vartheta \ . \end{aligned}\right\} \tag{2.93}$$

Note that the coordinate systems III. and X. have the subgroup structure SO(3,1) ⊃ SO(3). The characteristic operators are

$$I_1 = \mathbf{L}^2 \ , \qquad I_2 = L_3^2 \ . \tag{2.94}$$

The line element is $ds^2 = R^2[d\tau^2 + \sinh^2\tau(d\vartheta^2 + \sin^2\vartheta d\varphi^2)]$, and for the momentum operators we have

$$P_\vartheta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta\bigg) \ , \tag{2.95}$$

P_τ as in (2.33), P_φ as in (2.18), and for the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau^2} + 2\coth\tau\frac{\partial}{\partial\tau} + \frac{1}{\sinh^2\tau}\bigg(\frac{\partial}{\partial\vartheta} + \cot\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\bigg)\bigg] \\ &= \frac{1}{2MR^2}\bigg[P_\tau^2 + \frac{1}{\sinh^2\tau}\bigg(P_\vartheta^2 + \frac{P_\varphi^2}{\sin^2\vartheta}\bigg)\bigg] + \frac{\hbar^2}{8MR^2}\bigg(4 - \frac{1}{\sinh^2\tau}\bigg(+ \frac{1}{\sin^2\vartheta}\bigg)\bigg) \ . \end{aligned} \tag{2.96}$$

A potential separable in spherical coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\sinh^2\tau}\bigg(V_2(\vartheta) + \frac{V_3(\varphi)}{\sin^2\vartheta}\bigg) \ . \tag{2.97}$$

For the observables we get

$$I_1^V = \frac{1}{2M}\mathbf{L}^2 + V_2(\vartheta) + \frac{V_3(\varphi)}{\sin^2\vartheta} \ , \qquad I_2^V = \frac{1}{2M}L_3^2 + V_3(\varphi) \ . \tag{2.98}$$

11. *Equidistant-Cylindrical* coordinates are given by ($\tau_{1,2} > 0, \varphi \in [0, 2\pi)$)

$$\left.\begin{aligned} u_0 &= R\cosh\tau_1\cosh\tau_2 \ , \\ u_1 &= R\cosh\tau_1\sinh\tau_2\cos\varphi \ , \\ u_2 &= R\cosh\tau_1\sinh\tau_2\sin\varphi \ , \\ u_3 &= R\sinh\tau_1 \ . \end{aligned}\right\} \tag{2.99}$$

The characteristic operators are

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = L_3^2 \ . \tag{2.100}$$

Here $ds^2 = R^2[d\tau_1^2 + \cosh^2\tau_1(d\tau_2^2 + \sinh^2\tau_2 d\varphi^2)]$, and for the momentum operators we have

$$P_{\tau_2} = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\tau_2} + \frac{1}{2}\coth\tau_2\bigg) \ , \tag{2.101}$$

and P_{τ_1} and P_φ as in (2.46) and (2.18), respectively. For the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\partial^2}{\partial\tau_1^2} + 2\tanh\tau_1\frac{\partial}{\partial\tau_1} + \frac{1}{\cosh^2\tau_1}\bigg(\frac{\partial^2}{\partial\tau_2^2} + \tanh\tau_2\frac{\partial}{\partial\tau_2} + \frac{1}{\sinh^2\tau_2}\frac{\partial^2}{\partial\varphi^2}\bigg)\bigg] \\ &= \frac{1}{2MR^2}\bigg[P_{\tau_1}^2 + \frac{1}{\cosh^2\tau_1}\bigg(P_{\tau_2}^2 + \frac{P_\varphi^2}{\sinh^2\tau_2}\bigg)\bigg] + \frac{\hbar^2}{8MR^2}\bigg(4 + \frac{1}{\cosh^2\tau_1}\bigg(1 - \frac{1}{\sinh^2\tau_2}\bigg)\bigg) . \end{aligned} \tag{2.102}$$

A potential separable in equidistant-cylindrical coordinates must have the form

$$V(\mathrm{u}) = V_1(\tau) + \frac{1}{\cosh^2\tau_1}\bigg(V_2(\tau_2) + \frac{V_3(\varphi)}{\sinh^2\tau_2}\bigg) \ . \tag{2.103}$$

The observables then are

$$I_1^V = \frac{1}{2M}(K_1^2 + K_2^2 - L_3^2) + V_2(\tau_2) + \frac{V_3(\varphi)}{\sinh^2\tau_2} \ , \qquad I_2^V = \frac{1}{2M}L_3^2 + V_3(\varphi) \ . \tag{2.104}$$

12. *Equidistant* coordinates are given by ($\tau_{1,2,3} \in \mathbb{R}$)

$$\left.\begin{aligned} u_0 &= R\cosh\tau_1\cosh\tau_2\cosh\tau_3 \ , \\ u_1 &= R\cosh\tau_1\cosh\tau_2\sinh\tau_3 \ , \\ u_2 &= R\cosh\tau_1\sinh\tau_2 \ , \\ u_3 &= R\sinh\tau_1 \ . \end{aligned}\right\} \tag{2.105}$$

The characteristic operators have the form

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = K_1^2 \ . \tag{2.106}$$

Here $ds^2 = R^2[d\tau_1^2 + \cosh^2\tau_1(d\tau_2^2 + \cosh^2\tau_2 d\tau_3^2)]$, and for the momentum operators we have

$$P_{\tau_2} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau_2} + \frac{1}{2}\tanh\tau_2\right) \ , \tag{2.107}$$

and P_{τ_1} and P_{τ_3} as in (2.46) and (2.18). For the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\left[\frac{\partial^2}{\partial\tau_1^2} + 2\tanh\tau_1\frac{\partial}{\partial\tau_1} + \frac{1}{\cosh^2\tau_1}\left(\frac{\partial^2}{\partial\tau_2^2} + \tanh\tau_2\frac{\partial}{\partial\tau_2} + \frac{1}{\cosh^2\tau_2}\frac{\partial^2}{\partial\tau_3^2}\right)\right] \\ &= \frac{1}{2MR^2}\left[P_{\tau_1}^2 + \frac{1}{\cosh^2\tau_1}\left(P_{\tau_2}^2 + \frac{P_{\tau_3}^2}{\cosh^2\tau_2}\right)\right] + \frac{\hbar^2}{8MR^2}\left(4 + \frac{1}{\cosh^2\tau_1}\left(1 + \frac{1}{\cosh^2\tau_2}\right)\right) . \end{aligned} \tag{2.108}$$

A potential separable in equidistant coordinates must have the form

$$V(\mathbf{u}) = V_1(\tau_1) + \frac{1}{\cosh^2\tau_1}\left(V_2(\tau_2) + \frac{V_3(\tau_3)}{\cosh^2\tau_2}\right) \ . \tag{2.109}$$

The observables then are given by

$$I_1^V = \frac{1}{2M}(K_1^2 + K_2^2 - L_3^2) + V_2(\tau_2) + \frac{V_3(\tau_3)}{\cosh^2\tau_2} \ , \qquad I_2^V = \frac{1}{2M}K_1^2 + V_3(\tau_3) \ . \tag{2.110}$$

13. *Equidistant-Horicyclic* coordinates are defined by $(x, \tau \in \mathbb{R}, y > 0)$

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\left(y + \frac{x^2}{y} + \frac{1}{y}\right)\cosh\tau \ , \quad u_2 = R\frac{x}{y}\cosh\tau \ , \\ u_1 &= \frac{R}{2}\left(y + \frac{x^2}{y} - \frac{1}{y}\right)\cosh\tau \ , \quad u_3 = R\sinh\tau \ . \end{aligned}\right\} \tag{2.111}$$

Note that the coordinate systems IV.–IX. and XI.–XIII. have the subgroup structure $SO(3,1) \supset SO(2,1)$. The characteristic operators are

$$I_1 = K_1^2 + K_2^2 - L_3^2 \ , \qquad I_2 = (K_2 - L_3)^2 \ . \tag{2.112}$$

The line element is $ds^2 = R^2[d\tau^2 + \cosh^2\tau(dx^2 + dy^2)/y^2]$. For the momentum operators we have $P_x = -\mathrm{i}\hbar\partial_x$,

$$P_y = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial y} - \frac{1}{y}\right) \ , \tag{2.113}$$

P_τ as in (2.46), and the Hamiltonian has the form

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\left[\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{y^2}{\cosh^2\tau}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\right] \\ &= \frac{1}{2MR^2}\left[P_\tau^2 + \frac{1}{\cosh^2\tau}y(P_x^2 + P_y^2)y\right] + \frac{\hbar^2}{8MR^2}\left(4 + \frac{1}{\cosh^2\tau}\right) \ . \end{aligned} \tag{2.114}$$

A potential separable in equidistant-horicyclic coordinates must have the form

$$V(\mathbf{u}) = V_1(\tau) + \frac{1}{\cosh^2\tau}\Big[V_2(y) + y^2V_3(x)\Big] \ . \tag{2.115}$$

For the observables we then obtain

$$I_1^V = \frac{1}{2M}(K_1^2 + K_2^2 - L_3^2) + V_2(y) + y^2V_3(x) \ , \qquad I_2^V = \frac{1}{2M}(K_2 - L_3)^2 + V_3(x) \ . \tag{2.116}$$

14. *Horicyclic-Cylindrical* coordinates are defined by $(y, \varrho > 0, \varphi \in [0, 2\pi))$

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\left(y + \frac{\varrho^2}{y} + \frac{1}{y}\right) \quad , \quad u_1 = R\frac{\varrho\cos\varphi}{y} \quad , \\ u_3 &= \frac{R}{2}\left(y + \frac{\varrho^2}{y} - \frac{1}{y}\right) \quad , \quad u_2 = R\frac{\varrho\sin\varphi}{y} \quad . \end{aligned}\right\} \tag{2.117}$$

The characteristic operators are

$$I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2 \quad , \qquad I_2 = L_3^2 \quad . \tag{2.118}$$

Here $ds^2 = R^2(dy^2 + d\varrho^2 + \varrho^2 d\varphi^2)/y^2$, and the momentum operators P_φ and P_y are given by (2.18, 2.24) and

$$P_\varrho = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\varrho} + \frac{1}{2\varrho}\right) \quad . \tag{2.119}$$

For the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}y^2\left[\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \left(\frac{\partial^2}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right)\right] \\ &= \frac{1}{2MR^2}y\left(P_y^2 + P_\varrho^2 + \frac{P_\varphi^2}{\varrho^2}\right)y - \frac{\hbar^2 y^2}{8MR^2\varrho^2} + \frac{3\hbar^2}{8MR^2} \quad . \end{aligned} \tag{2.120}$$

A potential separable in horicyclic-cylindrical coordinates must have the form

$$V(\mathrm{u}) = V_1(y) + y^2\left(V_2(\varrho) + \frac{V_3(\varphi)}{\varrho^2}\right) \quad . \tag{2.121}$$

The observables then are given by

$$I_1^V = \frac{1}{2M}\Big[(K_1 + L_2)^2 + (K_2 - L_1)^2\Big] + V_2(\varrho) + \frac{V_3(\varphi)}{\varrho^2} \quad , \qquad I_2^V = \frac{1}{2M}L_3^2 + V_3(\varphi) \quad . \tag{2.122}$$

15. *Horicyclic-Elliptic* coordinates are given by $(y, \mu > 0, \nu \in (-\pi, \pi))$

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\left(y + \frac{\cosh^2\mu - \cos^2\nu}{y} + \frac{1}{y}\right) \quad , \quad u_1 = R\frac{\cosh\mu\cos\nu}{y} \quad , \\ u_3 &= \frac{R}{2}\left(y + \frac{\cosh^2\mu - \cos^2\nu}{y} - \frac{1}{y}\right) \quad , \quad u_2 = R\frac{\sinh\mu\sin\nu}{y} \quad . \end{aligned}\right\} \tag{2.123}$$

The characteristic operators are

$$I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2 \quad , \qquad I_2 = L_3^2 + (K_1 + L_2)^2 \quad . \tag{2.124}$$

Here $ds^2 = R^2[dy^2 + (\sinh^2\mu + \sin^2\nu)(d\mu^2 + d\nu^2)]/y^2$ and the momentum operators are given by

$$P_\mu = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu} + \frac{\sinh\mu\cosh\mu}{\sinh^2\mu + \sin^2\nu}\right) \quad , \qquad P_\nu = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\nu} + \frac{\sin\nu\cos\nu}{\sinh^2\mu + \sin^2\nu}\right) \quad , \tag{2.125}$$

and P_y as in (2.24). For the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}y^2\left[\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{1}{\sinh^2\mu + \sin^2\nu}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)\right] \\ &= \frac{1}{2MR^2}y\left(P_y^2 + \frac{1}{\sqrt{\sinh^2\mu + \sin^2\nu}}(P_\mu^2 + P_\nu^2)\frac{1}{\sqrt{\sinh^2\mu + \sin^2\nu}}\right)y + \frac{3\hbar^2}{8MR^2} \quad . \end{aligned} \tag{2.126}$$

A potential separable in horicyclic-elliptic coordinates must have the form

$$V(\mathrm{u}) = V_1(y) + y^2 \frac{V_2(\mu) + V_3(\nu)}{\sinh^2\mu + \sin^2\nu} \ . \tag{2.127}$$

For the second observable we then obtain

$$I_2^V = \frac{1}{2M}\Big[L_3^2 + (K_1 + L_2)^2\Big] + \frac{V_2(\mu) + V_3(\nu)}{\sinh^2\mu + \sin^2\nu} \ . \tag{2.128}$$

16. *Horicyclic-Parabolic* coordinates are given by $(y, \xi, \eta > 0)$

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\left(y + \frac{(\xi^2+\eta^2)^2}{y} + \frac{1}{y}\right) \ , \quad u_1 = R\frac{\eta^2 - \xi^2}{2y} \ , \\ u_3 &= \frac{R}{2}\left(y + \frac{(\xi^2+\eta^2)^2}{y} - \frac{1}{y}\right) \quad\ \ u_2 = R\frac{\xi\eta}{y} \ . \end{aligned}\right\} \tag{2.129}$$

Note that the coordinate systems II. and XIV.–XVI. have the subgroup structure $\mathrm{SO}(3,1) \supset E(2)$. The characteristic operators are

$$I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2 \ , \qquad I_2 = \{L_3, K_1\} + \{L_2, L_3\} \ . \tag{2.130}$$

Here $ds^2 = R^2[dy^2 + (\xi^2+\eta^2)(d\xi^2 + d\eta^2)]/y^2$, and the momentum operators are given by

$$P_\xi = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2+\eta^2}\right) \ , \qquad P_\eta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\nu} + \frac{\eta}{\xi^2+\eta^2}\right) \ , \tag{2.131}$$

and P_y as in (2.24). For the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}y^2\left[\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\right] \\ &= \frac{1}{2MR^2}y\left(P_y^2 + \frac{1}{\sqrt{\xi^2+\eta^2}}(P_\mu^2 + P_\nu^2)\frac{1}{\sqrt{\xi^2+\eta^2}}\right)y + \frac{3\hbar^2}{8MR^2} \ . \end{aligned} \tag{2.132}$$

A potential separable in horicyclic-parabolic coordinates must have the form

$$V(\mathrm{u}) = V_1(y) + y^2\frac{V_2(\eta) + V_3(\xi)}{\xi^2+\eta^2} \ , \tag{2.133}$$

and the second observable yields

$$I_2^V = \frac{1}{4M}(\{L_3, K_1\} + \{L_2, L_3\}) + \frac{V_2(\eta) + V_3(\xi)}{\xi^2+\eta^2} \ . \tag{2.134}$$

17. *Prolate Elliptic* coordinates have the form $[\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K), \beta \in [0, 4K'), \varphi \in [0, 2\pi)]$

$$\left.\begin{aligned} u_0 &= \ R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') \ , \\ u_1 &= \mathrm{i}R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\cos\varphi \ , \\ u_2 &= \mathrm{i}R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\sin\varphi \ , \\ u_3 &= \mathrm{i}R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k') \ . \end{aligned}\right\} \tag{2.135}$$

We do not state the algebraic form of the coordinates. It can be derived from the corresponding expressions of system IV. The characteristic operators are

$$I_1 = L_3^2 \ , \qquad I_2 = \mathbf{L}^2 - \frac{k'^2}{k^2}(K_3^2 + L_3^2) \ . \tag{2.136}$$

The line element is given by $ds^2 = R^2[(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)(d\alpha^2+d\beta^2) - \mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta d\varphi^2]$. Analogously as for the prolate elliptic system on the three-dimensional sphere [81] we can introduce a *rotated prolate elliptic*, also called *prolate elliptic II* system on $\Lambda^{(3)}$. Instead of a trigonometric rotation as for the case on the sphere we must consider in the present case a hyperbolic rotation. We set

$$\sinh^2 f = \frac{a_1-a_2}{a_2-a_3} = \frac{k'^2}{k^2}\ , \qquad \cosh^2 f = \frac{a_1-a_3}{a_2-a_3} = \frac{1}{k^2}\ , \tag{2.137}$$

and the rotated prolate elliptic system is then obtained by

$$\begin{pmatrix} u_0' \\ u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} \cosh f & 0 & 0 & \sinh f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh f & 0 & 0 & \cosh f \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_0\cosh f + u_3 \sinh f \\ u_1 \\ u_2 \\ u_0 \sinh f + u_3\cosh f \end{pmatrix}\ . \tag{2.138}$$

Explicitly this yields

$$\left.\begin{aligned} u_0' &= \frac{R}{a_2-a_3}\Big(\sqrt{(\varrho_1-a_3)(\varrho_2-a_3)} + \sqrt{(\varrho_1-a_2)(\varrho_2-a_2)}\Big) \\ &= R\bigg[\frac{1}{k}\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') \ + \mathrm{i}\frac{k'}{k}\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\bigg]\ , \\ u_1 &= R\sqrt{\frac{(\varrho_1-a_1)(a_1-\varrho_2)}{(a_1-a_2)(a_1-a_3)}} = \mathrm{i}R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\cos\varphi\ , \\ u_2 &= R\sqrt{\frac{(\varrho_1-a_1)(a_1-\varrho_2)}{(a_1-a_2)(a_1-a_3)}} = \mathrm{i}R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\sin\varphi\ , \\ u_3' &= \frac{R}{a_2-a_3}\left(\sqrt{\frac{a_1-a_2}{a_1-a_3}(\varrho_1-a_3)(\varrho_2-a_3)} + \sqrt{\frac{a_1-a_3}{a_1-a_2}(\varrho_1-a_2)(\varrho_2-a_2)}\right) \\ &= R\bigg[\frac{k'}{k}\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') + \frac{\mathrm{i}}{k}\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\bigg]\ . \end{aligned}\right\} \tag{2.139}$$

The rotated operators $\mathbf{L}$ and $\mathbf{K}$ are obtained via the matrix transformation

$$\begin{aligned} \begin{pmatrix} 0 & -L_1' & -L_2' & -L_3' \\ L_1' & 0 & K_3' & -K_2' \\ L_2' & -K_3' & 0 & K_1' \\ L_3' & K_2' & -K_1' & 0 \end{pmatrix} &= \begin{pmatrix} \cosh f & 0 & 0 & \sinh f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh f & 0 & 0 & \cosh f \end{pmatrix} \\ &\times \begin{pmatrix} 0 & -L_1 & -L_2 & -L_3 \\ L_1 & 0 & K_3 & -K_2 \\ L_2 & -K_3 & 0 & K_1 \\ L_3 & K_2 & -K_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cosh f & 0 & 0 & \sinh f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh f & 0 & 0 & \cosh f \end{pmatrix}\ , \end{aligned} \tag{2.140}$$

yielding

$$\left.\begin{aligned} L_1' &= \cosh f L_1 - \sinh f K_2\ , & K_1' &= \cosh f K_1 + \sinh f L_2\ , \\ L_2' &= \cosh f L_2 + \sinh f K_1\ , & K_2' &= \cosh f K_2 - \sinh f L_1\ , \\ L_3' &= L_3\ , & K_3' &= K_3\ . \end{aligned}\right\} \tag{2.141}$$

In the rotated prolate elliptic system we get for the second observable

$$I_2' \equiv \Lambda = \cosh 2f\mathbf{L}^2 - \tfrac{1}{2}\sinh 2f(\{K_2,L_1\} - \{K_1,L_2\})\ . \tag{2.142}$$

Note that for the case $k = k' = 1/\sqrt{2}$, i.e. $f = \pi/4$, we obtain

$$\begin{aligned} u_0' &= R(\sqrt{2}\,\mathrm{sn}\alpha\,\mathrm{dn}\beta + \mathrm{i}\,\mathrm{cn}\alpha\,\mathrm{cn}\beta)\ , \quad u_1 = \mathrm{i}R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\cos\varphi\ , \\ u_3' &= R(\,\mathrm{sn}\alpha\,\mathrm{dn}\beta + \mathrm{i}\sqrt{2}\,\mathrm{cn}\alpha\,\mathrm{cn}\beta)\ , \quad u_2 = \mathrm{i}R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\sin\varphi\ , \end{aligned} \tag{2.143}$$

and the rotated system (2.143) yields in the limit $R\to\infty$ *parabolic* coordinates in $\mathbb{R}^3$, c.f. [170] for the corresponding case of the sphere $S^{(3)}$. The momentum operators are given by

$$P_\alpha = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\alpha} - \frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - \frac{1}{2}\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha}{\mathrm{dn}\alpha}\bigg)\ , \tag{2.144}$$

$$P_\beta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\beta} - \frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} + \frac{1}{2}\frac{\mathrm{cn}\beta\,\mathrm{dn}\beta}{\mathrm{sn}\beta}\bigg)\ , \tag{2.145}$$

and P_φ as in (2.18). For the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2} \\ &\times\bigg[\frac{1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\bigg(\frac{\partial^2}{\partial\alpha^2} - \frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha}{\mathrm{dn}\alpha}\frac{\partial}{\partial\alpha} + \frac{\partial^2}{\partial\beta^2} - \frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta}{\mathrm{dn}\beta}\frac{\partial}{\partial\beta}\bigg) - \frac{1}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\frac{\partial^2}{\partial\varphi^2}\bigg] \\ &= \frac{1}{2MR^2}\Bigg[\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}}(P_\alpha^2 + P_\beta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}} - \frac{1}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}P_\varphi^2\Bigg] \\ &\qquad - \frac{\hbar^2}{8MR^2}\bigg(4 + \frac{1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\bigg(k^4\frac{\mathrm{sn}^2\alpha\,\mathrm{cn}^2\alpha}{\mathrm{dn}^2\alpha} + \frac{\mathrm{cn}^2\beta\,\mathrm{dn}^2\beta}{\mathrm{sn}^2\beta}\bigg)\bigg)\ . \end{aligned} \tag{2.146}$$

A potential separable in prolate elliptic coordinates must have the form

$$V(\mathbf{u}) = \frac{V_1(\alpha) + V_2(\beta)}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} + \frac{V_3(\varphi)}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\ . \tag{2.147}$$

The observables then are

$$I_1^V = \frac{1}{2M}L_3^2 + V_3(\varphi)\ , \tag{2.148}$$

$$I_2^V = \frac{1}{2M}\bigg[\mathbf{L}^2 - \frac{k'^2}{k^2}(K_3^2 + L_3^2)\bigg] + \frac{V_1(\alpha) + V_2(\beta)}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} + \frac{V_3(\varphi)}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\ . \tag{2.149}$$

18. *Oblate Elliptic* coordinates have the form ($\alpha\in(\mathrm{i}K', \mathrm{i}K' + 2K), \beta\in[0, 4K'), \varphi\in[0, 2\pi)$)

$$\left.\begin{aligned} u_0 &= R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k')\ , \\ u_1 &= \mathrm{i}R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\cos\varphi\ , \\ u_2 &= \mathrm{i}R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\sin\varphi\ , \\ u_3 &= R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\ . \end{aligned}\right\} \tag{2.150}$$

The characteristic operators are

$$I_1 = L_3^2\ , \qquad I_2 = \mathbf{L}^2 + k'^2(L_3^2 - K_1^2 - K_2^2)\ . \tag{2.151}$$

The line element is $ds^2 = R^2[(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)(d\alpha^2 + d\beta^2) - \mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta d\varphi^2]$, and the momentum operators are given by

$$P_\alpha = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\alpha} - \frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - \frac{1}{2}\frac{\mathrm{sn}\alpha\,\mathrm{dn}\alpha}{\mathrm{cn}\alpha}\bigg)\ , \tag{2.152}$$

$$P_\beta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\beta} - \frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - \frac{1}{2}\frac{\mathrm{sn}\beta\,\mathrm{dn}\beta}{\mathrm{cn}\beta}\bigg)\ , \tag{2.153}$$

and P_φ as in (2.18). For the Hamiltonian we obtain

$$\begin{aligned}
H_0 &= -\frac{\hbar^2}{2MR^2}\\
&\times\Bigg[\frac{1}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\bigg(\frac{\partial^2}{\partial\alpha^2}-\frac{\mathrm{sn}\alpha\,\mathrm{dn}\alpha}{\mathrm{cn}\alpha}\frac{\partial}{\partial\alpha}+\frac{\partial^2}{\partial\beta^2}-\frac{\mathrm{sn}\beta\,\mathrm{dn}\beta}{\mathrm{cn}\beta}\frac{\partial}{\partial\beta}\bigg)-\frac{1}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\frac{\partial^2}{\partial\varphi^2}\Bigg]\\
&=\frac{1}{2MR^2}\Bigg[\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}}(P_\alpha^2+P_\beta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}}-\frac{1}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}P_\varphi^2\Bigg]\\
&\qquad-\frac{\hbar^2}{8MR^2}\Bigg(4+\frac{1}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\bigg(\frac{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\alpha}{\mathrm{cn}^2\alpha}+\frac{\mathrm{sn}^2\beta\,\mathrm{dn}^2\beta}{\mathrm{cn}^2\beta}\bigg)\Bigg)\ . \qquad (2.154)
\end{aligned}$$

A potential separable in oblate elliptic coordinates must have the form

$$V(\mathrm{u})=\frac{V_1(\alpha)+V_2(\beta)}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}+\frac{V_3(\varphi)}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\ . \qquad (2.155)$$

The observables then are

$$I_1^V = \frac{1}{2M}L_3^2+V_3(\varphi)\ , \qquad (2.156)$$

$$I_2^V = \frac{1}{2M}\Big[\mathbf{L}^2+k'^2(L_3^2-K_1^2-K_2^2)\Big]+\frac{V_1(\alpha)+V_2(\beta)}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}+\frac{V_3(\varphi)}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\ . \qquad (2.157)$$

19. *Elliptic-Cylindrical* coordinates have the form ($\alpha\in(\mathrm{i}K',\mathrm{i}K'+2K),\beta\in[0,4K'),\tau\in\mathbb{R}$)

$$\left.\begin{aligned}
u_0 &= R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k')\cosh\tau\ ,\quad u_2=\mathrm{i}R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\ ,\\
u_1 &= R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k')\sinh\tau\ ,\quad u_3=\mathrm{i}R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\ .
\end{aligned}\right\} \qquad (2.158)$$

The characteristic operators are

$$I_1=K_1^2\ ,\qquad I_2=L_1^2+{k'}^2(L_2^2-K_3^2)\ . \qquad (2.159)$$

The line element is $ds^2=R^2[(k^2\,\mathrm{cn}^2\alpha+{k'}^2\,\mathrm{cn}^2\beta)(d\alpha^2+d\beta^2)+\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta d\tau^2]$, the momentum operators are given by

$$P_\alpha = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\alpha}-\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}+\frac{1}{2}\frac{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{\mathrm{sn}\alpha}\bigg)\ , \qquad (2.160)$$

$$P_\beta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\beta}-\frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}-\frac{1}{2}\frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta}{\mathrm{dn}\beta}\bigg)\ , \qquad (2.161)$$

and P_τ as in (2.18). For the Hamiltonian we obtain

$$\begin{aligned}
H_0 &= -\frac{\hbar^2}{2MR^2}\\
&\times\Bigg[\frac{1}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\bigg(\frac{\partial^2}{\partial\alpha^2}+\frac{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{\mathrm{sn}\alpha}\frac{\partial}{\partial\alpha}+\frac{\partial^2}{\partial\beta^2}-\frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta}{\mathrm{dn}\beta}\frac{\partial}{\partial\beta}\bigg)+\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\frac{\partial^2}{\partial\tau^2}\Bigg]\\
&=\frac{1}{2MR^2}\Bigg[\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}}(P_\alpha^2+P_\beta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}}+\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}P_\tau^2\Bigg]\\
&\qquad-\frac{\hbar^2}{8MR^2}\Bigg(4+\frac{1}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\bigg(\frac{\mathrm{cn}^2\alpha\,\mathrm{dn}^2\alpha}{\mathrm{sn}^2\alpha}+k'^2\frac{\mathrm{sn}^2\beta\,\mathrm{cn}^2\beta}{\mathrm{dn}^2\beta}\bigg)\Bigg)\ . \qquad (2.162)
\end{aligned}$$

A potential separable in elliptic-cylindrical coordinates must have the form

$$V(\mathrm{u})=\frac{V_1(\alpha)+V_2(\beta)}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}+\frac{V_3(\tau)}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\ . \tag{2.163}$$

The observables then are given by

$$I_1^V=\frac{1}{2M}K_1^2+V_3(\tau)\ , \tag{2.164}$$

$$I_2^V=\frac{1}{2M}\Big[L_1^2+k'^2(L_2^2-K_3^2)\Big]+\frac{V_1(\alpha)+V_2(\beta)}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}+\frac{V_3(\tau)}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\ . \tag{2.165}$$

20. *Hyperbolic-Cylindrical 1* coordinates have the form ($\mu\in(\mathrm{i}K',\mathrm{i}K'+2K),\eta\in[0,4K'),\tau\in\mathbb{R}$)

$$\left.\begin{aligned}u_0&=-R\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\eta,k')\cosh\tau\ ,\quad u_2=\mathrm{i}R\,\mathrm{sn}(\mu,k)\,\mathrm{dn}(\eta,k')\ ,\\ u_1&=-R\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\eta,k')\sinh\tau\ ,\quad u_3=\mathrm{i}R\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\eta,k')\ .\end{aligned}\right\} \tag{2.166}$$

We do not state the algebraic form of the coordinates. It can be derived from the corresponding expressions of system V. The characteristic operators are

$$I_1=K_1^2\ ,\qquad I_2=K_3^2-L_2^2+k^2(K_1^2-L_1^2)\ . \tag{2.167}$$

The line element is $ds^2=R^2[(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta)(d\mu^2+d\eta^2)+\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta d\tau^2]$, and the momentum operators are given by

$$P_\mu=\frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\mu}-\frac{k^2\,\mathrm{sn}\mu\,\mathrm{cn}\mu\,\mathrm{dn}\mu}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}-\frac{1}{2}\frac{\mathrm{sn}\mu\,\mathrm{dn}\mu}{\mathrm{cn}\mu}\bigg)\ , \tag{2.168}$$

$$P_\eta=\frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\eta}-\frac{k'^2\,\mathrm{sn}\eta\,\mathrm{cn}\eta\,\mathrm{dn}\eta}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}-\frac{1}{2}\frac{\mathrm{sn}\eta\,\mathrm{dn}\eta}{\mathrm{cn}\eta}\bigg)\ , \tag{2.169}$$

and P_τ as in (2.18). For the Hamiltonian we obtain

$$\begin{aligned}H_0&=-\frac{\hbar^2}{2MR^2}\\ &\quad\times\bigg[\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}\bigg(\frac{\partial^2}{\partial\mu^2}-\frac{\mathrm{sn}\mu\,\mathrm{dn}\mu}{\mathrm{cn}\mu}\frac{\partial}{\partial\mu}+\frac{\partial^2}{\partial\eta^2}-\frac{\mathrm{sn}\eta\,\mathrm{dn}\eta}{\mathrm{cn}\eta}\frac{\partial}{\partial\eta}\bigg)+\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\frac{\partial^2}{\partial\tau^2}\bigg]\\ &=\frac{1}{2MR^2}\bigg[\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}}(P_\mu^2+P_\eta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}}+\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}P_\tau^2\bigg]\\ &\qquad-\frac{\hbar^2}{8MR^2}\bigg(4+\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}\bigg(\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu}+\frac{\mathrm{sn}^2\eta\,\mathrm{dn}^2\eta}{\mathrm{cn}^2\eta}\bigg)\bigg)\ .\end{aligned} \tag{2.170}$$

A potential separable in hyperbolic-cylindrical 1 coordinates reads

$$V(\mathrm{u})=\frac{V_1(\mu)+V_2(\eta)}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}+\frac{V_3(\tau)}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\ , \tag{2.171}$$

and the observables then are given by

$$I_1^V=\frac{1}{2M}K_1^2+V_3(\tau)\ , \tag{2.172}$$

$$I_2^V=\frac{1}{2M}\Big[K_3^2-L_2^2+k^2(K_1^2-L_1^2)\Big]+\frac{V_1(\mu)+V_2(\eta)}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\eta}+\frac{V_3(\tau)}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\ . \tag{2.173}$$

21. *Hyperbolic-Cylindrical 2* coordinates have the form ($\mu \in (\mathrm{i}K', \mathrm{i}K' + 2K), \eta \in [0, 4K'), \tau \in \mathbb{R}$)

$$\left.\begin{aligned} u_0 &= -R\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\eta,k')\ , \quad & u_1 &= \mathrm{i}R\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\eta,k')\cos\varphi\ , \\ u_3 &= \mathrm{i}R\,\mathrm{sn}(\mu,k)\,\mathrm{dn}(\eta,k')\ , \quad & u_2 &= \mathrm{i}R\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\eta,k')\sin\varphi\ . \end{aligned}\right\} \tag{2.174}$$

The characteristic operators are

$$I_1 = L_3^2\ , \qquad I_2 = K_3^2 + L_3^2 - k^2(L_1^2 + L_2^2)\ . \tag{2.175}$$

The line element is $ds^2 = R^2[(k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta)(d\mu^2 + d\eta^2) - \mathrm{dn}^2\mu\,\mathrm{sn}^2\eta d\varphi^2]$, and the momentum operators are given by

$$\begin{aligned} P_\mu &= \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\mu} - \frac{k^2\,\mathrm{sn}\mu\,\mathrm{cn}\mu\,\mathrm{dn}\mu}{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta} - \frac{1}{2}\frac{k^2\,\mathrm{sn}\mu\,\mathrm{cn}\mu}{\mathrm{dn}\mu}\bigg)\ , & (2.176)\\ P_\eta &= \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\eta} - \frac{{k'}^2\,\mathrm{sn}\eta\,\mathrm{cn}\eta\,\mathrm{dn}\eta}{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta} + \frac{1}{2}\frac{\mathrm{cn}\eta\,\mathrm{dn}\eta}{\mathrm{sn}\eta}\bigg)\ , & (2.177) \end{aligned}$$

and P_φ as in (2.18). For the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2} \\ &\quad\times\bigg[\frac{1}{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta}\bigg(\frac{\partial^2}{\partial\mu^2} - \frac{k^2\,\mathrm{sn}\mu\,\mathrm{cn}\mu}{\mathrm{dn}\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\eta^2} + \frac{\mathrm{cn}\eta\,\mathrm{dn}\eta}{\mathrm{sn}\eta}\frac{\partial}{\partial\eta}\bigg) - \frac{1}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\frac{\partial^2}{\partial\varphi^2}\bigg] \\ &= \frac{1}{2MR^2}\bigg[\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta}}(P_\mu^2 + P_\eta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta}} - \frac{1}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}P_\varphi^2\bigg] \\ &\qquad - \frac{\hbar^2}{8MR^2}\bigg(4 + \frac{1}{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta}\bigg(k^4\frac{\mathrm{sn}^2\mu\,\mathrm{cn}^2\mu}{\mathrm{dn}^2\mu} + \frac{\mathrm{cn}^2\eta\,\mathrm{dn}^2\eta}{\mathrm{sn}^2\eta}\bigg)\bigg)\ . \end{aligned} \tag{2.178}$$

A potential separable in hyperbolic-cylindrical 2 coordinates must have the form

$$V(\mathrm{u}) = \frac{V_1(\mu) + V_2(\eta)}{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta} + \frac{V_3(\varphi)}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\ , \tag{2.179}$$

and for the observables we obtain

$$\begin{aligned} I_1^V &= \frac{1}{2M}L_3^2 + V_3(\varphi)\ , & (2.180)\\ I_2^V &= \frac{1}{2M}\Big[K_3^2 + L_3^2 - k^2(L_1^2 + L_2^2)\Big] + \frac{V_1(\mu) + V_2(\eta)}{k^2\,\mathrm{cn}^2\mu + {k'}^2\,\mathrm{cn}^2\eta} + \frac{V_3(\varphi)}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\ . & (2.181) \end{aligned}$$

22. *Semi-Hyperbolic* coordinates have the form ($\mu_{1,2} > 0, \varphi \in [0, 2\pi)$)

$$\left.\begin{aligned} u_0 &= \frac{R}{\sqrt{2}}\Big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} + \mu_1\mu_2 + 1\Big)^{1/2}\ , \quad & u_1 &= R\sqrt{\mu_1\mu_2}\,\cos\varphi\ , \\ u_3 &= \frac{R}{\sqrt{2}}\Big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} - \mu_1\mu_2 - 1\Big)^{1/2}\ , \quad & u_2 &= R\sqrt{\mu_1\mu_2}\,\sin\varphi\ . \end{aligned}\right\} \tag{2.182}$$

The characteristic operators are

$$I_1 = L_3^2\ , \qquad I_2 = \{K_1, L_2\} + \{K_2, L_1\}\ . \tag{2.183}$$

The line element has the form

$$ds^2 = R^2\left[\frac{\mu_1+\mu_2}{4}\left(\frac{d\mu_1^2}{P(\mu_1)} - \frac{d\mu_2^2}{P(\mu_2)}\right) + \mu_1\mu_2 d\varphi^2\right] , \tag{2.184}$$

the momentum operators are ($i = 1, 2$)

$$P_{\mu_i} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu_i} + \frac{1}{2(\mu_1+\mu_2)} - \frac{1}{4}\frac{P'(\mu_i)}{P(\mu_i)} + \frac{1}{2\mu_i}\right) , \tag{2.185}$$

with P_φ as in (2.18), and for the Hamiltonian we obtain

$$\begin{aligned}
H_0 &= -\frac{\hbar^2}{8MR^2}\left[\frac{4}{\mu_1+\mu_2}\left(P(\mu_1)\left(\frac{\partial^2}{\partial\mu_1^2} + \frac{P'(\mu_1)}{2P(\mu_1)}\frac{\partial}{\partial\mu_1}\right) - P(\mu_2)\left(\frac{\partial^2}{\partial\mu_2^2} + \frac{P'(\mu_2)}{2P(\mu_2)}\frac{\partial}{\partial\mu_2}\right)\right)\right. \\
&\qquad\qquad \left. + \frac{1}{\mu_1}\frac{\partial}{\partial\mu_1} + \frac{1}{\mu_2}\frac{\partial}{\partial\mu_2} + \frac{1}{\mu_1\mu_2}\frac{\partial^2}{\partial\varphi^2}\right] \\
&= \frac{1}{2MR^2}\left[\left(\sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}}P_{\mu_1}^2\sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}} - \sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}}P_{\mu_2}^2\sqrt{\frac{4P(\mu_2)}{\mu_1+\mu_2}}\right) + \frac{1}{\mu_1\mu_2}P_\varphi^2\right] \\
&\qquad + \frac{\hbar^2}{2MR^2}\frac{1}{\mu_1+\mu_2}\left(P''(\mu_1) - P''(\mu_2) - \frac{3P'^2(\mu_1)}{4P(\mu_1)} + \frac{3P'^2(\mu_2)}{4P(\mu_2)}\right) \\
&\qquad - \frac{\hbar^2}{8MR^2(\mu_1+\mu_2)}\left(\frac{P(\mu_1)}{\mu_1^2} - \frac{P(\mu_2)}{\mu_2^2}\right) .
\end{aligned} \tag{2.186}$$

A potential separable in semi-hyperbolic coordinates must have the form

$$V(\mathrm{u}) = \frac{V_1(\mu_1) + V_2(\mu_2)}{\mu_1+\mu_2} + \frac{V_3(\varphi)}{\mu_1\mu_2} . \tag{2.187}$$

The observables then are

$$I_1^V = \frac{1}{2M}L_3^2 + V_3(\varphi) , \tag{2.188}$$

$$I_2^V = \frac{1}{4M}(\{K_1, L_2\} + \{K_2, L_1\}) + \frac{V_1(\mu_1) + V_2(\mu_2)}{\mu_1+\mu_2} + \frac{V_3(\varphi)}{\mu_1\mu_2} . \tag{2.189}$$

23. *Elliptic-Parabolic 1* coordinates are defined by ($a, \varrho \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$)

$$\left.\begin{aligned}
u_0 &= R\frac{\cosh^2 a + \cos^2\vartheta + \varrho^2}{2\cosh a\cos\vartheta} , & u_1 &= R\frac{\varrho}{\cosh a\cos\vartheta} , \\
u_3 &= R\frac{\cosh^2 a + \cos^2\vartheta - \varrho^2 - 2}{2\cosh a\cos\vartheta} , & u_2 &= R\tanh a\tan\vartheta .
\end{aligned}\right\} \tag{2.190}$$

The characteristic operators are

$$I_1 = (K_1 + L_2)^2 , \tag{2.191}$$

$$I_2 = 2K_1^2 + K_2^2 + K_3^2 + L_1^2 - \{K_1, L_2\} - \{K_2, L_1\} . \tag{2.192}$$

The line element has the form

$$ds^2 = R^2\frac{(\cosh^2 a - \cos^2\vartheta)(da^2 + d\vartheta^2) + d\varrho^2}{\cosh^2 a\cos^2\vartheta} , \tag{2.193}$$

the momentum operators are

$$P_a = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial a} + \frac{\sinh a\cosh a}{\cosh^2 a - \cos^2\vartheta} - \frac{3}{2}\tanh a\right) , \tag{2.194}$$

$$P_\vartheta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial \vartheta} + \frac{\sin\vartheta\cos\vartheta}{\cosh^2 a - \cos^2\vartheta} + \frac{3}{2}\tan\vartheta\right) , \tag{2.195}$$

with $P_\varrho = -\mathrm{i}\hbar\partial_\varrho$, and the Hamiltonian has the form

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\cosh^2 a\cos^2\vartheta \\ &\quad\times\left[\frac{1}{\cosh^2 a - \cos^2\vartheta}\left(\frac{\partial^2}{\partial a^2} - 3\tanh a\frac{\partial}{\partial a} + \frac{\partial^2}{\partial\vartheta^2} + 3\tan\vartheta\frac{\partial^2}{\partial\vartheta^2}\right) + \frac{\partial^2}{\partial\varrho^2}\right] \\ &= \frac{1}{2MR^2}\left[\frac{\cosh a\cos\vartheta}{\sqrt{\cosh^2 a - \cos^2\vartheta}}(P_a^2 + P_\vartheta^2)\frac{\cosh a\cos\vartheta}{\sqrt{\cosh^2 a - \cos^2\vartheta}} + \cosh^2 a\cos^2\vartheta P_\varrho^2\right] + \frac{3\hbar^2}{8MR^2} . \end{aligned} \tag{2.196}$$

A potential separable in elliptic-parabolic 1 coordinates must have the form

$$V(\mathrm{u}) = \frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\Big[V_1(a) + V_2(\vartheta)\Big] + \cosh^2 a\cos^2\vartheta\, V_3(\varrho) . \tag{2.197}$$

The corresponding observables are

$$I_1^V = \frac{1}{2M}(K_1 + L_2)^2 + V_3(\varrho) , \tag{2.198}$$

$$\begin{aligned} I_2^V &= \frac{1}{4M}\Big[2K_1^2 + K_2^2 + K_3^2 + L_1^2 - \{K_1, L_2\} - \{K_2, L_1\}\Big] \\ &\quad + \frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\Big[V_1(a) + V_2(\vartheta)\Big] + \cosh^2 a\cos^2\vartheta\, V_3(\varrho) . \end{aligned} \tag{2.199}$$

24. *Hyperbolic-Parabolic 1* coordinates have the form ($b > 0, \varrho \in \mathbb{R}, \vartheta \in (0, \pi)$)

$$\left.\begin{aligned} u_0 &= R\frac{\sinh^2 b - \sin^2\vartheta + \varrho^2 + 2}{2\sinh b\sin\vartheta} , & u_1 &= \frac{R\varrho}{\sinh b\sin\vartheta} , \\ u_3 &= R\frac{\sinh^2 b - \sin^2\vartheta - \varrho^2}{2\sinh b\sin\vartheta} , & u_2 &= R\coth b\cot\vartheta . \end{aligned}\right\} \tag{2.200}$$

The characteristic operators are

$$I_1 = (K_1 + L_2)^2 , \tag{2.201}$$

$$I_2 = 2L_2^2 + L_1^2 + K_2^2 - K_3^2 - \{L_1, K_2\} - \{K_1, L_2\} . \tag{2.202}$$

with the line element

$$ds^2 = R^2\frac{(\sinh^2 b + \sin^2\vartheta)(db^2 + d\vartheta^2) + d\varrho^2}{\sinh^2 b\sin^2\vartheta} , \tag{2.203}$$

the momentum operators are

$$P_b = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial b} + \frac{\sinh b\cosh b}{\sinh^2 b + \sin^2\vartheta} - \frac{3}{2}\coth b\right) , \tag{2.204}$$

$$P_\vartheta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial \vartheta} + \frac{\sin\vartheta\cos\vartheta}{\sinh^2 b + \sin^2\vartheta} - \frac{3}{2}\cot\vartheta\right) , \tag{2.205}$$

with $P_\varrho = -\mathrm{i}\hbar\partial_\varrho$, and the Hamiltonian is given by

$$H_0 = -\frac{\hbar^2}{2MR^2}\sinh^2 b\sin^2\vartheta\left[\frac{1}{\sinh^2 b+\sin^2\vartheta}\left(\frac{\partial^2}{\partial b^2}-3\coth b\frac{\partial}{\partial b}+\frac{\partial^2}{\partial\vartheta^2}-3\cot\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{\partial^2}{\partial\varrho^2}\right]$$
$$=\frac{1}{2MR^2}\left[\frac{\sinh b\sin\vartheta}{\sqrt{\sinh^2 b+\sin^2\vartheta}}(P_b^2+P_\vartheta^2)\frac{\sinh b\sin\vartheta}{\sqrt{\sinh^2 b+\sin^2\vartheta}}+\sinh^2 b\sin^2\vartheta P_\varrho^2\right]+\frac{3\hbar^2}{8MR^2}\ . \tag{2.206}$$

A potential separable in hyperbolic-parabolic 1 coordinates reads

$$V(\mathrm{u}) = \frac{\sinh^2 b\sin^2\vartheta}{\sinh^2 b+\sin^2\vartheta}\Big[V_1(b)+V_2(\vartheta)\Big]+\sinh^2 b\sin^2\vartheta\, V_3(\varrho)\ , \tag{2.207}$$

and for the corresponding observables we obtain

$$I_1^V = \frac{1}{2M}(K_1+L_2)^2+V_3(\varrho)\ , \tag{2.208}$$
$$I_2^V = \frac{1}{4M}\Big[2L_2^2+L_1^2+K_2^2-K_3^2-\{K_2,L_1\}-\{K_1,L_2\}\Big]$$
$$+\frac{\sinh^2 b\sin^2\vartheta}{\sinh^2 b+\sin^2\vartheta}\Big[V_1(b)+V_2(\vartheta)\Big]+\sinh^2 b\sin^2\vartheta\, V_3(\varrho)\ . \tag{2.209}$$

25. *Elliptic-Parabolic 2* coordinates are $(a>0,\vartheta\in(0,\pi/2),\varphi\in[0,2\pi))$

$$\left.\begin{aligned} u_0 &= R\frac{\cos^2\vartheta+\cosh^2 a}{2\cosh a\cos\vartheta}\ , & u_1 &= R\tanh a\tan\vartheta\cos\varphi\ ,\\ u_3 &= R\frac{\sinh^2 a-\sin^2\vartheta}{2\cosh a\cos\vartheta}\ , & u_2 &= R\tanh a\tan\vartheta\sin\varphi\ . \end{aligned}\right\} \tag{2.210}$$

The characteristic operators are

$$I_1 = L_3^2\ ,\qquad I_2 = 2\mathbf{L}^2-\{L_2,K_1\}-\{L_1,K_2\}\ , \tag{2.211}$$

with the line element

$$ds^2 = R^2\frac{(\cosh^2 a-\cos^2\vartheta)(da^2+d\vartheta^2)+\sinh^2 a\sin^2\vartheta d\varphi^2}{\cosh^2 a\cos^2\vartheta}\ , \tag{2.212}$$

the momentum operators are

$$P_a = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial a}+\frac{\sinh a\cosh a}{\cosh^2 a-\cos^2\vartheta}+\frac{1}{2}(\coth a-3\tanh a)\right)\ , \tag{2.213}$$
$$P_\vartheta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\vartheta}+\frac{\sin\vartheta\cos\vartheta}{\cosh^2 a-\cos^2\vartheta}+\frac{1}{2}(\cot\vartheta+3\tan\vartheta)\right)\ , \tag{2.214}$$

with P_φ as in (2.18), and the Hamiltonian has the form

$$H_0 = -\frac{\hbar^2}{2MR^2}\cosh^2 a\cos^2\vartheta$$
$$\times\left[\frac{1}{\cosh^2 a-\cos^2\vartheta}\left(\frac{\partial^2}{\partial a^2}+(\coth a-3\tanh a)\frac{\partial}{\partial a}+\frac{\partial^2}{\partial\vartheta^2}+(\cot\vartheta+3\tan\vartheta)\frac{\partial}{\partial\vartheta}\right)+\frac{\partial^2}{\partial\varphi^2}\right]$$
$$=\frac{1}{2MR^2}\left[\frac{\cosh a\cos\vartheta}{\sqrt{\cosh^2 a-\cos^2\vartheta}}(P_a^2+P_\vartheta^2)\frac{\cosh a\cos\vartheta}{\sqrt{\cosh^2 a-\cos^2\vartheta}}+\cosh^2 a\cos^2\vartheta P_\varphi^2\right]$$
$$-\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\sinh^2 a\sin^2\vartheta}+\frac{\hbar^2}{2MR^2}\ . \tag{2.215}$$

A potential separable in elliptic-parabolic 2 coordinates must have the form

$$V(\mathrm{u}) = \frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\Big[V_1(a)+V_2(\vartheta)\Big]+\coth^2 a\cot^2\vartheta\,V_3(\varphi)\ . \tag{2.216}$$

The corresponding observables have the form

$$I_1^V = \frac{1}{2M}L_3^2+V_3(\varphi)\ , \tag{2.217}$$

$$\begin{aligned} I_2^V &= \frac{1}{4M}\Big[2\mathbf{L}^2-\{L_2,K_1\}-\{L_1,K_2\}\Big] \\ &\qquad +\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\Big[V_1(a)+V_2(\vartheta)\Big]+\coth^2 a\cot^2\vartheta\,V_3(\varphi)\ . \end{aligned} \tag{2.218}$$

26. *Hyperbolic-Parabolic 2* coordinates are $(b>0,\vartheta\in(0,\pi/2),\varphi\in[0,2\pi))$

$$\left.\begin{aligned} u_0 &= R\frac{\cosh^2 b+\cos^2\vartheta}{2\sinh b\sin\vartheta}\ ,\quad u_1 = R\coth b\cot\vartheta\cos\varphi\ , \\ u_3 &= R\frac{\sin^2\vartheta-\sinh^2 b}{2\sinh b\sin\vartheta}\ ,\quad u_2 = R\coth b\cot\vartheta\sin\varphi\ . \end{aligned}\right\} \tag{2.219}$$

with the characteristic operators

$$I_1 = L_3^2\ ,\qquad I_2 = \{K_1,L_2\}+\{K_2,L_1\}-K_1^2-K_2^2\ , \tag{2.220}$$

the line element

$$ds^2 = R^2\frac{(\sinh^2 b+\sin^2\vartheta)(db^2+d\vartheta^2)+\cosh^2 b\cos^2\vartheta d\varphi^2}{\sinh^2 b\sin^2\vartheta}\ , \tag{2.221}$$

the momentum operators

$$P_b = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial b}+\frac{\sinh b\cosh b}{\sinh^2 b+\sin^2\vartheta}+\frac{1}{2}(\tanh b-3\coth b)\bigg)\ , \tag{2.222}$$

$$P_\vartheta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\vartheta}+\frac{\sin\vartheta\cos\vartheta}{\sinh^2 b+\sin^2\vartheta}-\frac{1}{2}(\tan\vartheta+3\cot\vartheta)\bigg)\ , \tag{2.223}$$

with P_φ as in (2.18), and for the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\sinh^2 b\sin^2\vartheta \\ &\quad\times\bigg[\frac{1}{\sinh^2 b+\sin^2\vartheta}\bigg(\frac{\partial^2}{\partial b^2}+(\tanh b-3\coth b)\frac{\partial}{\partial b}+\frac{\partial^2}{\partial\vartheta^2}-(\tan\vartheta+3\cot\vartheta)\frac{\partial}{\partial\vartheta}\bigg)+\frac{\partial^2}{\partial\varphi^2}\bigg] \\ &= \frac{1}{2MR^2}\bigg[\frac{\sinh b\sin\vartheta}{\sqrt{\sinh^2 b+\sin^2\vartheta}}(P_b^2+P_\vartheta^2)\frac{\sinh b\sin\vartheta}{\sqrt{\sinh^2 b+\sin^2\vartheta}}+\sinh^2 b\sin^2\vartheta P_\varphi^2\bigg] \\ &\qquad +\frac{\hbar^2}{8MR^2}\frac{\sin^2\vartheta-\sinh^2 b-1}{\cosh^2 b\cos^2\vartheta}+\frac{\hbar^2}{2MR^2}\ . \end{aligned} \tag{2.224}$$

A potential separable in hyperbolic-parabolic 2 coordinates must have the form

$$V(\mathrm{u}) = \frac{\sinh^2 b\sin^2\vartheta}{\sinh^2 b+\sin^2\vartheta}\Big[V_1(b)+V_2(\vartheta)\Big]+\tanh^2 b\tan^2\vartheta\,V_3(\varphi)\ . \tag{2.225}$$

For the corresponding observables we obtain

$$I_1 = \frac{1}{2M} L_3^2 + V_3(\varphi) \ , \tag{2.226}$$

$$\begin{aligned} I_2 &= \frac{1}{4M}(\{K_1, L_2\} + \{K_2, L_1\} - K_1^2 - K_2^2) \\ &\qquad + \frac{\sinh^2 b \sin^2 \vartheta}{\sinh^2 b + \sin^2 \vartheta}\Big[V_1(b) + V_2(\vartheta)\Big] + \tanh^2 b \tan^2 \vartheta \, V_3(\varphi) \ . \end{aligned} \tag{2.227}$$

27. *Semi-Circular-Parabolic coordinates* are defined by ($\varrho \in \mathbb{R}, \xi, \eta > 0$)

$$\left.\begin{aligned} u_0 &= R\frac{(\eta^2-\xi^2)^2 + 4\varrho^2 + 4}{8\xi\eta} \ , \quad u_1 = R\frac{\eta^2-\xi^2}{2\xi\eta} \ , \\ u_3 &= R\frac{(\eta^2-\xi^2)^2 + 4\varrho^2 - 4}{8\xi\eta} \ , \quad u_2 = R\frac{\varrho}{\xi\eta} \ . \end{aligned}\right\} \tag{2.228}$$

The characteristic operators are

$$I_1 = (K_1 + L_2)^2 \ , \qquad I_2 = \{L_3, K_1 + L_2\} + \{K_3, K_2 - L_1\} \ , \tag{2.229}$$

the line element has the form

$$ds^2 = R^2 \frac{(\xi^2 + \eta^2)(d\xi^2 + d\eta^2) + d\varrho^2}{\xi^2\eta^2} \ , \tag{2.230}$$

and the momentum operators are

$$P_\xi = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2+\eta^2} - \frac{3}{2\xi}\bigg) \ , \quad P_\eta = \frac{\hbar}{\mathrm{i}}\bigg(\frac{\partial}{\partial\eta} + \frac{\eta}{\xi^2+\eta^2} - \frac{3}{2\eta}\bigg) \ , \tag{2.231}$$

with $P_\varrho = -\mathrm{i}\hbar\partial_\varrho$. For the Hamiltonian we get

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\bigg[\frac{\xi^2\eta^2}{\xi^2+\eta^2}\bigg(\frac{\partial}{\partial 2}\xi - \frac{3}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\eta^2} - \frac{3}{\eta}\frac{\partial}{\partial\eta}\bigg) + \xi^2\eta^2\frac{\partial^2}{\partial\varrho^2}\bigg] \\ &= \frac{1}{2MR^2}\bigg[\frac{\xi\eta}{\sqrt{\xi^2+\eta^2}}(P_\xi^2 + P_\eta^2)\frac{\xi\eta}{\sqrt{\xi^2+\eta^2}} + \xi^2\eta^2 P_\varrho^2\bigg] + \frac{3\hbar^2}{8MR^2} \ . \end{aligned} \tag{2.232}$$

A potential separable in semi-circular-parabolic coordinates must have the form

$$V(\mathrm{u}) = \frac{\xi^2\eta^2}{\xi^2+\eta^2}\Big[V_1(\xi) + V_2(\eta)\Big] + \xi^2\eta^2 \, V_3(\varrho) \ , \tag{2.233}$$

and for the observables we obtain

$$I_1^V = \frac{1}{2M}(K_1 + L_2)^2 + V_3(\varrho) \ , \tag{2.234}$$

$$\begin{aligned} I_2^V &= \frac{1}{4M}(\{L_3, K_1 + L_2\} + \{K_3, K_2 - L_1\}) \\ &\qquad + \frac{\xi^2\eta^2}{\xi^2+\eta^2}\Big[V_1(\xi) + V_2(\eta)\Big] + \xi^2\eta^2 \, V_3(\varrho) \ . \end{aligned} \tag{2.235}$$

28. *Ellipsoidal* coordinates have the form ($0 < 1 < \varrho_3 < b < \varrho_2 < a < \varrho_1$)

$$\left.\begin{aligned} u_0^2 &= R^2\frac{\varrho_1\varrho_2\varrho_3}{ab} \ , \\ u_1^2 &= R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)} \ , \\ u_2^2 &= -R^2\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b)}{(a-b)(b-1)b} \ , \\ u_3^2 &= R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a} \ . \end{aligned}\right\} \tag{2.236}$$

The characteristic operators are

$$I_1 = abK_1^2 + aK_2^2 + bK_3^2 \ , \tag{2.237}$$

$$I_2 = (a+b)K_1^2 + (a+1)K_2^2 + (b+1)K_3^2 - aL_3^2 - bL_2^2 - L_1^2 \ . \tag{2.238}$$

The line element is given by

$$ds^2 = \frac{1}{4}\left[\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{P(\varrho_1)}\mathrm{d}\varrho_1^2 + \frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{P(\varrho_2)}\mathrm{d}\varrho_2^2 + \frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{P(\varrho_3)}\mathrm{d}\varrho_3^2\right] \ , \tag{2.239}$$

where we have set for the *characteristic polynomial*

$$P(\varrho) = (\varrho-a)(\varrho-b)(\varrho-1)\varrho \ . \tag{2.240}$$

The momentum operators have the form

$$P_{\varrho_i} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\varrho_i} + \frac{1}{2}\frac{1}{\varrho_i-\varrho_j} + \frac{1}{2}\frac{1}{\varrho_i-\varrho_k} - \frac{1}{4}\frac{P'(\varrho_i)}{P(\varrho_i)}\right) \ , \tag{2.241}$$

with i,j,k cyclic, and for the Hamiltonian we obtain ($g_{aa} = h_{ac}h_{ac} \equiv h_a^2, g^{aa} = (h^a)^2$)

$$H_0 = -\frac{\hbar^2}{2MR^2}\sum_{\substack{i,j,k=1,2,3\\ i\neq j\neq k\neq i \ \text{cyclic}}} \frac{4\sqrt{P(\varrho_i)}}{(\varrho_i-\varrho_j)(\varrho_i-\varrho_k)}\frac{\partial}{\partial\varrho_i}\sqrt{P(\varrho_i)}\frac{\partial}{\partial\varrho_i} \tag{2.242}$$

$$= \frac{1}{2MR^2}\left[\sum_{i=1}^{3} h^i P_{\varrho_i}^2 h^i + \Delta V(\boldsymbol{\varrho})\right] \ . \tag{2.243}$$

The quantum potential ΔV is given by

$$\Delta V(\boldsymbol{\varrho}) = \frac{\hbar^2}{8MR^2}\Big[(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)(\Gamma_1^2+2\Gamma_1') + (\varrho_2-\varrho_3)(\varrho_2-\varrho_1)(\Gamma_1^2+2\Gamma_1') + (\varrho_3-\varrho_1)(\varrho_3-\varrho_2)(\Gamma_1^2+2\Gamma_1')\Big] \ , \tag{2.244}$$

where $\Gamma_i = f_i'/f_i, f_i^2 = P(\varrho_i)$. A potential separable in ellipsoidal coordinates must have the form

$$V(\mathrm{u}) = \frac{(\varrho_2-\varrho_3)V_1(\varrho_1) + (\varrho_1-\varrho_3)V_2(\varrho_2) + (\varrho_1-\varrho_2)V_3(\varrho_3)}{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)} \ . \tag{2.245}$$

29. *Hyperboloidal* coordinates have the form ($\varrho_3 < 0 < 1 < b < \varrho_2 < a < \varrho_1$)

$$\left.\begin{aligned} u_0^2 &= -R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)} \ , \\ u_1^2 &= -R^2\frac{\varrho_1\varrho_2\varrho_3}{ab} \ , \\ u_2^2 &= -R^2\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b)}{(a-b)(b-1)b} \ , \\ u_3^2 &= R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a} \ . \end{aligned}\right\} \tag{2.246}$$

The characteristic operators are

$$I_1 = abK_1^2 - aL_3^2 - bL_2^2 \ , \tag{2.247}$$

$$I_2 = (a+b)K_1^2 - (a+1)L_3^2 - (b+1)L_2^2 + aK_2^2 + bK_3^2 - L_1^2 \ . \tag{2.248}$$

The characteristic polynomial $P(\varrho)$, the line element ds^2, the momentum operators P_{ϱ_i}, and the Hamiltonian H have the same form as in (2.240–2.242), with the appropriate changes, respectively, and a potential separable in hyperboloidal coordinates must have the same form as in (2.245).

30. *Paraboloidal* coordinates are given by ($\varrho_3 < 0 < \varrho_2 < 1 < \varrho_1, a = b^* = \alpha + \mathrm{i}\beta, \alpha, \beta \in \mathbb{R}$)

$$\left.\begin{aligned} (u_1 + \mathrm{i}u_0)^2 &= 2R^2\frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{(a-b)(b-1)b} \ , \\ u_2^2 &= R^2\frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)(b-1)} \ , \\ u_3^2 &= -R^2\frac{\varrho_1\varrho_2\varrho_3}{ab} \ . \end{aligned}\right\} \tag{2.249}$$

The characteristic operators are

$$I_1 = -(\alpha^2 + \beta^2)L_1^2 + \alpha(K_3^2 - L_2^2) - \beta\{K_3, L_2\} \ , \tag{2.250}$$

$$I_2 = -2\alpha L_1^2 + (\alpha + 1)(K_3^2 - L_2^2) + \alpha(K_2^2 - L_3^2) + \beta(\{K_2, L_3\} - \{K_3, L_2\}) \ . \tag{2.251}$$

The characteristic polynomial $P(\varrho)$, the line element, the momentum operators P_{ϱ_i}, and the Hamiltonian H have the same form as in (2.240–2.242), with the appropriate changes, respectively, and a potential separable in paraboloidal coordinates must have the same form as in (2.245).

31. The coordinates of *System XXXI.* are given by ($0 < \varrho_3 < 1 < \varrho_2 < a < \varrho_1$)

$$\left.\begin{aligned} (u_0 + u_1)^2 &= R^2\frac{\varrho_1\varrho_2\varrho_3}{a} \ , \\ (u_0^2 - u_1^2) &= R^2\frac{a(\varrho_1\varrho_2 + \varrho_1\varrho_3 + \varrho_2\varrho_3) - (a+1)\varrho_1\varrho_2\varrho_3}{a^2} \ , \\ u_2^2 &= -R^2\frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)} \ , \\ u_3^2 &= R^2\frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{a^2(a-1)} \ . \end{aligned}\right\} \tag{2.252}$$

The characteristic operators are

$$I_1 = (K_3 + L_2)^2 - a(K_2 + L_3)^2 + aK_1^2 \ , \tag{2.253}$$

$$I_2 = (a+1)K_1^2 + K_3^2 - L_2^2 + a(L_3^2 - K_2^2) + (K_2 + L_3)^2 + (K_3 + L_2)^2 \ . \tag{2.254}$$

The line element, the momentum operators P_{ϱ_i}, and the Hamiltonian H have the same form as in (2.239–2.242), with the appropriate changes, respectively. The characteristic polynomial has the form

$$P(\varrho) = (\varrho - a)(\varrho - 1)\varrho^2 \ , \tag{2.255}$$

and a potential separable in coordinates XXXI must have the same form as in (2.245).

32. The coordinates of *System XXXII.* are given by ($-\varrho_3 < 0 < 1 < \varrho_2 < a < \varrho_1$)

$$\left.\begin{aligned} (u_0 + u_1)^2 &= -R^2\frac{\varrho_1\varrho_2\varrho_3}{a} \ , \\ (u_0^2 - u_1^2) &= R^2\frac{a(\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3) - (a+1)\varrho_1\varrho_2\varrho_3}{a^2} \ , \\ u_2^2 &= -R^2\frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)} \ , \\ u_3^2 &= R^2\frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{a^2(a-1)} \ . \end{aligned}\right\} \tag{2.256}$$

The characteristic operators are

$$I_1 = -(K_3 + L_2)^2 + a(K_2 + L_3)^2 + aK_1^2 \ , \tag{2.257}$$

$$I_2 = (a+1)K_1^2 - K_3^2 + L_2^2 + a(K_2^2 - L_3^2) - (K_2 + L_3)^2 - (K_3 + L_2)^2 \ . \tag{2.258}$$

The line element, the momentum operators P_{ϱ_i}, and the Hamiltonian H have the same form as in (2.239–2.242), with the appropriate changes, respectively. The characteristic polynomial has the form as in system XXXI., and a potential separable in system XXXII. must have the same form as in (2.245).

33. The coordinates of *System XXXIII.* are given by ($\varrho_3 < -1 < 0 < \varrho_2 < a < \varrho_1$)

$$\left.\begin{aligned}
(u_0+u_1)^2 &= -R^2\frac{\varrho_1\varrho_2\varrho_3}{a} \ , \\
(u_0^2-u_1^2) &= R^2\frac{a(\varrho_1\varrho_3+\varrho_1\varrho_2+\varrho_2\varrho_3)-(a-1)\varrho_1\varrho_2\varrho_3}{a^2} \ , \\
u_2^2 &= R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a+1)} \ , \\
u_3^2 &= -R^2\frac{(\varrho_1+1)(\varrho_2+1)(\varrho_3+1)}{(a+1)} \ .
\end{aligned}\right\} \tag{2.259}$$

The characteristic operators are

$$I_1 = aK_1^2-(K_2+L_3)^2+a(K_2+L_3)^2 \ , \tag{2.260}$$
$$I_2 = (a-1)K_1^2-K_2^2+L_3^2+a(L_2^2-K_3^2)-(K_2+L_3)^2+(K_3+L_2)^2 \ . \tag{2.261}$$

The line element, the momentum operators P_{ϱ_i}, and the Hamiltonian H have the same form as in (2.239–2.242), with the appropriate changes, respectively. The characteristic polynomial has the form

$$P(\varrho) = (\varrho-a)(\varrho+1)\varrho^2 \ , \tag{2.262}$$

and a potential separable in system XXXIII. must have the same form as in (2.245).

34. The coordinates of *System XXXIV.* are given by ($\varrho_3 < 0 < \varrho_2 < 1 < \varrho_1$)

$$\left.\begin{aligned}
(u_0-u_1)^2 &= -R^2\varrho_1\varrho_2\varrho_3 \ , \\
2u_2(u_1-u_0) &= R^2(\varrho_1\varrho_2+\varrho_2\varrho_3+\varrho_1\varrho_3-\varrho_1\varrho\varrho_3) \ , \ , \\
u_1^2+u_2^2-u_0^2 &= R^2(-\varrho_1\varrho_2\varrho_3+\varrho_1\varrho_2+\varrho_2\varrho_3+\varrho_1\varrho_3-\varrho_1-\varrho_2-\varrho_3) \ , \\
u_3^2 &= R^2(\varrho_1-1)(\varrho_2-1)(\varrho_3-1) \ .
\end{aligned}\right\} \tag{2.263}$$

The characteristic operators are

$$I_1 = (L_2-K_3)^2-K_1(K_2-L_3)-(K_2-L_3)K_1 \ , \tag{2.264}$$
$$I_2 = L_2^2-K_3^2-L_1^2-(L_2-K_3)^2-\{L_1,L_2-K_3\} \ . \tag{2.265}$$

The line element, the momentum operators P_{ϱ_i}, and the Hamiltonian H have the same form as in (2.239–2.242), with the appropriate changes, respectively. The characteristic polynomial has the form

$$P(\varrho) = (\varrho-1)\varrho^3 \ , \tag{2.266}$$

and a potential separable in system XXXIV. must have the same form as in (2.245).

Table 2.3 summarizes our enumeration of the coordinate systems according to our findings. The potentials $V_1,\dots,V_{20}$ refer to sections 3,4 and 5. By the the notion "limiting systems" we mean the emerging coordinate system in $\mathbb{R}^3$, as $R\to\infty$.

Table 2.3: Coordinate Systems on the Three-Dimensional Hyperboloid

Coordinate System Observables I_1, I_2	Coordinates	Separates Potential	Limiting Systems
I. Cylindrical $\tau_{1,2} \in \mathbb{R}, \varphi \in [0, 2\pi)$ $I_1 = K_3^2$ $I_2 = L_3^2$	$u_0 = R \cosh \tau_1 \cosh \tau_2$ $u_1 = R \sinh \tau_1 \cos \varphi$ $u_2 = R \sinh \tau_1 \sin \varphi$ $u_3 = R \cosh \tau_1 \sinh \tau_2$	V_1	Circular-Polar
II. Horicyclic $x_{1,2} \in \mathbb{R}, y > 0$ $I_1 = (K_1 + L_2)^2$ $I_2 = (K_2 - L_1)^2$	$u_0 = R\big[y + (x_1^2 + x_2^2)/y + 1/y\big]/2$ $u_1 = R\,x_1/2y$ $u_2 = R\,x_2/2y$ $u_3 = R\big[y + (x_1^2 + x_2^2)/y - 1/y\big]/2$	V_8, V_9, V_{10}	Cartesian
III. Sphero-Elliptic $\tau > 0, \tilde{\alpha} \in [-K, K]$ $\tilde{\beta} \in [-2K', 2K']$ $I_1 = \mathbf{L}^2, I_2 = L_1^2 + {k'}^2 L_2^2$	$u_0 = R \cosh \tau$ $u_1 = R \sinh \tau \,\mathrm{sn}\tilde{\alpha}\,\mathrm{dn}\tilde{\beta}$ $u_2 = R \sinh \tau \,\mathrm{cn}\tilde{\alpha}\,\mathrm{cn}\tilde{\beta}$ $u_3 = R \sinh \tau \,\mathrm{dn}\tilde{\alpha}\,\mathrm{sn}\tilde{\beta}$	V_1, V_2, V_7 $V_7'^{*}, V_{13}^{*}$	Sphero-Conical Sphero Conical II*
IV. Equidistant-Elliptic $\tau \in \mathbb{R}, \alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\beta \in [0, 4K')$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = L_3^2 + \sinh^2 f K_1^2$	$u_0 = R \cosh \tau \,\mathrm{sn}\alpha\,\mathrm{dn}\beta$ $u_1 = \mathrm{i}R \cosh \tau \,\mathrm{cn}\alpha\,\mathrm{cn}\beta$ $u_2 = \mathrm{i}R \cosh \tau \,\mathrm{dn}\alpha\,\mathrm{sn}\beta$ $u_3 = R \sinh \tau$	V_1, V_4^+, V_{14} V_{15}^+	Circular-Elliptic Circular-Elliptic II+
V. Equidistant-Hyperbolic $\tau \in \mathbb{R}, \mu \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\eta \in [0, 4K')$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = K_1^2 - \sin^2 \alpha L_3^2$	$u_0 = -R \cosh \tau \,\mathrm{cn}\mu\,\mathrm{cn}\eta$ $u_1 = \mathrm{i}R \cosh \tau \,\mathrm{sn}\mu\,\mathrm{dn}\eta$ $u_2 = \mathrm{i}R \cosh \tau \,\mathrm{dn}\mu\,\mathrm{sn}\eta$ $u_3 = R \sinh \tau$	V_1, V_{14}	Cartesian
VI. Equidistant-Semi-Hyperbolic $\tau \in \mathbb{R}, \mu_{1,2} > 0$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = \{L_3, K_2\}$	$u_0 = \frac{R}{\sqrt{2}} \cosh \tau \big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} + \mu_1\mu_2 + 1\big)^{1/2}$ $u_1 = \frac{R}{\sqrt{2}} \cosh \tau \big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} - \mu_1\mu_2 - 1\big)^{1/2}$ $u_2 = R \cosh \tau \sqrt{\mu_1\mu_2}$ $u_3 = R \sinh \tau$	V_4, V_{15}	Circular-Parabolic
VII. Equidistant-Elliptic-Parabolic $\tau, a \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = (K_2 - L_3)^2 + K_1^2$	$u_0 = R \cosh \tau \frac{\cosh^2 a + \cos^2 \vartheta}{2 \cosh a \cos \vartheta}$ $u_1 = R \cosh \tau \frac{\sinh^2 a - \sin^2 \vartheta}{2 \cosh a \cos \vartheta}$ $u_2 = R \cosh \tau \tan \vartheta \tanh a$ $u_3 = R \sinh \tau$	V_4, V_{15}, V_{17}	Circular-Parabolic
VIII. Equidistant-Hyperbolic-Parabolic $\tau \in \mathbb{R}, b > 0, \vartheta \in (0, \pi)$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = (K_2 - L_3)^2 - K_1^2$	$u_0 = R \cosh \tau \,\frac{\cosh^2 b + \cos^2 \vartheta}{2 \sinh b \sin \vartheta}$ $u_2 = R \cosh \tau \,\frac{\sinh^2 b - \sin^2 \vartheta}{2 \sinh b \sin \vartheta}$ $u_1 = R \cosh \tau \cot \vartheta \coth b$ $u_3 = R \sinh \tau$	V_{17}	Cartesian
IX. Equidistant-Semi-Circular-Parabolic ($\tau \in \mathbb{R}, \xi, \eta > 0$) $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = \{K_1, K_2\} - \{K_1, L_3\}$	$u_0 = R \cosh \tau \,\frac{(\xi^2 + \eta^2)^2 + 4}{8\xi\eta}$ $u_1 = R \cosh \tau \,\frac{(\xi^2 + \eta^2)^2 - 4}{8\xi\eta}$ $u_2 = R \cosh \tau \,\frac{\eta^2 - \xi^2}{2\xi\eta}$ $u_3 = R \sinh \tau$	V_{16}, V_{17}, V_{18}	Cartesian

* after rotation with $I_2' = \cos 2f L_3^2 - \frac{1}{2} \sin 2f \{L_1, L_3\}$; + after rotation with $I_2' = \cosh 2f L_3^2 - \frac{1}{2} \sinh 2f \{K_2, L_3\}$

Table 2.3 (cont.) Coordinate System	Coordinates	Separates Potential	Limiting Systems
X. Spherical $\tau > 0, \vartheta \in (0,\pi), \varphi \in [0,2\pi)$ $I_1 = \mathbf{L}^2$ $I_2 = L_3^2$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\sin\vartheta\cos\varphi$ $u_2 = R\sinh\tau\sin\vartheta\sin\varphi$ $u_3 = R\sinh\tau\cos\vartheta$	V_1, V_2, V_3 V_5, V_6, V_7 V_7', V_{13}	Spherical
XI. Equidistant-Cylindrical $\tau_{1,2} \in \mathbb{R}, \varphi \in [0,2\pi)$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = L_3^2$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2\cos\varphi$ $u_2 = R\cosh\tau_1\sinh\tau_2\sin\varphi$ $u_3 = R\sinh\tau_1$	V_1, V_3, V_4 V_5, V_{14}, V_{15}	Circular-Polar
XII. Equidistant $\tau_{1,2,3} \in \mathbb{R}$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = K_1^2$	$u_0 = R\cosh\tau_1\cosh\tau_2\cosh\tau_3$ $u_1 = R\cosh\tau_1\cosh\tau_2\sinh\tau_3$ $u_2 = R\cosh\tau_1\sinh\tau_2$ $u_3 = R\sinh\tau_1$	V_1, V_{14}, V_{17} V_{18}	Cartesian
XIII. Equidistant-Horicyclic $\tau, x \in \mathbb{R}, y > 0$ $I_1 = K_1^2 + K_2^2 - L_3^2$ $I_2 = (K_2 - L_3)^2$	$u_0 = R\cosh\tau(y + x^2/y + 1/y)/2$ $u_1 = R\sinh\tau$ $u_2 = R\cosh\tau x/y$ $u_3 = R\cosh\tau(y + x^2/y - 1/y)/2$	V_{16}, V_{17}	Cartesian
XIV. Horicyclic-Cylindrical $y, \varrho > 0, \varphi \in [0,2\pi)$ $I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2$ $I_2 = L_3^2$	$u_0 = R(y + \varrho^2/y + 1/y)/2$ $u_1 = R\,\varrho\cos\varphi/y$ $u_2 = R\,\varrho\sin\varphi/y$ $u_3 = R(y + \varrho^2/y - 1/y)/2$	V_9, V_{11}	Circular-Polar
XV. Horicyclic-Elliptic $y, \mu > 0, \nu \in (-\pi,\pi)$ $I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2$ $I_2 = L_3^2 + (K_1 + L_2)^2$	$u_0 = R[y + (\cosh^2\mu - \sin^2\nu)/y + 1/y]/2$ $u_1 = R\cosh\mu\cos\nu/y$ $u_2 = R\sinh\mu\sin\nu/y$ $u_3 = R[y + (\cosh^2\mu - \sin^2\nu)/y - 1/y]/2$	V_9, V_{11}^*	Cartesian
XVI. Horicyclic-Parabolic $y, \eta > 0, \xi \in \mathbb{R}$ $I_1 = (K_1 + L_2)^2 + (K_2 - L_1)^2$ $I_2 = \{L_3, K_1 + L_2\}$	$u_0 = R[y + (\xi^2 + \eta^2)^2/y + 1/y]/2$ $u_1 = R(\eta^2 - \xi^2)/2y$ $u_2 = R\xi\eta/y$ $u_3 = R[y + (\xi^2 + \eta^2)^2/y - 1/y]/2$	V_{10}, V_{11}, V_{12}	Circular-Parabolic
XVII. Prolate Elliptic $\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\beta \in [0,4K'), \varphi \in [0,2\pi)$ $I_1 = L_3^2$ $I_2 = \mathbf{L}^2 - (k'^2/k^2)(K_3^2 + L_3^2)$	$u_0 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$ $u_1 = \mathrm{i}R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\cos\varphi$ $u_2 = \mathrm{i}R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\sin\varphi$ $u_3 = \mathrm{i}R\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$	V_1, V_2^*, V_3 V_5, V_6^*	Prolate-Spheroidal Prolate-Spheroidal II* (Parabolic)
XVIII. Oblate Elliptic $\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\beta \in [0,4K'), \varphi \in [0,2\pi)$ $I_1 = L_3^2$ $I_2 = \mathbf{L}^2 + k'^2(L_3^2 - K_1^2 - K_2^2)$	$u_0 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta$ $u_1 = \mathrm{i}R\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\cos\varphi$ $u_2 = \mathrm{i}R\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\sin\varphi$ $u_3 = \mathrm{i}R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$	V_1, V_3, V_5	Oblate-Spheroidal
XIX. Elliptic-Cylindrical $\tau \in \mathbb{R}, \alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K), \beta \in [0,4K')$ $I_1 = K_1^2$ $I_2 = L_1^2 + k'^2(L_2^2 - K_3^2)$	$u_0 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\cosh\tau$ $u_1 = R\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\sinh\tau$ $u_2 = \mathrm{i}R\,\mathrm{dn}\alpha\,\mathrm{sn}\beta$ $u_3 = \mathrm{i}R\,\mathrm{cn}\alpha\,\mathrm{cn}\beta$	V_1	Circular-Elliptic
XX. Hyperbolic-Cylindrical 1 $\tau \in \mathbb{R}, \mu \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\eta \in [0,4K')$ $I_1 = K_1^2, I_2 = K_3^2 - L_2^2 + k^2(K_1^2 - L_1^2)$	$u_0 = -R\,\mathrm{cn}\mu\,\mathrm{cn}\eta\cosh\tau$ $u_1 = -R\,\mathrm{cn}\mu\,\mathrm{cn}\eta\sinh\tau$ $u_2 = \mathrm{i}R\,\mathrm{sn}\mu\,\mathrm{dn}\eta$ $u_3 = \mathrm{i}R\,\mathrm{dn}\mu\,\mathrm{sn}\eta$	V_1	Cartesian

* after rotation with $I_2' = \cosh 2f\mathbf{L}^2 - \frac{1}{2}\sinh 2f(\{K_2, L_1\} - \{K_1, L_2\})$

Table 2.3 (cont.) Coordinate System	Coordinates	Separates Potential	Limiting Systems
XXI. Hyperbolic-Cylindrical 2 $\mu \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\eta \in [0, 4K'), \varphi \in [0, 2\pi)$ $I_1 = L_3^2, I_2 = K_3^2 + L_3^2 - k^2(L_1^2 + L_2^2)$	$u_0 = -R\,\mathrm{cn}\mu\,\mathrm{cn}\eta$ $u_1 = \mathrm{i}R\,\mathrm{sn}\mu\,\mathrm{dn}\eta\cos\varphi$ $u_2 = \mathrm{i}R\,\mathrm{sn}\mu\,\mathrm{dn}\eta\sin\varphi$ $u_3 = \mathrm{i}R\,\mathrm{dn}\mu\,\mathrm{sn}\eta$	V_1, V_3, V_5	Circular-Polar
XXII. Semi-Hyperbolic $\mu_{1,2} > 0, \varphi \in [0, 2\pi)$ $I_1 = L_3^2$ $I_2 = \{K_1, L_2\} + \{K_2, L_1\}$	$u_0 = \frac{R}{\sqrt{2}}\big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} + \mu_1\mu_2 + 1\big)^{1/2}$ $u_1 = R\sqrt{\mu_1\mu_2}\,\cos\varphi$ $u_2 = R\sqrt{\mu_1\mu_2}\,\sin\varphi$ $u_3 = \frac{R}{\sqrt{2}}\big(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} - \mu_1\mu_2 - 1\big)^{1/2}$	V_2, V_6 V_{19}, V_{20}	Parabolic Circular-Polar
XXIII. Elliptic-Parabolic 1 $a, \varrho \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$ $I_1 = (K_1 + L_2)^2$ $I_2 = 2K_1^2 + K_2^2 + K_3^2 + L_1^2$ $-\{K_1, L_2\} - \{K_2, L_1\}$	$u_0 = R\,\frac{\cosh^2 a + \cos^2\vartheta + \varrho^2}{2\cosh a\cos\vartheta}$ $u_1 = R\,\varrho/\cosh a\cos\vartheta$ $u_2 = R\tanh a\tan\vartheta$ $u_3 = R\,\frac{\cosh^2 a + \cos^2\vartheta - \varrho^2 - 2}{2\cosh a\cos\vartheta}$	$V_9', V_9^{(\omega=0)}$	Cartesian
XXIV. Hyperbolic-Parabolic 1 $b > 0, \varrho \in \mathbb{R}, \vartheta \in (0, \pi)$ $I_1 = (K_1 + L_2)^2$ $I_2 = 2L_2^2 + L_1^2 + K_2^2 - K_3^2$ $-\{K_2, L_1\} - \{K_1, L_2\}$	$u_0 = R\,\frac{\sinh^2 b - \sin^2\vartheta + \varrho^2 + 2}{2\sinh b\sin\vartheta}$ $u_1 = R\,\varrho/\sinh b\sin\vartheta$ $u_2 = R\coth b\cot\vartheta$ $u_3 = R\,\frac{\sinh^2 b - \sin^2\vartheta - \varrho^2}{2\sinh b\sin\vartheta}$	$V_9', V_9^{(\omega=0)}$	Cartesian
XXV. Elliptic-Parabolic 2 $a > 0, \vartheta \in (0, \pi/2), \varphi \in [0, 2\pi)$ $I_1 = L_3^2$ $I_2 = 2\mathbf{L}^2 - \{L_2, K_1\} - \{L_1, K_2\}$	$u_0 = R\,\frac{\cos^2\vartheta + \cosh^2 a}{2\cosh a\cos\vartheta}$ $u_1 = R\tanh a\tan\vartheta\cos\varphi$ $u_2 = R\tanh a\tan\vartheta\sin\varphi$ $u_3 = R\,\frac{\sinh^2 a - \sin^2\vartheta}{2\cosh a\cos\vartheta}$	$V_2, V_9', V_9^{(\omega=0)}$	Parabolic
XXVI. Hyperbolic-Parabolic 2 $b > 0, \vartheta \in (0, \pi/2), \varphi \in [0, 2\pi)$ $I_1 = L_3^2$ $I_2 = \{K_1, L_2\} + \{K_2, L_1\} - K_1^2 - K_2^2$	$u_0 = R\,\frac{\cosh^2 b + \cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1 = R\coth b\cot\vartheta\cos\varphi$ $u_2 = R\coth b\cot\vartheta\sin\varphi$ $u_3 = R\,\frac{\sin^2\vartheta - \sinh^2 b}{2\sinh b\sin\vartheta}$	$V_9', V_9^{(\omega=0)}$	Circular-Polar
XXVII. Semi-Circular-Parabolic $\varrho \in \mathbb{R}, \xi, \eta > 0$ $I_1 = (K_1 + L_2)^2$ $I_2 = \{L_3, K_1 + L_2\} + \{K_3, K_2 - L_1\}$	$u_0 = R\,\frac{(\eta^2 - \xi^2)^2 + 4\varrho^2 + 4}{8\xi\eta}$ $u_1 = R\,\frac{\eta^2 - \xi^2}{2\xi\eta}$ $u_2 = R\,\frac{\varrho}{\xi\eta}$ $u_3 = R\,\frac{(\eta^2 - \xi^2)^2 + 4\varrho^2 - 4}{8\xi\eta}$	V_8	Cartesian
XXVIII. Ellipsoidal $0 < 1 < \varrho_3 < b < \varrho_2 < a < \varrho_1$ $I_1 = abK_1^2 + aK_2^2 + bK_3^2$ $I_2 = (a+b)K_1^2 + (a+1)K_2^2 + (b+1)K_3^2$ $-aL_3^2 - bL_2^2 - L_1^2$	$u_0^2 = R^2\,\frac{\varrho_1\varrho_2\varrho_3}{ab}$ $u_1^2 = R^2\,\frac{(\varrho_1 - 1)(\varrho_2 - 1)(\varrho_3 - 1)}{(a-1)(b-1)}$ $u_2^2 = -R^2\,\frac{(\varrho_1 - b)(\varrho_2 - b)(\varrho_3 - b)}{(a-b)(b-1)b}$ $u_3^2 = R^2\,\frac{(\varrho_1 - a)(\varrho_2 - a)(\varrho_3 - a)}{(a-b)(a-1)a}$	V_1	Ellipsoidal

Table 2.3 (cont.) Coordinate System	Coordinates	Separates Potential	Limiting Systems
XXIX. Hyperboloidal $\varrho_3 < 0 < 1 < b < \varrho_2 < a < \varrho_1$ $I_1 = abK_1^2 - aL_3^3 - bL_2^2$ $I_2 = (a+b)K_1^2 - (a+1)L_3^2 - (b+1)L_2^2$ $+aK_2^2 + bK_3^2 - L_1^2$	$u_0^2 = -R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}$ $u_1^2 = -R^2\frac{\varrho_1\varrho_2\varrho_3}{ab}$ $u_2^2 = -R^2\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b)}{(a-b)(b-1)b}$ $u_3^2 = R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a}$	V_1	Cartesian
XXX. Paraboloidal $\varrho_3 < 0 < \varrho_2 < 1 < \varrho_1$ $a = b^* = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$ $I_1 = -\|a\|^2L_1^2 + \alpha(K_3^2 - L_2^2) - \beta\{K_3, L_2\}$ $I_2 = -2\alpha L_1^2 + (\alpha+1)(K_3^2 - L_2^2)$ $+\alpha(K_2^2 - L_3^2) + \beta(\{K_2, L_3\} - \{K_3, L_2\})$	$(u_1 + iu_0)^2 = 2R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(b-1)b}$ $u_2^2 = R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}$ $u_3^2 = -R^2\frac{\varrho_1\varrho_2\varrho_3}{ab}$	$\frac{\alpha}{u_2^2} + \frac{\beta}{u_3^2}$	Paraboloidal
XXXI. $0 < \varrho_3 < 1 < \varrho_2 < a < \varrho_1$ $I_1 = (K_3 + L_2)^2 - a(K_2 + L_3)^2 + aK_1^2$ $I_2 = (a+1)K_1^2 + K_3^2 - L_2^2 + a(L_3^2 - K_2^2)$ $+(K_2 + L_3)^2 + (K_3 + L_2)^2$	$(u_0 + u_1)^2 = R^2\frac{\varrho_1\varrho_2\varrho_3}{a}$ $(u_0^2 - u_1^2)$ $= R^2\frac{a(\varrho_1\varrho_2 + \varrho_1\varrho_3 + \varrho_2\varrho_3) - (a+1)\varrho_1\varrho_2\varrho_3}{a^2}$ $u_2^2 = -R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)}$ $u_3^2 = R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a-1)}$	V_{21}	–
XXXII. $-\varrho_3 < 0 < 1 < \varrho_2 < a < \varrho_1$ $I_1 = -(K_3 + L_2)^2 + a(K_2 + L_3)^2 + aK_1^2$ $I_2 = (a+1)K_1^2 - K_3^2 + L_2^2 + a(K_2^2 - L_3^2)$ $-(K_2 + L_3)^2 - (K_3 + L_2)^2$	$(u_0 + u_1)^2 = -R^2\frac{\varrho_1\varrho_2\varrho_3}{a}$ $(u_0^2 - u_1^2)$ $= R^2\frac{a(\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3) - (a+1)\varrho_1\varrho_2\varrho_3}{a^2}$ $u_2^2 = -R^2\frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)}$ $u_3^2 = R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a-1)}$	V_{21}	–
XXXIII. $\varrho_3 < -1 < 0 < \varrho_2 < a < \varrho_1$ $I_1 = aK_1^2 - (K_2 + L_3)^2 + a(K_2 + L_3)^2$ $I_2 = (a-1)K_1^2 - K_2^2 + L_3^2 + a(L_2^2 - K_3^2)$ $-(K_2 + L_3)^2 + (K_3 + L_2)^2$	$(u_0 + u_1)^2 = -R^2\frac{\varrho_1\varrho_2\varrho_3}{a}$ $(u_0^2 - u_1^2)$ $= R^2\frac{a(\varrho_1\varrho_3 + \varrho_1\varrho_2 + \varrho_2\varrho_3) - (a-1)\varrho_1\varrho_2\varrho_3}{a^2}$ $u_2^2 = R^2\frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a+1)}$ $u_3^2 = -R^2\frac{(\varrho_1+1)(\varrho_2+1)(\varrho_3+1)}{(a+1)}$	V_{21}	–
XXXIV. $\varrho_3 < 0 < \varrho_2 < 1 < \varrho_1$ $I_1 = (L_2 - K_3)^2 - \{K_1, K_2 - L_3\}$ $I_2 = L_2^2 - K_3^2 - L_1^2$ $-(L_2 - K_3)^2 - \{L_1, L_2 - K_3\}$	$(u_0 - u_1)^2 = -R^2\varrho_1\varrho_2\varrho_3$ $2u_2(u_1 - u_0)$ $= R^2(\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3 - \varrho_1\varrho_2\varrho_3)$ $u_1^2 + u_2^2 - u_0^2 = R^2(-\varrho_1\varrho_2\varrho_3$ $+\varrho_1\varrho_2 + \varrho_2\varrho_3 + \varrho_1\varrho_3 - \varrho_1 - \varrho_2 - \varrho_3)$ $u_3^2 = R^2(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)$	$\frac{\beta}{u_3^2}$	–

3 Path Integral Formulation of the Maximally Super-Integrable Potentials on $\Lambda^{(3)}$

In table 3.1 we list the super-integrable potentials on the three-dimensional hyperboloid together with the separating coordinate systems. The cases where an explicit path integration is possible are underlined.

3.1 The Higgs-Oscillator.

We consider the generalized Higgs-oscillator on the hyperboloid ($k_{1,2,3} > 0$)

$$V_1(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2 + u_2^2 + u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\bigg(\frac{k_1^2 - \frac{1}{4}}{u_1^2} + \frac{k_2^2 - \frac{1}{4}}{u_2^2} + \frac{k_3^2 - \frac{1}{4}}{u_3^2}\bigg) \ , \tag{3.1}$$

which in the 14 separating coordinate systems has the form

Cylindrical ($\tau_{1,2} > 0, \varphi \in (0, \pi/2)$) :

$$V_1(\mathrm{u}) = \frac{M}{2}\omega^2 R^2\bigg(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2}\bigg) + \frac{\hbar^2}{2MR^2}\bigg(\frac{1}{\sinh^2\tau_1}\bigg(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\bigg) + \frac{k_3^2 - \frac{1}{4}}{\cosh^2\tau_1 \sinh^2\tau_2}\bigg) \tag{3.2}$$

Sphero-Elliptic ($\tau > 0, \tilde\alpha \in (0, K), \tilde\beta \in (0, K')$) :

$$= \frac{M}{2}\omega^2 R^2 \tanh^2\tau + \frac{\hbar^2}{2MR^2\sinh^2\tau}\bigg(\frac{k_1^2 - \frac{1}{4}}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} + \frac{k_3^2 - \frac{1}{4}}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\bigg) \tag{3.3}$$

Equidistant-Elliptic ($\tau > 0, \alpha \in (\mathrm{i}K', \mathrm{i}K' + K), \beta \in (0, K')$) :

$$= \frac{M}{2}\omega^2 R^2\bigg(1 - \frac{1}{\cosh^2\tau}\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\bigg) - \frac{\hbar^2}{2MR^2}\bigg[\frac{1}{\cosh^2\tau}\bigg(\frac{k_1^2 - \frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\bigg) - \frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau}\bigg] \tag{3.4}$$

Equidistant-Hyperbolic ($\tau > 0, \mu \in (\mathrm{i}K', \mathrm{i}K' + K), \eta \in (0, K')$) :

$$= \frac{M}{2}\omega^2 R^2\bigg(1 - \frac{1}{\cosh^2\tau}\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\bigg) - \frac{\hbar^2}{2MR^2}\bigg[\frac{1}{\cosh^2\tau}\bigg(\frac{k_1^2 - \frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\bigg) - \frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau}\bigg] \tag{3.5}$$

Spherical ($\tau > 0, \vartheta \in (0, \pi/2), \varphi \in (0, \pi/2)$) :

$$= \frac{M}{2}\omega^2 R^2 \tanh^2\tau + \frac{\hbar^2}{2MR^2\sinh^2\tau}\bigg(\frac{1}{\sin^2\vartheta}\bigg(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\bigg) + \frac{k_3^2 - \frac{1}{4}}{\cos^2\vartheta}\bigg) \tag{3.6}$$

Equidistant-Cylindrical ($\tau_{1,2} > 0, \varphi \in (0, \pi/2)$) :

$$= \frac{M}{2}\omega^2 R^2\bigg(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2}\bigg) + \frac{\hbar^2}{2MR^2}\bigg(\frac{1}{\cosh^2\tau_1 \sinh^2\tau_2}\bigg(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\bigg) + \frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau_1}\bigg) \tag{3.7}$$

Equidistant ($\tau_{1,2,3} > 0$) :

$$= \frac{M}{2}\omega^2 R^2\bigg(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2 \cosh^2\tau_3}\bigg)$$

Table 3.1: Maximally Superintegrable Potentials on $\Lambda^{(3)}$

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_1(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right)$	Cylindrical Sphero-Elliptic Equidistant-Elliptic Equidistant-Hyperbolic Spherical Equidistant-Cylindrical Equidistant Prolate Elliptic Oblate Elliptic Elliptic-Cylindrical Hyperbolic-Cylindrical 1 Hyperbolic-Cylindrical 2 Ellipsoidal Hyperboloidal	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_1(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$ $I_4 = \frac{1}{2M}K_3^2-\frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_2}+\frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{\sinh^2\tau_2}$ $I_5 = \frac{1}{2M}(L_1^2+k'^2L_2^2) + \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)$
$V_2(\mathrm{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}}-1\right)+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\right)$ $\mathbf{A} = \frac{1}{2R}(\mathbf{L}\times\mathbf{K}-\mathbf{K}\times\mathbf{L})-\frac{\alpha\mathbf{u}}{\lvert\mathbf{u}\rvert}, \quad \mathbf{u}=(u_1,u_2,u_3)$	Sphero-Elliptic Spherical Prolate Elliptic II Semi-Hyperbolic Elliptic Parabolic II	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_2(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_4 = \frac{1}{2M}(L_1^2+k'^2L_2^2) + \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_1^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}\right)$ $I_5 = \frac{1}{2M}A_3+\frac{\hbar^2}{2M\sinh^2\tau\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$

Table 3.1 (cont.): Maximally Superintegrable Potentials on $\Lambda^{(3)}$

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_3(\mathrm{u})=\frac{\hbar^2}{2M}\bigg[-\frac{k_0^2-\frac14}{u_0^2}+\frac{k_3^2-\frac14}{u_3^2}$ $+\frac{1}{\sqrt{u_1^2+u_2^2}}\bigg(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\bigg)\bigg]$	Spherical	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_3(\mathrm{u})$
	Equidistant-Cylindrical	$I_2=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\bigg(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\bigg)$
	Prolate Elliptic	$I_3=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\bigg(\frac{1}{4\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\bigg)$
	Oblate Elliptic	$I_4=\frac{1}{2M}(K_1^2+K_2^2-L_3^2)+\frac{\hbar^2}{8M\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\bigg)$
	Hyperbolic-Cylindrical 2	$I_5=\frac{1}{2M}\Big[K_3^2+L_3^2-k^2(L_1^2+L_2^2)\Big]$ $-\frac{\hbar^2}{2M}\bigg[\frac{1}{4\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\bigg(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\bigg)$
Minimally Super-Integrable Potential on $\Lambda^{(3)}$ (with Maximally Super-Integrable Analogue in $\mathbb{R}^3$)		
$V_4(\mathrm{u})=\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\bigg(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\bigg)+k_3u_3$	Equidistant-Elliptic*	$I_1=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_4(\mathrm{u})$
	Equidistant Semi-Hyperbolic	$I_2=\frac{1}{2M}(K_1^2+K_2^2-L_3^2)+\frac{\hbar^2}{8M\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\bigg)$
	Equidistant-Elliptic-Parabolic	$I_3=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\bigg(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\bigg)$
	Equidistant-Cylindrical	$I_4=\frac{1}{2M}\big[(K_2-L_3)^2+K_1^2\big]+\frac{\hbar^2}{2M}\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\sin^2\vartheta}+\frac{k_2^2-\frac14}{\sinh^2 a}\bigg)$

* after appropriate rotation, $\sin^2 f=k^2$.

$$+\frac{\hbar^2}{2MR^2}\bigg[\frac{1}{\cosh^2\tau_1}\bigg(\frac{k_1^2-\frac14}{\cosh^2\tau_2\sinh^2\tau_3}+\frac{k_2^2-\frac14}{\sinh^2\tau_2}\bigg)+\frac{k_3^2-\frac14}{\sinh^2\tau_1}\bigg] \tag{3.8}$$

Prolate Elliptic $(\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi/2))$:

$$=\frac{M}{2}\omega^2R^2\bigg(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\bigg)-\frac{\hbar^2}{2MR^2}\bigg[\frac{1}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)+\frac{k_3^2-\frac14}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg] \tag{3.9}$$

Oblate Elliptic $(\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi/2))$:

$$=\frac{M}{2}\omega^2R^2\bigg(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\bigg)-\frac{\hbar^2}{2MR^2}\bigg[\frac{1}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\bigg] \tag{3.10}$$

Elliptic-Cylindrical $(\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\tau>0)$:

$$=\frac{M}{2}\omega^2R^2\bigg(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta\cosh^2\tau}\bigg)$$
$$+\frac{\hbar^2}{2MR^2}\bigg(\frac{k_1^2-\frac14}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta\sinh^2\tau}-\frac{k_2^2-\frac14}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}-\frac{k_3^2-\frac14}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg) \tag{3.11}$$

Hyperbolic-Cylindrical 1 $(\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K'),\tau>0)$:

$$=\frac{M}{2}\omega^2R^2\bigg(1-\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta\cosh^2\tau}\bigg)$$
$$+\frac{\hbar^2}{2MR^2}\bigg(\frac{k_1^2-\frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta\sinh^2\tau}-\frac{k_2^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}-\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\bigg) \tag{3.12}$$

Hyperbolic-Cylindrical 2 $(\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K'),\varphi\in(0,\pi/2))$:

$$=\frac{M}{2}\omega^2R^2\bigg(1-\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\bigg)-\frac{\hbar^2}{2MR^2}\bigg[\frac{1}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\bigg] \tag{3.13}$$

Ellipsoidal $(a_{ij}=a_i-a_j,a_1=0,a_2=1,a_3=b,a_4=a)$:

$$=\frac{M}{2}\omega^2R^2\bigg[a_{14}a_{24}a_{34}\bigg(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\frac{1}{\varrho_3-a_4}$$
$$+\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\frac{1}{\varrho_2-a_4}+\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\frac{1}{\varrho_1-a_4}\bigg)-1\bigg]$$
$$+\frac{\hbar^2}{2MR^2}\bigg\{\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\bigg[a_{31}a_{21}a_{41}\frac{k_1^2-\frac14}{\varrho_3-a_1}+a_{12}a_{32}a_{42}\frac{k_2^2-\frac14}{\varrho_3-a_2}+a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_3-a_3}\bigg]$$
$$+\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\bigg[a_{31}a_{21}a_{41}\frac{k_1^2-\frac14}{\varrho_2-a_1}+a_{12}a_{32}a_{42}\frac{k_2^2-\frac14}{\varrho_2-a_2}+a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_2-a_3}\bigg]$$
$$+\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\bigg[a_{31}a_{21}a_{41}\frac{k_1^2-\frac14}{\varrho_1-a_1}+a_{12}a_{32}a_{42}\frac{k_2^2-\frac14}{\varrho_1-a_2}+a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_1-a_3}\bigg]\bigg\} \tag{3.14}$$

Hyperboloidal $(a_{ij}=a_i-a_j,a_1=0,a_2=1,a_3=b,a_4=a)$:

$$=\frac{M}{2}\omega^2R^2\bigg[a_{14}a_{24}a_{34}\bigg(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\frac{1}{\varrho_3-a_4}$$
$$+\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\frac{1}{\varrho_2-a_4}+\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\frac{1}{\varrho_1-a_4}\bigg)-1\bigg]$$
$$-\frac{\hbar^2}{2MR^2}\bigg\{\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\bigg[a_{31}a_{21}a_{41}\frac{k_2^2-\frac14}{\varrho_3-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac14}{\varrho_3-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_3-a_3}\bigg]$$
$$-\frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\bigg[a_{31}a_{21}a_{41}\frac{k_2^2-\frac14}{\varrho_2-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac14}{\varrho_2-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_2-a_3}\bigg]$$

$$
-\frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\left[a_{31}a_{21}a_{41}\frac{k_2^2-\frac14}{\varrho_1-a_1}+a_{12}a_{32}a_{42}\frac{k_1^2-\frac14}{\varrho_1-a_2}-a_{13}a_{23}a_{43}\frac{k_3^2-\frac14}{\varrho_1-a_3}\right]\Biggr\}\ . \tag{3.15}
$$

The five constants of motion have the form

$$
\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_1(\mathrm{u})\ ,\\
I_2&=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\ ,\\
I_3&=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right)\ ,\\
I_4&=\frac{1}{2M}K_3^2-\frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_2}+\frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{\sinh^2\tau_2}\ ,\\
I_5&=\frac{1}{2M}(L_1^2+k'^2L_2^2)+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)\ .
\end{aligned}\right\} \tag{3.16}
$$

In the following we do not display all path integral representations for all potentials in all separable coordinate systems. In order not to blow up the length of the paper too much we display explicitly only those path integral representations, where an analytic solution is available.

3.1.1 Pure Oscillator Case.

The pure oscillator allows an explicit solution in five coordinate systems. The solution in the first system, the cylindrical coordinates will be discussed in some more detail, including the statement of the wave-functions. In the statement of the corresponding Green functions we always set $E'=E+\hbar^2/2MR^2+M\omega^2R^2/2$. The propagator, the Green function, and the wave-functions for this special case are denoted by $K^{(\omega)}(T)$, $G^{(\omega)}(E)$, and $\Psi^{(\omega)}(\mathrm{u})$, respectively.

Cylindrical Coordinates. We start with the cylindrical system. To evaluate the path integral we separate off in the first step the φ- and τ_2- path integration which correspond to the path integral for circular waves and a symmetric Rosen-Morse potential [132], the latter giving two contributions due to its discrete and continuous spectrum. This property is accompanied with the range of the two coordinates according to $\varphi\in[0,2\pi)$ and $\tau_2\in\mathrm{I\!R}$. The remaining path integral in the variable $\tau_1>0$ is a path integral for the general modified Pöschl-Teller potential. Therefore we have

$$
\begin{aligned}
K^{(\omega)}(\mathrm{u}'',\mathrm{u}';T)&=\frac{1}{R^3}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\sinh\tau_1\cosh\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2+\sinh^2\tau_1\dot\varphi^2-\left(1-\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right)\right.\right.\\
&\qquad\qquad\left.\left.-\frac{\hbar^2}{8MR^2}\left(4+\frac{1}{\cosh^2\tau_1}-\frac{1}{\sinh^2\tau_1}\right)\right]dt\right\}
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
&=\frac{\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]}{(\sinh\tau_1'\sinh\tau_1''\cosh\tau_1'\cosh\tau_1'')^{1/2}}\sum_{j\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}j(\varphi''-\varphi')}}{2\pi R}\\
&\quad\times\left(\sum_{m_2=0}^{N_{m_2}}(m_2-\nu-\tfrac12)\frac{\Gamma(2\nu-m_2)}{m_2!}P_{\nu-1/2}^{m_2-\nu+1/2}(\tanh\tau_2'')P_{\nu-1/2}^{m_2-\nu+1/2}(\tanh\tau_2')\right.
\end{aligned}
$$

$$\times \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot\tau_1^2 - \frac{\hbar^2}{2MR^2}\left(\frac{j^2-\frac14}{\sinh^2\tau_1} - \frac{(m_2-\nu+\frac12)^2-\frac14}{\cosh^2\tau_1}\right)\right]dt\right\}$$

$$+\int_{\mathbb{R}}\sum_{\epsilon=\pm}\frac{dk\,k\sinh\pi k}{2(\cos^2\pi\nu+\sinh^2\pi k)}P_{\nu-1/2}^{\mathrm{i}k}(\epsilon\tanh\tau_2'')P_{\nu-1/2}^{-\mathrm{i}k}(\epsilon\tanh\tau_2')$$

$$\left.\times \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot\tau_1^2 - \frac{\hbar^2}{2MR^2}\left(\frac{j^2-\frac14}{\sinh^2\tau_1} + \frac{k^2+\frac14}{\cosh^2\tau_1}\right)\right]dt\right\}\right) \tag{3.18}$$

$$= \sum_{j\in\mathbb{Z}}\left(\sum_{m_2=0}^{N_{m_2}}\left\{\sum_{m_1=0}^{N_{m_1}} \mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi^{(\omega)}_{m_1m_2j}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(\omega)}_{m_1m_2j}(\tau_1',\tau_2',\varphi';R)\right.\right.$$

$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(\omega)}_{pm_2j}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(\omega)*}_{pm_2j}(\tau_1',\tau_2',\varphi';R)\right\}$$

$$\left.+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(\omega)}_{pkj}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(\omega)*}_{pkj}(\tau_1',\tau_2',\varphi';R)\right) \ . \tag{3.19}$$

The discrete wave-functions are given by ($\lambda_2 = m_1-\nu+1/2, \nu^2=\omega^2M^2R^4/\hbar^2+1/4$)

$$\Psi^{(\omega)}_{m_1m_2j}(\tau_1,\tau_2,\varphi;R) = (2\pi\sinh\tau_1\cosh\tau_1)^{-1/2}S_{m_1}^{(|j|,\lambda_2)}(\tau_1;R)\psi_{m_2}^{(\nu)}(\tau_2)\,\mathrm{e}^{\mathrm{i}j\varphi} \ , \tag{3.20}$$

where

$$S_{m_1}^{(|j|,\lambda_2)}(\tau_1;R) = \frac{1}{\Gamma(1+|j|)}\left[\frac{2(\lambda_2-|j|-2m_1-1)\Gamma(m_1+1+|j|)\Gamma(\lambda_2-m_1)}{R^3\Gamma(\lambda_2-|j|-n)m_1!}\right]^{1/2}$$
$$\times(\sinh\tau_1)^{1/2+|j|}(\cosh\tau_1)^{2m_1+1/2-\lambda_2}{}_2F_1(-m_1,\lambda_2-m_1;1+|j|;\tanh^2\tau_1) \ , \tag{3.21}$$

$$\psi_{m_2}^{(\nu)}(\tau_2) = \sqrt{(m_2-\nu-\tfrac12)\frac{\Gamma(2\nu-m_2)}{m_2!}}\,P_{\nu-1/2}^{m_2-\nu+1/2}(\tanh\tau_2) \ , \tag{3.22}$$

and the discrete spectrum has the form

$$E_N = -\frac{\hbar^2}{2MR^2}\left[\left(N-\nu+\frac32\right)^2-1\right]+\frac{M}{2}\omega^2R^2 \ , \qquad N=2m_1+m_2+|j| \ . \tag{3.23}$$

$P_\nu^\mu(z)$ are Legendre functions [57, p.998]. Only a finite number of energy-levels can exist with $m_1=0,1,\dots,N_{m_1}=[(\nu-m_2-|j|-3/2)/2]$, $N=0,\dots,N_{Max}=[\nu-3/2]$, and $[x]$ denotes the integer part of x. In the flat space limit we obtain ($R\to\infty$)

$$E_N = \hbar\omega(N+\tfrac32) \ , \qquad N\in\mathbb{N}_0 \ , \tag{3.24}$$

which is the proper feature for a three-dimensional isotropic oscillator. Note that the Legendre functions of the discrete spectrum can also be expressed as Gegenbauer polynomials.

The continuous wave-functions consist of two contributions, first where the quantum number corresponding to τ_2 is discrete, second where it is continuous. For the first set of continuous states we obtain

$$\Psi^{(\omega)}_{pm_2j}(\tau_1,\tau_2,\varphi;R) = (2\pi\sinh\tau_1\cosh\tau_1)^{-1/2}S_p^{(|j|,\lambda_2)}(\tau_1;R)\psi_{m_2}^{(\nu)}(\tau_2)\,\mathrm{e}^{\mathrm{i}j\varphi} \ , \tag{3.25}$$

where

$$S_p^{(|j|,\lambda_2)}(\tau_1;R) = \frac{1}{\Gamma(1+|j|)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\left(\frac{\lambda_2-|j|+1-\mathrm{i}p}{2}\right)\Gamma\left(\frac{|j|-\lambda_2+1-\mathrm{i}p}{2}\right)$$
$$\times(\tanh\tau_1)^{1/2+|j|}(\cosh\tau_1)^{\mathrm{i}p}{}_2F_1\left(\frac{\lambda_2+|j|+1-\mathrm{i}p}{2},\frac{1+|j|-\lambda_2-\mathrm{i}p}{2};1+|j|;\tanh^2\tau_1\right) \ . \tag{3.26}$$

The continuous spectrum has the form

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)+\frac{M}{2}\omega^2R^2\ . \tag{3.27}$$

In the limiting case $\omega\to 0$ we obtain

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)\ , \tag{3.28}$$

which corresponds to the case where just a radial part is present which has the same feature as the spectrum of the free motion on $\Lambda^{(3)}$. For the second set of continuous states we obtain

$$\Psi^{(\omega)}_{pkj}(\tau_1,\tau_2,\varphi;R)=(2\pi\sinh\tau_1\cosh\tau_1)^{-1/2}S_p^{(|j|,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\nu)}(\tau_2)\,\mathrm{e}^{\mathrm{i}j\varphi}\ , \tag{3.29}$$

where

$$\begin{aligned}S_p^{(|j|,\mathrm{i}k)}(\tau_1;R)&=\frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\bigg(\frac{\mathrm{i}k\mp k_3+1-\mathrm{i}p}{2}\bigg)\Gamma\bigg(\frac{\lambda_1-\mathrm{i}k+1-\mathrm{i}p}{2}\bigg)\\&\quad\times(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{\mathrm{i}p}{}_2F_1\bigg(\frac{\mathrm{i}k+\lambda_1+1-\mathrm{i}p}{2},\frac{1+\lambda_1-\mathrm{i}k-\mathrm{i}p}{2};1+\lambda_1;\tanh^2\tau_1\bigg)\ ,\end{aligned} \tag{3.30}$$

$$\psi_k^{(\nu)}(\tau_2)=\sqrt{\frac{k\sinh\pi k}{2(\cos^2\pi\nu+\sinh^2\pi k)}}\,\Big[P^{\mathrm{i}k}_{\nu-1/2}(\tanh\tau_2)+P^{\mathrm{i}k}_{\nu-1/2}(-\tanh\tau_2)\Big]\ , \tag{3.31}$$

with the same spectrum as before.
The Green function in these coordinates is given by

$$\begin{aligned}G^{(\omega)}(\mathrm{u}'',\mathrm{u}';E)&=(\sinh\tau_1'\cosh\tau_1'\sinh\tau_1''\cosh\tau_1'')^{-1/2}\sum_{j\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}j(\varphi''-\varphi')}}{2\pi R}\\&\quad\times\Bigg\{\sum_{m_2=0}^{N_{m_2}}\psi^{(\nu)}_{m_2}(\tau_2')\psi^{(\nu)}_{m_2}(\tau_2'')G^{(|j|,\lambda_2)}_{mPT}(\tau_1'',\tau_1';E')+\int_0^\infty dk\,\psi_k^{(\nu)*}(\tau_2')\psi_k^{(\nu)}(\tau_2'')G^{(|j|,\mathrm{i}k)}_{mPT}(\tau_1'',\tau_1';E')\Bigg\}\ ,\end{aligned} \tag{3.32}$$

where $G^{(\kappa,\lambda)}_{mPT}(E)$ is the Green function (A.34) of the modified Pöschl-Teller potential.

Sphero-Elliptic and Spherical Coordinates. For the path integrals of the pure oscillator in sphero-elliptic and spherical coordinates we can separate off in the $(\tilde\alpha,\tilde\beta)$, respectively the (ϑ,φ) path integration, very easily because we only have to deal with the free motion on the sphere $S^{(2)}$. The remaining path integration over $\tau>0$ is of the modified Pöschl-Teller type. We obtain, e.g., in sphero-elliptic coordinates

$$\begin{aligned}&K^{(\omega)}(\mathrm{u}'',\mathrm{u}';T)\\&=\frac{1}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\tilde\alpha(t')=\tilde\alpha'}^{\tilde\alpha(t'')=\tilde\alpha''}\mathcal{D}\tilde\alpha(t)\int\limits_{\tilde\beta(t')=\tilde\beta'}^{\tilde\beta(t'')=\tilde\beta''}\mathcal{D}\tilde\beta(t)(k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta)\\&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta)(\dot{\tilde\alpha}^2+\dot{\tilde\beta}^2)-\omega^2\tanh^2\tau\Big)-\frac{\hbar^2}{2MR^2}\right]dt\right\}\end{aligned} \tag{3.33}$$

$$\begin{aligned}&=\sum_{lhkq}\Bigg\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi^{(\omega)}_{nlhkq}(\tau'',\tilde\alpha'',\tilde\beta'';R)\Psi^{(\omega)}_{nlhkq}(\tau',\tilde\alpha',\tilde\beta';R)\\&\qquad\qquad+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(\omega)}_{plhkq}(\tau'',\tilde\alpha'',\tilde\beta'';R)\Psi^{(\omega)*}_{plhkq}(\tau',\tilde\alpha',\tilde\beta';R)\Bigg\}\ .\end{aligned} \tag{3.34}$$

The bound state wave-functions are given by $[k, q = \pm 1, h + \tilde{h} = l(l+1), l = 0, 1, \dots]$

$$\Psi^{(\omega)}_{nlhkq}(\tau, \tilde{\alpha}, \tilde{\beta}; R) = (\sinh\tau)^{-1} S^{(l,\nu)}_n(\tau; R)\Lambda^k_{lh}(\tilde{\alpha})\Lambda^q_{l\tilde{h}}(\tilde{\beta}) \ , \tag{3.35}$$

where

$$\begin{aligned} S^{(l,\nu)}_n(\tau; R) = \frac{1}{l!}\left[\frac{2(\nu-l-2n-1)(n+l)!\Gamma(\nu-l)}{R^3\Gamma(\nu-l-n)n!}\right]^{1/2} \\ \times(\sinh\tau)^{l+1/2}(\cosh\tau)^{2n+1/2-\nu}{}_2F_1(-l, \nu-n; 1+l; \tanh^2\tau) \ . \end{aligned} \tag{3.36}$$

For the *periodic Lamé polynomials* $\Lambda^k_{lh}(z)$ we have adopted the notation of [167]. An alternative notation is due to Lukàčs [138]. The spectrum is the same as in the previous case with the principal quantum number $N = 2n + l = 0, 1, \dots$. The continuous wave-functions have the form

$$\Psi^{(\omega)}_{plhkq}(\tau, \tilde{\alpha}, \tilde{\beta}; R) = (\sinh\tau)^{-1} S^{(l,\nu)}_p(\tau; R)\Lambda^k_{lh}(\tilde{\alpha})\Lambda^q_{l\tilde{h}}(\tilde{\beta}) \ , \tag{3.37}$$

where

$$\begin{aligned} S^{(l,\nu)}_p(\tau; R) = \frac{1}{l!}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\Gamma\left(\frac{\nu-l+1-\mathrm{i}p}{2}\right)\Gamma\left(\frac{l-\nu+1-\mathrm{i}p}{2}\right) \\ \times(\tanh\tau)^{l+1/2}(\cosh\tau)^{\mathrm{i}p}{}_2F_1\left(\frac{\nu+l+1-\mathrm{i}p}{2}, \frac{l-\nu+1-\mathrm{i}p}{2}; 1+l; \tanh^2\tau\right) \ , \end{aligned} \tag{3.38}$$

with E_p as in (3.27). In the case of spherical coordinates the Lamé polynomials degenerate into spherical harmonics which yields

$$\Psi^{(\omega)}_{nlm}(\tau, \vartheta, \varphi; R) = (\sinh\tau)^{-1} S^{(l,\nu)}_n(\tau; R) Y^m_l(\vartheta, \varphi) \ , \tag{3.39}$$

$$\Psi^{(\omega)}_{plm}(\tau, \vartheta, \varphi; R) = (\sinh\tau)^{-1} S^{(l,\nu)}_p(\tau; R) Y^m_l(\vartheta, \varphi) \ . \tag{3.40}$$

For the Green function in sphero-elliptic, respectively spherical coordinates, we obtain

$$\begin{aligned} &G^{(\omega)}(\mathrm{u}'', \mathrm{u}'; E) \\ &= (R\sinh\tau'\sinh\tau'')^{-1}\left(\begin{array}{c} \sum_{lhkq}\Lambda^{k\,*}_{lh}(\tilde{\alpha}')\Lambda^{q\,*}_{l\tilde{h}}(\tilde{\beta}')\Lambda^k_{lh}(\tilde{\alpha}'')\Lambda^q_{l\tilde{h}}(\tilde{\beta}'') \\ \sum_{lm} Y^{m\,*}_l(\vartheta', \varphi')Y^m_l(\vartheta'', \varphi'') \end{array}\right) G^{(l,\nu)}_{mPT}(\tau'', \tau'; E') \ . \end{aligned} \tag{3.41}$$

Equidistant-Cylindrical Coordinates. In equidistant-cylindrical coordinates the φ path integration gives circular waves with $\varphi \in [0, 2\pi)$. The emerging $\tau_2 > 0$ path integral is again of the modified Pöschl-Teller type, and the τ_1 path integral has the structure of the symmetric Rosen-Morse potential with $\tau_1 \in \mathbb{R}$. Therefore $(\lambda_2 = 2m_2 + |j| - \nu + 1, \nu^2 = M^2\omega^2R^4/\hbar^2 + 1/4)$

$$\begin{aligned} K^{(\omega)}(\mathrm{u}'', \mathrm{u}'; T) &= \frac{1}{R^3}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\sinh\tau_2\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) \\ &\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot{\tau}_1^2 + \cosh^2\tau_1(\dot{\tau}_2^2 + \sinh^2\tau_2\dot{\varphi}^2) - \left(1 - \frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right)\right.\right. \\ &\left.\left.\qquad - \frac{\hbar^2}{8MR^2}\left(4 + \frac{1}{\cosh^2\tau_1}\left(1 - \frac{1}{\sinh^2\tau_2}\right)\right)\right]dt\right\} \end{aligned} \tag{3.42}$$

$$= \sum_{j\in\mathbb{Z}}\left\{\sum_{m_2=0}^{N_{m_2}}\left[\sum_{m_1=0}^{N_{m_1}}\mathrm{e}^{-\mathrm{i}\hbar E_N T/\hbar}\Psi^{(\omega)}_{m_1m_2j}(\tau_1'', \tau_2'', \varphi''; R)\Psi^{(\omega)}_{m_1m_2j}(\tau_1', \tau_2', \varphi'; R)\right.\right.$$

$$+\int_0^\infty dp\, e^{-iE_pT/\hbar}\Psi^{(\omega)}_{pm_2j}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(\omega)*}_{pm_2j}(\tau_1',\tau_2',\varphi';R)\Bigg]$$

$$+\int_0^\infty dk\int_0^\infty dp\, e^{-iE_pT/\hbar}\Psi^{(\omega)}_{pkm}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(\omega)*}_{pkm}(\tau_1',\tau_2',\varphi';R)\Bigg\}\ . \tag{3.43}$$

We obtain one set of bound state wave-functions and two sets of scattering wave-functions. The bound state wave-functions are ($\lambda_2 = m_1 - \nu + 1/2$)

$$\Psi^{(\omega)}_{m_1m_2j}(\tau_1,\tau_2,\varphi;R) = (2\pi\cosh^2\tau_1\sinh\tau_2)^{-1/2}S^{(\lambda_2)}_{m_1}(\tau_1;R)\psi^{(|j|,\nu)}_{m_2}(\tau_2)\,e^{ij\varphi}\ , \tag{3.44}$$

where ($m_1 = 0,\dots,N_{m_1} < \nu - 1/2, N = 0,\dots,N_{Max} = [\nu - 3/2]$

$$S^{(\lambda_2)}_{m_1}(\tau_1) = \sqrt{(m_1-\nu-\tfrac{1}{2})\frac{\Gamma(2\nu-m_1)}{m_1!R^3}}\,P^{m_1-\nu+1/2}_{\nu-1/2}(\tanh\tau_1)\ , \tag{3.45}$$

$$\psi^{(|j|,\nu)}_{m_2}(\tau_2) = \frac{1}{\Gamma(1+|j|)}\bigg[\frac{2(\nu-|j|-2m_2-1)\Gamma(m_2+1+|j|)\Gamma(\nu-m_2)}{\Gamma(\nu-|j|-m_2)m_2!}\bigg]^{1/2}$$
$$\times(\sinh\tau_2)^{1/2+|j|}(\cosh\tau_2)^{m_2+1/2-\nu}{}_2F_1(-m_2,\nu-m_2;1+|j|;\tanh^2\tau_2)\ , \tag{3.46}$$

and E_N as in (3.23). The first set of the continuous spectrum has the form

$$\Psi^{(\omega)}_{pm_2j}(\tau_1,\tau_2,\varphi;R) = (2\pi\cosh^2\tau_1\sinh\tau_2)^{-1/2}S^{(\lambda_2)}_{p}(\tau_1;R)\psi^{(|j|,\nu)}_{m_2}(\tau_2)\,e^{ij\varphi}\ , \tag{3.47}$$

where

$$S^{(\lambda_2)}_{p}(\tau_1;R) = R^{-3/2}\sqrt{\frac{p\sinh\pi p}{2(\cos^2\pi\lambda_2+\sinh^2\pi p)}}\Big[P^{ip}_{\lambda_2-1/2}(\tanh\tau_1)+P^{ip}_{\lambda_2-1/2}(-\tanh\tau_1)\Big]\ . \tag{3.48}$$

The second set of the continuous spectrum is given by

$$\Psi^{(\omega)}_{pkj}(\tau_1,\tau_2,\varphi;R) = (2\pi\cosh^2\tau_1\sinh\tau_2)^{-1/2}S^{(ik)}_{p}(\tau_1;R)\psi^{(|j|,\nu)}_{k}(\tau_2)\,e^{ij\varphi}\ , \tag{3.49}$$

where

$$S^{(ik)}_{p}(\tau_1;R) = R^{-3/2}\sqrt{\frac{p\sinh\pi p}{2(\cosh^2\pi k+\sinh^2\pi p)}}\Big[P^{ip}_{ik-1/2}(\tanh\tau_1)+P^{ip}_{ik-1/2}(-\tanh\tau_1)\Big]\ , \tag{3.50}$$

$$\psi^{(|j|,\nu)}_{k}(\tau_2) = \frac{1}{\Gamma(1+|j|)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\,\Gamma\bigg(\frac{\nu\mp k_1+1-ik}{2}\bigg)\Gamma\bigg(\frac{|j|-\nu+1-ik}{2}\bigg)$$
$$\times(\tanh\tau_2)^{1/2+|j|}(\cosh\tau_2)^{ik}{}_2F_1\bigg(\frac{\nu+|j|+1-ik}{2},\frac{1+|j|-\nu-ik}{2};1+|j|;\tanh^2\tau_2\bigg)\ , \tag{3.51}$$

with the same energy-spectrum as for the first set.
The corresponding Green function has the form (E' as before)

$$G^{(\omega)}(\mathrm{u}'',\mathrm{u}';E) = \frac{M}{\hbar^2R}(\cosh^2\tau_1'\sinh\tau_2'\cosh^2\tau_1''\sinh\tau_2'')^{-1/2}\sum_{j\in\mathbb{Z}}\frac{e^{ij(\varphi''-\varphi')}}{2\pi}$$
$$\times\Bigg\{\sum_{m_2=0}^{N_{m_2}}\psi^{(|j|,\nu)}_{m_2}(\tau_2')\psi^{(|j|,\nu)}_{m_2}(\tau_2'')\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}-\lambda_2+\frac{1}{2}\bigg)\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}+\lambda_2+\frac{1}{2}\bigg)$$
$$\times P^{-\sqrt{-2MR^2E'}/\hbar}_{\lambda_2-1/2}(\tanh\tau_{1,<})P^{-\sqrt{-2MR^2E'}/\hbar}_{\lambda_2-1/2}(-\tanh\tau_{1,>})$$

$$+\int_0^\infty dk\,\psi_k^{(|j|,\nu)*}(\tau_2')\psi_k^{(|j|,\nu)}(\tau_2'')\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}-\mathrm{i}k+\frac{1}{2}\bigg)\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}+\mathrm{i}k+\frac{1}{2}\bigg)$$
$$\times P_{\mathrm{i}k-1/2}^{-\sqrt{-2MR^2E'}/\hbar}(\tanh\tau_{1,<})P_{\mathrm{i}k-1/2}^{-\sqrt{-2MR^2E'}/\hbar}(-\tanh\tau_{1,>})\bigg\}\ . \tag{3.52}$$

Equidistant Coordinates. The last system, where an explicit solution is possible is the equidistant system. From its path integral representation we see that we have three interrelated symmetric Rosen-Morse potentials with $\tau_{1,2,3}\in\mathbb{R}$. Therefore we obtain

$$K^{(\omega)}(\mathrm{u}'',\mathrm{u}';T)=\frac{1}{R^3}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\cosh\tau_2\int\limits_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''}\mathcal{D}\tau_3(t)$$
$$\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\bigg(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\cosh^2\tau_1\dot\tau_3^2)-\bigg(1-\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2\cosh^2\tau_3}\bigg)\bigg)$$
$$-\frac{\hbar^2}{2MR^2}\bigg(4+\frac{1}{\cosh^2\tau_1}\bigg(1+\frac{1}{\cosh^2\tau_2}\bigg)\bigg)\Bigg]dt\Bigg\} \tag{3.53}$$
$$=\sum_{m_3=0}^{N_{m_3}}\Bigg\{\sum_{m_2=0}^{N_{m_2}}\Bigg[\sum_{m_1=0}^{N_{m_1}}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{m_1m_2m_3}^{(\omega)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{m_1m_2m_3}^{(\omega)}(\tau_1',\tau_2',\tau_3';R)$$
$$+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pm_2m_3}^{(\omega)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pm_2m_3}^{(\omega)*}(\tau_1',\tau_2',\tau_3';R)\Bigg]$$
$$+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pkm_3}^{(\omega)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pkm_3}^{(\omega)*}(\tau_1',\tau_2',\tau_3';R)\Bigg\}$$
$$+\int_0^\infty d\varrho\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pk\varrho}^{(\omega)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pk\varrho}^{(\omega)*}(\tau_1',\tau_2',\tau_3';R)\ . \tag{3.54}$$

We obtain one set of bound state wave-functions and three sets of scattering wave-functions. The bound state wave-functions are ($\lambda_1=m_3-\nu+1/2$, $\lambda_2=m_2-\lambda_1+1/2$)

$$\Psi_{m_1m_2m_3}^{(\omega)}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_{m_1}^{(\lambda_2)}(\tau_1;R)\psi_{m_2}^{(\lambda_1)}(\tau_2)\psi_{m_3}^{(\nu)}(\tau_3)\ , \tag{3.55}$$

where ($m_3=0,\dots,N_{m_3}<\nu-\frac{1}{2}$, $m_2=0,\dots,N_{m_2}<\lambda_1-\frac{1}{2}$, $N=0,\dots,N_{Max}=[\nu-3/2]$

$$S_{m_1}^{(\lambda_2)}(\tau_1)=\sqrt{(m_1-\lambda_2-\tfrac{1}{2})\frac{\Gamma(2+\lambda_2-m_1)}{R^3m_1!}}\,P_{\lambda_2-1/2}^{m_1-\lambda_2+1/2}(\tanh\tau_1)\ , \tag{3.56}$$
$$\psi_{m_2}^{(\lambda_1)}(\tau_2)=\sqrt{(m_2-\lambda_1-\tfrac{1}{2})\frac{\Gamma(2+\lambda_1-m_2)}{m_2!}}\,P_{\lambda_1-1/2}^{m_2-\lambda_1+1/2}(\tanh\tau_2)\ , \tag{3.57}$$
$$\psi_{m_3}^{(\nu)}(\tau_3)=\sqrt{(m_3-\nu-\tfrac{1}{2})\frac{\Gamma(2+\nu-m_3)}{m_3!}}\,P_{\nu-1/2}^{m_3-\nu+1/2}(\tanh\tau_3)\ , \tag{3.58}$$

and the discrete spectrum has the form of E_N as in (3.23) with $N=m_1+m_2+m_3$. The first of the three sets of continuous states is given by

$$\Psi_{pm_2m_3}^{(\omega)}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\lambda_2)}(\tau_1;R)\psi_{m_2}^{(\lambda_1)}(\tau_2)\psi_{m_3}^{(\nu)}(\tau_3)\ , \tag{3.59}$$

where

$$S_p^{(\lambda_2)}(\tau_1;R)=R^{-3/2}\sqrt{\frac{p\sinh\pi p}{2(\cos^2\pi\lambda_2+\sinh^2\pi p)}}\Big[P_{\lambda_2-1/2}^{\mathrm{i}p}(\tanh\tau_1)+P_{\lambda_2-1/2}^{\mathrm{i}p}(-\tanh\tau_1)\Big]\ , \tag{3.60}$$

with the $\psi^{(\lambda_1)}_{m_2}(\tau_2), \psi^{(\nu)}_{m_3}(\tau_3)$ as in (3.57,3.58). The second has the form

$$\Psi^{(\omega)}_{pkm_3}(\tau_1,\tau_2,\tau_3;R) = (\cosh^2\tau_1\cosh\tau_2)^{-1/2} S^{(\mathrm{i}k)}_p(\tau_1;R)\psi^{(\lambda_1)}_k(\tau_2)\psi^{(\nu)}_{m_3}(\tau_3) \ , \tag{3.61}$$

where

$$S^{(\mathrm{i}k)}_p = R^{-3/2}\sqrt{\frac{p\sinh\pi p}{2(\cosh^2\pi k+\sinh^2\pi p)}}\Big[P^{\mathrm{i}p}_{\mathrm{i}k-1/2}(\tanh\tau_1)+P^{\mathrm{i}p}_{\mathrm{i}k-1/2}(-\tanh\tau_1)\Big] \ , \tag{3.62}$$

$$\psi^{(\lambda_1)}_k(\tau_2) = \sqrt{\frac{k\sinh\pi k}{2(\cos^2\pi\lambda_1+\sinh^2\pi k)}}\Big[P^{\mathrm{i}k}_{\lambda_1-1/2}(\tanh\tau_2)+P^{\mathrm{i}k}_{\lambda_1-1/2}(-\tanh\tau_2)\Big] \ , \tag{3.63}$$

with the $\psi^{(\nu)}_{m_3}(\tau_3)$ as in (3.58).The third set finally is given by

$$\Psi^{(\omega)}_{pk\varrho}(\tau_1,\tau_2,\tau_3;R) = (\cosh^2\tau_1\cosh\tau_2)^{-1/2} S^{(\mathrm{i}k)}_p(\tau_1;R)\psi^{(\mathrm{i}\varrho)}_k(\tau_2)\psi^{(\nu)}_{\varrho}(\tau_3) \ , \tag{3.64}$$

where

$$\psi^{(\mathrm{i}\varrho)}_k(\tau_2) = \sqrt{\frac{k\sinh\pi k}{2(\cosh^2\pi\varrho+\sinh^2\pi k)}}\Big[P^{\mathrm{i}k}_{\mathrm{i}\varrho-1/2}(\tanh\tau_2)+P^{\mathrm{i}k}_{\mathrm{i}\varrho-1/2}(-\tanh\tau_2)\Big] \ , \tag{3.65}$$

$$\psi^{(\nu)}_{\varrho}(\tau_3) = \sqrt{\frac{\varrho\sinh\pi\varrho}{2(\cos^2\pi\nu+\sinh^2\pi\varrho)}}\Big[P^{\mathrm{i}\varrho}_{\nu-1/2}(\tanh\tau_3)+P^{\mathrm{i}\varrho}_{\nu-1/2}(-\tanh\tau_3)\Big] \ . \tag{3.66}$$

The spectra in all sets are as in (3.27).
The corresponding Green function has the form (E' as before)

$$\begin{aligned}
G^{(\omega)}(\mathrm{u}'',\mathrm{u}';E) &= \frac{M}{\hbar^2 R}(\cosh^2\tau_1'\cosh\tau_2'\cosh^2\tau_1''\cosh\tau_2'')^{-1/2}\\
&\times\Bigg\{\sum_{m_3=0}^{N_{m_3}}\psi^{(\nu)}_{m_3}(\tau_3')\psi^{(\nu)}_{m_3}(\tau_3'')\Bigg[\sum_{m_2=0}^{N_{m_2}}\psi^{(\lambda_1)}_{m_2}(\tau_2')\psi^{(\lambda_1)}_{m_2}(\tau_2'')\\
&\qquad\times\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}-\lambda_2+\frac{1}{2}\bigg)\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}+\lambda_2+\frac{1}{2}\bigg)\\
&\qquad\times P^{-\sqrt{-2MR^2E'}/\hbar}_{\lambda_2-1/2}(\tanh\tau_{1,<})P^{-\sqrt{-2MR^2E'}/\hbar}_{\lambda_2-1/2}(-\tanh\tau_{1,>}) \ .\\
&\quad+\int_0^\infty dk\,\psi^{(\lambda_1)\,*}_k(\tau_2')\psi^{(\lambda_1)}_k(\tau_2'')\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}-\mathrm{i}k+\frac{1}{2}\bigg)\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}+\mathrm{i}k+\frac{1}{2}\bigg)\\
&\qquad\times P^{-\sqrt{-2MR^2E'}/\hbar}_{\mathrm{i}k-1/2}(\tanh\tau_{1,<})P^{-\sqrt{-2MR^2E'}/\hbar}_{\mathrm{i}k-1/2}(-\tanh\tau_{1,>})\Bigg]\\
&+\int_0^\infty dk\int_0^\infty d\varrho\,\psi^{(\mathrm{i}\varrho)\,*}_k(\tau_2')\psi^{(\mathrm{i}\varrho)}_k(\tau_2'')\psi^{(\nu)\,*}_{\varrho}(\tau_3')\psi^{(\nu)}_{\varrho}(\tau_3'')\\
&\quad\times\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}-\mathrm{i}k+\frac{1}{2}\bigg)\Gamma\bigg(\frac{1}{\hbar}\sqrt{-2MR^2E'}+\mathrm{i}k+\frac{1}{2}\bigg)\\
&\quad\times P^{-\sqrt{-2MR^2E'}/\hbar}_{\mathrm{i}k-1/2}(\tanh\tau_{1,<})P^{-\sqrt{-2MR^2E'}/\hbar}_{\mathrm{i}k-1/2}(-\tanh\tau_{1,>})\Bigg\} \ .
\end{aligned} \tag{3.67}$$

3.1.2 General Case.

The generalized oscillator allows an explicit solution in five coordinate systems. The principal difference to the pure oscillator case lies in the fact that we must deal with interrelated (modified) Pöschl-Teller potentials in all variables. In the following the wave-functions are only explicitly presented if they have been not stated before, otherwise we refer to already stated ones.

We also do not repeat the statement of the corresponding Green functions which are easy to write down. In the relevant coordinates τ and τ_1, respectively, they are always of a form of the modified Pöschl-Teller type Green function $G^{(\kappa,\lambda)}_{mPT}(E)$ of (A.34). We just have to insert E' as indicated in the last subsection, and for the parameters (κ,λ) the parameters of the corresponding radial function $S^{(\kappa,\lambda)}_{n/p}$, respectively.

Cylindrical Coordinates. For the oscillator on $\Lambda^{(3)}$ we obtain the following path integral representation ($\lambda_1 = 2m \mp k_1 \mp k_2 + 1, \lambda_2 = 2l \mp k_3 - \nu + 1, \nu^2 = M^2\omega^2 R^4/\hbar^2 + 1/4$):

$$K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T) = R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]$$
$$\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\sinh\tau_1\cosh\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2+\sinh^2\tau_1\dot\varphi^2+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau_1}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)\right)+\frac{1}{\cosh^2\tau_1}\left(\frac{k_3^2-\frac14}{\sinh^2\tau_2}+\frac14\right)\right]dt\right\} \tag{3.68}$$

$$= \sum_{m=0}^\infty\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi^{(V_1)}_{nlm}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_1)}_{nlm}(\tau_1',\tau_2',\varphi';R)\right.\right.$$
$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(V_1)}_{plm}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_1)\,*}_{plm}(\tau_1',\tau_2',\varphi';R)\right]$$
$$\left.+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^{(V_1)}_{pkm}(\tau_1'',\tau_2'',\varphi'';R)\Psi^{(V_1)\,*}_{pkm}(\tau_1',\tau_2',\varphi';R)\right\}\ . \tag{3.69}$$

The bound state wave-functions are given by, with $m = 0,\dots,N_m = [\frac12(2l \mp k_2 - \nu)], l = 0,\dots,N_l = [(\nu \mp k_3 - 1)/2], N = 0,\dots,N_{Max} = [\frac12(\nu \mp k_1 \mp k_2 \mp k_3)]$:

$$\Psi^{(V_1)}_{nlm}(\tau_1,\tau_2,\varphi;R) = (\sinh\tau_1\cosh\tau_1)^{-1/2}S^{(\lambda_1,\lambda_2)}_n(\tau_1;R)\psi^{(\pm k_3,\nu)}_l(\tau_2)\phi^{(\pm k_2,\pm k_1)}_m(\varphi)\ , \tag{3.70}$$

where

$$S^{(\lambda_1,\lambda_2)}_n(\tau_1;R) = \frac{1}{\Gamma(1+\lambda_1)}\left[\frac{2(\lambda_2-\lambda_1-2n-1)\Gamma(n+1+\lambda_1)\Gamma(\lambda_2-n)}{R^3\Gamma(\lambda_2-\lambda_1-n)n!}\right]^{1/2}$$
$$\times(\sinh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{2n+1/2-\lambda_2}{}_2F_1(-n,\lambda_2-n;1+\lambda_1;\tanh^2\tau_1)\ , \tag{3.71}$$
$$\psi^{(\pm k_3,\nu)}_l(\tau_2) = \frac{1}{\Gamma(1\pm k_3)}\left[\frac{2(\nu\mp k_3-2l-1)\Gamma(l+1\pm k_3)\Gamma(\nu-l)}{\Gamma(\nu\mp k_3-l)l!}\right]^{1/2}$$
$$\times(\sinh\tau_2)^{1/2\pm k_3}(\cosh\tau_2)^{2l+1/2-\nu}{}_2F_1(-l,\nu-l;1\pm k_3;\tanh^2\tau_2) \tag{3.72}$$
$$\phi^{(\pm k_2,\pm k_1)}_m(\varphi) = \left[2(1+2m\pm k_1\pm k_2)\frac{m!\Gamma(m\pm k_1\pm k_2+1)}{\Gamma(1+m\pm k_1)\Gamma(1+m\pm k_2)}\right]^{1/2}$$
$$\times(\sin\varphi)^{1/2\pm k_2}(\cos\varphi)^{1/2\pm k_1}P^{(\pm k_2,\pm k_1)}_m(\cos 2\varphi)\ . \tag{3.73}$$

The bound state energy spectrum is given by ($N = m + l + n + 3$ is the principal quantum number)

$$E_N = -\frac{\hbar^2}{2MR^2}\Big[(2N \pm k_1 \pm k_2 \pm k_3 - \nu)^2 - 1\Big] + \frac{M}{2}\omega^2 R^2 \ . \tag{3.74}$$

In the limit $R \to \infty$ we obtain

$$E_N \simeq \hbar\omega(2N \pm k_1 \pm k_2 \pm k_3) \ , \qquad N \in \mathbb{N}_0 \ , \tag{3.75}$$

which gives the correct spectrum for the corresponding super-integrable flat space oscillator, i.e., the generalized oscillator in $\mathbb{R}^3$ [80].
For the first set of continuous states we find

$$\Psi^{(V_1)}_{plm}(\tau_1,\tau_2,\varphi;R) = (\sinh\tau_1\cosh\tau_1)^{-1/2} S_p^{(\lambda_1,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_3,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ , \tag{3.76}$$

where

$$\begin{aligned} S_p^{(\lambda_1,\lambda_2)}(\tau_1;R) &= \frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2 R^3}}\,\Gamma\bigg(\frac{\lambda_2-\lambda_1+1-\mathrm{i}p}{2}\bigg)\Gamma\bigg(\frac{\lambda_1-\lambda_2+1-\mathrm{i}p}{2}\bigg)\\ &\quad\times(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{\mathrm{i}p}{}_2F_1\bigg(\frac{\lambda_2+\lambda_1+1-\mathrm{i}p}{2},\frac{1+\lambda_1-\lambda_2-\mathrm{i}p}{2};1+\lambda_1;\tanh^2\tau_1\bigg) \ , \end{aligned} \tag{3.77}$$

with the $\psi_l^{(\pm k_3,\nu)}(\tau_2)$ and $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.72, 3.73), and the continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2MR^2}(p^2+1) + \frac{M}{2}\omega^2 R^2 \ . \tag{3.78}$$

In the limiting case $\omega \to 0$ we obtain E_p as in (3.28) which corresponds to the case where just a radial part is present and has the same feature as the spectrum of the free motion on $\Lambda^{(3)}$, i.e., there is no discrete spectrum in this case.
For the second set of continuous states we find

$$\Psi^{(V_1)}_{pkm}(\tau_1,\tau_2,\varphi;R) = (\sinh\tau_1\cosh\tau_1)^{-1/2} S_p^{(\lambda_1,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\pm k_3,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ , \tag{3.79}$$

where

$$\begin{aligned} S_p^{(\lambda_1,\mathrm{i}k)}(\tau_1;R) &= \frac{1}{\Gamma(1+\lambda_1)}\sqrt{\frac{p\sinh\pi p}{2\pi^2 R^3}}\,\Gamma\bigg(\frac{\mathrm{i}k-\lambda_1+1-\mathrm{i}p}{2}\bigg)\Gamma\bigg(\frac{\lambda_1-\mathrm{i}k+1-\mathrm{i}p}{2}\bigg)\\ &\quad\times(\tanh\tau_1)^{1/2+\lambda_1}(\cosh\tau_1)^{\mathrm{i}p}{}_2F_1\bigg(\frac{\mathrm{i}k+\lambda_1+1-\mathrm{i}p}{2},\frac{1+\lambda_1-\mathrm{i}k-\mathrm{i}p}{2};1+\lambda_1;\tanh^2\tau_1\bigg) \ , \end{aligned} \tag{3.80}$$

$$\begin{aligned} \psi_k^{(\pm k_3,\nu)}(\tau_2) &= \frac{1}{\Gamma(1\pm k_3)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\,\Gamma\bigg(\frac{\nu\mp k_3+1-\mathrm{i}k}{2}\bigg)\Gamma\bigg(\frac{\pm k_3-\nu+1-\mathrm{i}k}{2}\bigg)\\ &\quad\times(\tanh\tau_2)^{1/2\pm k_3}(\cosh\tau_2)^{\mathrm{i}k}{}_2F_1\bigg(\frac{\nu\pm k_3+1-\mathrm{i}k}{2},\frac{1\pm k_3-\nu-\mathrm{i}k}{2};1\pm k_3;\tanh^2\tau_2\bigg) \ , \end{aligned} \tag{3.81}$$

with the $\phi_m^{(\pm k_1,\pm k_2)}(\varphi)$ as in (3.73).

Sphero-Elliptic Coordinates. In sphero-elliptic coordinates we have the path integral representation ($\lambda_2 = 2(l+m+1) \pm k_3, \nu^2 = M^2\omega^2 R^4/\hbar^2 + 1/4$)

$$K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T)$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\tilde\alpha(t')=\tilde\alpha'}^{\tilde\alpha(t'')=\tilde\alpha''} \mathcal{D}\tilde\alpha(t) \int\limits_{\tilde\beta(t')=\tilde\beta'}^{\tilde\beta(t'')=\tilde\beta''} \mathcal{D}\tilde\beta(t)(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau^2 + \sinh^2\tau(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)(\dot{\tilde\alpha}^2 + \dot{\tilde\beta}^2) - \omega^2\tanh^2\tau\Big)\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} + \frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)\right]dt\right\} \tag{3.82}$$

$$= \sum_{lh}\left\{\sum_{n=0}^{N_n} \mathrm{e}^{-\mathrm{i}E_N T/\hbar}\Psi_{nlh}^{(V_1)}(\tau'',\tilde\alpha'',\tilde\beta'';R)\Psi_{nlh}^{(V_1)}(\tau',\tilde\alpha',\tilde\beta';R)\right.$$

$$\left. + \int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plh}^{(V_1)}(\tau'',\tilde\alpha'',\tilde\beta'';R)\Psi_{plh}^{(V_1)\,*}(\tau',\tilde\alpha',\tilde\beta';R)\right\}\ . \tag{3.83}$$

The bound state wave-functions are given by

$$\Psi_{nlh}^{(V_1)}(\tau,\tilde\alpha,\tilde\beta;R) = (\sinh^2\tau\mathrm{sn}\tilde\alpha\mathrm{cn}\tilde\alpha\mathrm{dn}\tilde\alpha\mathrm{sn}\tilde\beta\mathrm{cn}\tilde\beta\mathrm{dn}\tilde\beta)^{-1/2} S_n^{(\lambda_2,\nu)}(\tau;R)\,\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)\ , \tag{3.84}$$

where $N=0,\dots,N_{Max}=[\frac12(\nu\pm k_1\pm k_2\pm k_3)]$

$$S_n^{(\lambda_2,\nu)}(\tau;R) = \frac{1}{\Gamma(1+\lambda_2)}\left[\frac{2(\nu-\lambda_2-2n-1)\Gamma(n+1+\lambda_2)\Gamma(\nu-n)}{R^3\Gamma(\nu-\lambda_2-n)n!}\right]^{1/2}$$
$$\times(\sinh\tau)^{\lambda_2+1/2}(\cosh\tau)^{2n+1/2-\nu}{}_2F_1(-n,\nu-n;1+\lambda_2;\tanh^2\tau)\ , \tag{3.85}$$

with the same energy-spectrum as in the previous case. The wave-functions $\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)$ have been determined in [81] and correspond to the free wave-function on the six-dimensional sphere in a cylindric-elliptic coordinate system. They are not explicitly known yet, and therefore the above solution in sphero-conical coordinates remains on a somewhat formal level. We present it for completeness, though. The continuous spectrum has the form

$$\Psi_{plh}^{(V_1)}(\tau,\tilde\alpha,\tilde\beta;R) = (\sinh^2\tau\mathrm{sn}\tilde\alpha\mathrm{cn}\tilde\alpha\mathrm{dn}\tilde\alpha\mathrm{sn}\tilde\beta\mathrm{cn}\tilde\beta\mathrm{dn}\tilde\beta)^{-1/2} S_p^{(\lambda_2,\nu)}(\tau;R)\,\Xi_{lh}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha,\tilde\beta)\ , \tag{3.86}$$

where

$$S_p^{(\lambda_2,\nu)}(\tau;R) = \frac{1}{\Gamma(1+\lambda_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^3}}\,\Gamma\Big(\frac{\nu-\lambda_2+1-\mathrm{i}p}{2}\Big)\Gamma\Big(\frac{\lambda_2-\nu+1-\mathrm{i}p}{2}\Big)$$
$$\times(\tanh\tau)^{\lambda_2+1/2}(\cosh\tau)^{\mathrm{i}p}{}_2F_1\Big(\frac{\nu+\lambda_2+1-\mathrm{i}p}{2},\frac{\lambda_2-\nu+1-\mathrm{i}p}{2};1+\lambda_2;\tanh^2\tau\Big)\ , \tag{3.87}$$

and E_p as in the previous case.

Spherical Coordinates. In spherical coordinates we have the path integral representation ($\lambda_1 = 2m\mp k_1\mp k_2+1, \lambda_2 = 2l\mp k_3+\lambda_1+1, \nu^2 = M^2\omega^2R^4/\hbar^2+1/4$)

$$K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T)$$
$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\omega^2\tanh^2\tau\Big)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\right)\right]dt\right\} \quad (3.88)$$

$$=\sum_{m=0}^\infty\sum_{l=0}^\infty\left\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{nlm}^{(V_1)}(\tau'',\vartheta'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau',\vartheta',\varphi';R)\right.$$

$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_1)\,*}(\tau',\vartheta',\varphi';R)\right\}\ . \quad (3.89)$$

The bound state wave-functions are given by $N=0,\dots,N_{Max}=[\frac12(\nu\mp k_1\mp k_2\mp k_3)]$

$$\Psi_{nlm}^{(V_1)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_n^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \quad (3.90)$$

where

$$\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)=\left[2(1+2l\pm k_3+\lambda_1)\frac{l!\Gamma(l+\lambda_1\pm k_3+1)}{\Gamma(1+l\pm k_3)\Gamma(1+l+\lambda_1)}\right]^{1/2}$$

$$\times(\sin\vartheta)^{1/2+\lambda_1}(\cos\vartheta)^{1/2\pm k_3}P_l^{(\lambda_1,\pm k_3)}(\cos 2\vartheta)\ . \quad (3.91)$$

The scattering states are

$$\Psi_{plm}^{(V_1)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_p^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ . \quad (3.92)$$

Here denote:

- the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ are the same wave-functions as in (3.73), and
- the wave-functions $S_n^{(\lambda_2,\nu)}(\tau;R)$ and $S_p^{(\lambda_2,\nu)}(\tau;R)$ are the same wave-functions as in (3.85, 3.87), respectively.

The discrete and continuous energy spectra E_N and E_p are, of course, the same as in (3.74, 3.78), respectively.

Equidistant-Cylindrical Coordinates. In equidistant cylindrical coordinates we obtain the path integral solution ($\lambda_1=2m\mp k_1\mp k_2+1,\lambda_2=2l+\lambda_1-\nu+1,\nu^2=M^2\omega^2R^4/\hbar^2+1/4$)

$$K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T)=R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\right)\right]$$

$$\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\sinh\tau_2\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\sinh^2\tau_2\dot\varphi^2)+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\Big)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1}\left(\frac{1}{\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\right)+\frac14\right)+\frac{k_3^2-\frac14}{\sinh^2\tau_1}\right)\right]dt\right\} \quad (3.93)$$

$$=\sum_{m=0}^\infty\left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}\hbar E_NT/\hbar}\Psi_{nlm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\varphi';R)\right.\right.$$

$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{plm}^{(V_1)\,*}(\tau_1',\tau_2',\varphi';R)\right]$$

$$\left.+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pkm}^{(V_1)}(\tau_1'',\tau_2'',\varphi'';R)\Psi_{pkm}^{(V_1)*}(\tau_1',\tau_2',\varphi';R)\right\}\ . \quad (3.94)$$

We obtain one set of bound state wave-functions and two sets of scattering wave-functions. The bound state wave-functions are ($l+m=0,\dots,N_l=[\frac{1}{2}[(\nu\mp k_1\mp k_2-2)], N=0,\dots,N_{Max}=[\frac{1}{2}(\nu\mp k_1\mp k_2\mp k_3)]$

$$\Psi_{nlm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \tag{3.95}$$

and E_N as in (3.74). The two sets of continuous states are

$$\Psi_{pkm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \tag{3.96}$$
$$\Psi_{plm}^{(V_1)}(\tau_1,\tau_2,\varphi;R)=(\cosh^2\tau_1\sinh\tau_2)^{-1/2}S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\lambda_1,\nu)}(\tau_2)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \tag{3.97}$$

and E_p as in (3.27). Here denote:

- the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ are the same wave-functions as in (3.73),
- the wave-functions $\psi_{l,k}^{(\lambda_1,\nu)}(\tau_2)$ are the same wave-functions as in (3.72, 3.81) with $\pm k_3\to\lambda_1$,
- the wave-functions $S_{n,p}^{(\pm k_3,\lambda_2)}(\tau_1;R)$ are the same wave-functions as in (3.71, 3.77) with $\lambda_1\to\pm k_3$, respectively.

Equidistant Coordinates. As the last system where an explicit solution is possible we consider the equidistant system and obtain the path integral solution ($\lambda_1=2m\mp k_1-\nu+1,\lambda_2=2l\mp k_2-\lambda_1+1,\nu^2=M^2\omega^2R^4/\hbar^2+1/4$)

$$\begin{aligned}
K^{(V_1)}(\mathrm{u}'',\mathrm{u}';T)&=R^{-3}\exp\bigg[-\frac{\mathrm{i}}{\hbar}T\bigg(\frac{\hbar^2}{2MR^2}+\frac{M}{2}R^2\omega^2\bigg)\bigg]\\
&\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\cosh\tau_2\int\limits_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''}\mathcal{D}\tau_3(t)\\
&\times\exp\bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg[\frac{M}{2}R^2\bigg(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\cosh^2\tau_1\dot\tau_3^2)+\frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2\cosh^2\tau_3}\bigg)\\
&\qquad-\frac{\hbar^2}{2MR^2}\bigg(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\bigg(\frac{1}{\cosh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\sinh^2\tau_3}+\frac14\bigg)+\frac{k_2^2-\frac14}{\sinh^2\tau_2}+\frac14\bigg)\bigg)\bigg]dt\bigg\}
\end{aligned}\tag{3.98}$$

$$\begin{aligned}
&=\sum_{m=0}^{N_m}\Bigg\{\sum_{l=0}^{N_l}\Bigg[\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{nlm}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{nlm}^{(V_1)}(\tau_1',\tau_2',\tau_3';R)\\
&\qquad+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{plm}^{(V_1)*}(\tau_1',\tau_2',\tau_3';R)\Bigg]\\
&\qquad+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pkm}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pkm}^{(V_1)*}(\tau_1',\tau_2',\tau_3';R)\Bigg\}\\
&+\int_0^\infty d\varrho\int_0^\infty dk\int_0^\infty dp\ \mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pk\varrho}^{(V_1)}(\tau_1'',\tau_2'',\tau_3'';R)\Psi_{pk\varrho}^{(V_1)*}(\tau_1',\tau_2',\tau_3';R)\ .
\end{aligned}\tag{3.99}$$

We obtain one set of bound state wave-functions and three sets of scattering wave-functions. The bound state wave-functions are ($m=0,\dots,N_m<(\nu\mp k_1-1)/2, l=0,\dots,N_l<(\lambda_1\mp k_2-1)/2, n=0,\dots,N_n<(\lambda_2\mp k_3-1)/2$)

$$\Psi_{nlm}^{(V_1)}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_n^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)\psi_m^{(\pm k_1,\nu)}(\tau_3)\ , \tag{3.100}$$

and E_N as in (3.74). The three sets of continuous states are

$$\Psi^{(V_1)}_{plm}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\pm k_3,\lambda_2)}(\tau_1;R)\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)\psi_m^{(\pm k_1,\nu)}(\tau_3)\ , \tag{3.101}$$

$$\Psi^{(V_1)}_{mkp}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\pm k_2,\lambda_1)}(\tau_2)\psi_m^{(\pm k_1,\nu)}(\tau_3)\ , \tag{3.102}$$

$$\Psi^{(V_1)}_{\varrho kp}(\tau_1,\tau_2,\tau_3;R)=(\cosh^2\tau_1\cosh\tau_2)^{-1/2}S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)\psi_k^{(\pm k_2,\mathrm{i}\varrho)}(\tau_2)\psi_\varrho^{(\pm k_1,\nu)}(\tau_3)\ , \tag{3.103}$$

and E_p as in (3.27). Here denote:

- the wave-functions $\psi_m^{(\pm k_1,\nu)}(\tau_3)$ are the same wave-functions as in (3.72) with $\tau_2\to\tau_3$, $l\to m$ and $\pm k_3\to\pm k_1$,
- the wave-functions $\psi_\varrho^{(\pm k_1,\nu)}(\tau_3)$ are the same wave-functions as in (3.81) with $\tau_2\to\tau_3$, $k\to\varrho$ and $\pm k_3\to\pm k_1$,
- the wave-functions $\psi_l^{(\pm k_2,\lambda_1)}(\tau_2)$ are the same wave-functions as in (3.71) with $\tau_1\to\tau_2$, $n\to l$, $(\lambda_1,\lambda_2)\to(\pm k_2,\lambda_1)$ and $R=1$,
- the wave-functions $\psi_k^{(\pm k_2,\lambda_1)}(\tau_2)$ are the same wave-functions as in (3.77) with $\tau_1\to\tau_2$, $p\to k$, $(\lambda_1,\lambda_2)\to(\pm k_2,\lambda_1)$ and $R=1$,
- the wave-functions $\psi_k^{(\pm k_2,\mathrm{i}\varrho)}(\tau_2)$ are the same wave-functions as in (3.80) with $\tau_1\to\tau_2$, $(p,k)\to(k,\varrho)$, $\lambda_1\to\pm k_2$ and $R=1$,
- the wave-functions $S_{n,p}^{(\pm k_3,\lambda_2)}(\tau_1;R)$ and $S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1;R)$ are the same wave-functions as in (3.71, 3.77) and (3.80) with $\lambda_1\to\pm k_3$, respectively.

Let us remark that the wave-functions have been normalized in the domains $\varphi\in(0,\pi/2),\vartheta\in(0,\pi/2)$ and $\tau>0$ in the spherical and in $\tau_{1,2,3}>0$ in the equidistant system. The positive sign for the k_i has to be taken whenever $k_i\geq\frac12$ $(i=1,2,3)$, i.e., the potential term is repulsive at the origin, and the motion takes only place in the denoted domains. If $0<|k_i|<\frac12$, i.e., the potential term is attractive at the origin, both the positive and the negative sign must be taken into account in the solution. This is indicated by the notion $\pm k_i$ in the formulæ. It has also the consequence that for each k_i the motion can take place in the entire domains of the variables on $\Lambda^{(3)}$. In the present case this means that we must, e.g., in the equidistant system distinguish eight cases: i) $\tau_{1,2,3}>0$, ii) $\tau_{1,2}>0$, $\tau_2\in\mathbb{R}$, iii) $\tau_1\in\mathbb{R}$, $\tau_{2,3}>0$, iv) $\tau_2\in\mathbb{R}$, $\tau_{1,3}>0$, v) $\tau_{1,2}\in\mathbb{R}$, $\tau_2>0$, vi) $\tau_1>0$, $\tau_{2,3}\in\mathbb{R}$, vii) $\tau_2>0$, $\tau_{1,3}\in\mathbb{R}$ and viii) $\tau_{1,2,3}\in\mathbb{R}$. In polar coordinates the same feature is recovered by the observation that the Pöschl-Teller barriers are absent for $|k_i|<\frac12$.

In elliptic coordinates this feature is taken into account in the following way: Due to $\alpha\in(\mathrm{i}K',\mathrm{i}K'+K)$, we have $\mathrm{sn}(\alpha,k),\mathrm{i}\,\mathrm{cn}(\alpha,k)>k'/k$, $\mathrm{i}\,\mathrm{dn}(\alpha,k)\geq0$, and we see that for $\alpha\in(\mathrm{i}K',\mathrm{i}K'+K)$ and $\beta\in(K',4K')$ we get $u_0\geq0$ and the variables u_1,u_2,u_3 change their signs in the eight domains, i.e., $\beta\in(0,K'),\beta\in(K',2K'),\beta\in(2K',3K')$ and $\beta\in(3K',4K')$. We then have for $\alpha\neq0$

$$\begin{aligned}\mathrm{sn}(0,k')=\ \mathrm{sn}(2K',k')=\ \mathrm{sn}(4K',k')=0\ ,\\ \mathrm{cn}(K',k')=\ \mathrm{cn}(3K',k')=0\ ,\end{aligned} \tag{3.104}$$

and $\mathrm{dn}(\beta,k')>0$, $\beta\in[0,4K')$. For convenience, we have made the choice $\beta\in(0,K')$, and the same is true in all following systems. The situation is similar in the hyperbolic system, where we choose $\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K')$. In the sphero-elliptic system we must choose for the same reasons $\tilde\alpha\in(0,K)$ and $\tilde\beta\in(0,K')$.

3.2 The Coulomb Potential.

We consider the Coulomb potential on the three-dimensional hyperboloid ($k_{1,2}>0$)

$$V_2(\mathrm{u})=-\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}}-1\right)+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\right)\ , \tag{3.105}$$

which in the five separating coordinate systems has the form

Sphero-Elliptic ($\tau>0,\tilde\alpha\in(0,K),\tilde\beta\in(0,K')$) :

$$V_2(\mathrm{u})=-\frac{\alpha}{R}(\coth\tau-1)+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}\right) \tag{3.106}$$

Spherical ($\tau>0,\vartheta\in(0,\pi),\varphi\in(0,\pi/2)$) :

$$=-\frac{\alpha}{R}(\coth\tau-1)+\frac{\hbar^2}{2MR^2\sinh^2\tau\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right) \tag{3.107}$$

Prolate Elliptic II ($\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi/2)$) :

$$=-\frac{\alpha}{R}\left(\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\beta-k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}-1\right)+\frac{\hbar^2}{2MR^2\,\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right) \tag{3.108}$$

Semi-Hyperbolic ($\mu_{1,2}>0,\varphi\in(0,\pi/2)$) :

$$=-\frac{\alpha}{R}\left(\frac{\sqrt{1+\mu_1^2}+\sqrt{1+\mu_2^2}}{\mu_1+\mu_2}-1\right)+\frac{\hbar^2}{2MR^2\mu_1\mu_2}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right) \tag{3.109}$$

Elliptic-Parabolic 2 ($a>0,\vartheta\in(0,\pi),\varphi\in(0,\pi/2)$) :

$$=-\frac{\alpha}{R}\left(\frac{\cosh^2a+\cos^2\vartheta}{\cosh^2a-\cos^2\vartheta}-1\right)+\frac{\hbar^2}{2MR^2}\coth^2a\cot^2\vartheta\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\ . \tag{3.110}$$

For the five observables we find

$$\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_2(\mathrm{u})\ ,\\
I_2&=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\ ,\\
I_3&=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\ ,\\
I_4&=\frac{1}{2M}(L_1^2+k'^2L_2^2)+\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}\right)\ ,\\
I_5&=\frac{1}{2M}A_3+\frac{\hbar^2}{2M\sinh^2\tau\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\ .
\end{aligned}\right\} \tag{3.111}$$

Here A_3 denotes the third component of the Pauli-Lenz-Runge vector on the hyperboloid [165, 170], i.e.,

$$\mathbf{A}=\frac{1}{2R}(\mathbf{L}\times\mathbf{K}-\mathbf{K}\times\mathbf{L})-\frac{\alpha\mathbf{u}}{\sqrt{u_1^2+u_2^2+u_3^2}}\ ,\qquad \mathbf{u}=(u_1,u_2,u_3)\ . \tag{3.112}$$

The path integral for the Coulomb potential on $\Lambda^{(3)}$ can be explicitly evaluated in three coordinate systems which will be discussed in the following. In the prolate-elliptic II and the semi-hyperbolic system no explicit solution is known.

3.2.1 Sphero-Elliptic Coordinates.

The $\tilde\alpha$- and $\tilde\beta$-path integrations can be separated off without difficulty and we obtain

$$K^{(V_2)}(\tau'',\tau',\tilde\alpha'',\tilde\alpha',\tilde\beta'',\tilde\beta';T)=R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]$$

$$\times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh^2\tau \int\limits_{\tilde\alpha(t')=\tilde\alpha'}^{\tilde\alpha(t'')=\tilde\alpha''} \mathcal{D}\tilde\alpha(t) \int\limits_{\tilde\beta(t')=\tilde\beta'}^{\tilde\beta(t'')=\tilde\beta''} \mathcal{D}\tilde\beta(t)(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau^2 + \sinh^2\tau(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)(\dot{\tilde\alpha}^2 + \dot{\tilde\beta}^2)\Big) + \frac{\alpha}{R}\coth\tau \right.\right.$$
$$\left.\left. -\frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} - \frac14\right)\right]dt\right\} \qquad (3.113)$$

$$= \sum_{lh\pm}(\sinh^2\tau'\,\mathrm{sn}\tilde\alpha'\,\mathrm{cn}\tilde\alpha'\,\mathrm{dn}\tilde\alpha'\,\mathrm{sn}\tilde\beta'\,\mathrm{cn}\tilde\beta'\,\mathrm{dn}\tilde\beta'\sinh^2\tau''\,\mathrm{sn}\tilde\alpha''\,\mathrm{cn}\tilde\alpha''\,\mathrm{dn}\tilde\alpha''\,\mathrm{sn}\tilde\beta''\,\mathrm{cn}\tilde\beta''\,\mathrm{dn}\tilde\beta'')^{-1/2}$$
$$\times \Xi_{lh}^{(\pm k_1,\pm k_2,\pm\frac12)*}(\tilde\alpha',\tilde\beta')\Xi_{lh}^{(\pm k_1,\pm k_2,\pm\frac12)}(\tilde\alpha'',\tilde\beta'')K_{lm}^{(V_2)}(\tau'',\tau';T)\ , \qquad (3.114)$$

with the remaining path integral (l,h as in subsection 3.1.2, $\lambda_2 = 2m + l \mp k_1 \mp k_2 + 3/2$)

$$K_{lm}^{(V_2)}(\tau'',\tau';T) = R^{-1}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]$$
$$\times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{M}{2}R^2\dot\tau^2 + \frac{\alpha}{R}\coth\tau - \frac{\hbar^2}{2MR^2}\frac{\lambda_2^2-\frac14}{\sinh^2\tau}\right)dt\right]\ . \qquad (3.115)$$

This path integral is solved in a similar way as in the two-dimensional case [7, 65, 82], which we do not repeat. Hence, we obtain for the radial path integral $K_{lm}^{(V_2)}(T)$

$$K_{lm}^{(V_2)}(\tau'',\tau';T)$$
$$= \sum_{N=0}^{N_{Max}} \mathrm{e}^{-\mathrm{i}E_nT/\hbar}S_N^{(V_2)*}(\tau';R)S_N^{(V_2)}(\tau'';R) + \int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}S_p^{(V_2)*}(\tau';R)S_p^{(V_2)}(\tau'';R)\ . \qquad (3.116)$$

The bound state wave-functions and the energy-spectrum are given by

$$S_N^{(V_2)}(\tau;R) = \frac{2^{\lambda_2+\frac12}}{\Gamma(2\lambda_2+\frac12)}\left[\frac{\sigma_N^2-\tilde N^2}{R^3\tilde N^2}\frac{\Gamma(\tilde N+\lambda_2+\frac12)\Gamma(\sigma_N+\lambda_2+\frac12)}{\Gamma(\tilde N-\lambda_2)\,\Gamma(\sigma_N-\lambda_2)}\right]^{1/2}$$
$$\times(\sinh\tau)^{\lambda_2+1/2}\mathrm{e}^{-\tau(\sigma_N-\tilde N)}{}_2F_1\left(-n, \lambda_2+\frac12+\sigma_N; 2\lambda_2+1; \frac{2}{1+\coth\tau}\right)\ , \qquad (3.117)$$

$$E_N = \frac{\alpha}{R} - \hbar^2\frac{\tilde N^2-1}{2MR^2} - \frac{M\alpha^2}{2\hbar^2\tilde N^2}\ . \qquad (3.118)$$

Here we have abbreviated $a = \hbar^2/M\alpha$ (the Bohr radius), $\sigma_N = aR/\tilde N$, $\tilde N = N \mp k_1 \mp k_2 + 2$, $N = n + l + 2m$, $N = 0, 1, 2, \dots, N_{Max} < \sqrt{R/a}$. The continuous spectrum has the form

$$S_p^{(V_2)}(\tau;R) = \frac{2^{(\mathrm{i}/2)(p-\tilde p)+\lambda_2+\frac12}}{\pi\Gamma(2\lambda_2+1)}\sqrt{\frac{p\sinh\pi p}{2R^3}}\,\Gamma\left(\lambda_2+\frac12+\frac{\mathrm{i}}{2}(\tilde p-p)\right)\Gamma\left(\lambda_2+\frac12-\frac{\mathrm{i}}{2}(\tilde p+p)\right)$$
$$\times(\sinh\tau)^{\lambda_2+1/2}\exp\left[\tau\left(\frac{\mathrm{i}}{2}(\tilde p+p)-\lambda_2-\frac12\right)\right]$$
$$\times{}_2F_1\left(\lambda_2+\frac12+\frac{\mathrm{i}}{2}(\tilde p-p), \lambda_2+\frac12-\frac{\mathrm{i}}{2}(\tilde p+p); 2\lambda_2+1; \frac{2}{1+\coth\tau}\right)\ , \qquad (3.119)$$

$\tilde p = \sqrt{2MR^2(E_p - \alpha\hbar^2/R)}\Big/\hbar$, and E_p as in (3.28). The complete wave functions of the generalized Coulomb problem on the three-dimensional pseudosphere in spherical coordinates are thus given by

$$\Psi_{nlh}^{(V_2)}(\tau,\tilde\alpha,\tilde\beta;R) = (\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2}S_N^{(V_2)}(\tau;R)\,\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac12)}(\tilde\alpha,\tilde\beta)\ , \qquad (3.120)$$
$$\Psi_{plh}^{(V_2)}(\tau,\tilde\alpha,\tilde\beta;R) = (\sinh^2\tau\,\mathrm{sn}\tilde\alpha\,\mathrm{cn}\tilde\alpha\,\mathrm{dn}\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{cn}\tilde\beta\,\mathrm{dn}\tilde\beta)^{-1/2}S_p^{(V_2)}(\tau;R)\,\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac12)}(\tilde\alpha,\tilde\beta)\ . \qquad (3.121)$$

Let us note that in the pure Coulomb case, the path integral evaluation is almost the same with only minor differences. The wave-functions $\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac{1}{2})}$ are replaced by the wave-functions of the free motion on the sphere $S^{(2)}$, i.e., together with the notation $k,q=\pm 1, h+\tilde{h}=l(l+1)$

$$\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac{1}{2})}(\tilde{\alpha},\tilde{\beta}) \to \Lambda_{lh}^{k}(\tilde{\alpha})\Lambda_{l\tilde{h}}^{q}(\tilde{\beta}) \ . \tag{3.122}$$

The quantum number λ_2 yields the usual angular momentum number $l\in\mathbb{N}_0$. The discrete spectrum has the same form as (3.118), however with the principal quantum number N now given by $N=n+l+1$, therefore giving degeneracies with respect to the quantum number m. Everything else remains the same.

3.2.2 Spherical Coordinates.

The separation of the Coulomb problem in spherical coordinates is similarly done as for the sphero-elliptic one, and we have

$$\begin{aligned}
K^{(V_2)}(\tau'',\tau',\vartheta'',\vartheta',\varphi'',\varphi';T) &= R^{-3}\exp\bigg[-\frac{\mathrm{i}}{\hbar}T\bigg(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\bigg)\bigg]\\
&\times\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\times\exp\bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)+\frac{\alpha}{R}\coth\tau\\
&\qquad-\frac{\hbar^2}{2MR^2\sinh^2\tau}\bigg(\frac{1}{\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}-\frac14\bigg)-\frac14\bigg)\bigg]dt\bigg\}
\end{aligned}\tag{3.123}$$

$$\begin{aligned}
&=\sum_{m=0}^\infty\sum_{l=0}^\infty\bigg\{\sum_{N=0}^{N_{Max}}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{Nlm}^{(V_2)}(\tau'',\vartheta'',\varphi'';R)\Psi_{Nlm}^{(V_2)\,*}(\tau',\vartheta',\varphi';R)\\
&\qquad+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_2)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_2)\,*}(\tau',\vartheta',\varphi';R)\bigg\} \ .
\end{aligned}\tag{3.124}$$

The bound and continuous wave-functions are given by

$$\begin{aligned}
\Psi_{Nlm}^{(V_2)}(\tau,\vartheta,\varphi;R) &= (\sinh\tau)^{-1}S_N^{(V_2)}(\tau;R)\\
&\times\sqrt{(l+\lambda_1+\tfrac12)\frac{\Gamma(l+\lambda_1+1)}{l!}}P_{l+\lambda_1}^{-\lambda_1}(\cos\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ ,
\end{aligned}\tag{3.125}$$

$$\begin{aligned}
\Psi_{plm}^{(V_2)}(\tau,\vartheta,\varphi;R) &= (\sinh\tau)^{-1}S_p^{(V_2)}(\tau;R)\\
&\times\sqrt{(l+\lambda_1+\tfrac12)\frac{\Gamma(l+\lambda_1+1)}{l!}}P_{l+\lambda_1}^{-\lambda_1}(\cos\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ ,
\end{aligned}\tag{3.126}$$

with $\lambda_1=2m\mp k_1\mp k_2+1, \lambda_2=l+\lambda_1+\frac12, N=N\mp k_1\mp k_2+2$, the wave-functions $S_N^{(V_2}(\tau;R), S_p^{(V_2}(\tau;R)$ (3.117, 3.119), with the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.73), and the energy-spectra (3.118, 3.28), respectively.

In the case of the pure Coulomb problem the angular wave-functions are just the spherical harmonics Y_l^m on $S^{(2)}$, i.e., we obtain for wave-functions in this case

$$\Psi_{Nlm}^{(V_2)}(\tau,\vartheta,\varphi;R)=(\sinh\tau)^{-1}S_N^{(V_2)}(\tau;R)Y_l^m(\vartheta,\varphi) \ , \tag{3.127}$$

$$\Psi_{plm}^{(V_2)}(\tau,\vartheta,\varphi;R)=(\sinh\tau)^{-1}S_p^{(V_2)}(\tau;R)Y_l^m(\vartheta,\varphi) \ , \tag{3.128}$$

together with the principal quantum number $N = n + l + 1 = 0, \dots, N_{Max} < \sqrt{R/a} - \lambda_1 - 1/2$. In [65] it was shown that the $S_N^{(V_2)}(\tau;R)$ and $S_p^{(V_2)}(\tau;R)$ yield the correct radial wave-functions in $\mathbb{R}^3$, as $R \to \infty$.

The Green function of the Coulomb problem in sphero-elliptic and spherical coordinates can be derived by means of the Green function of the modified Pöschl-Teller potential [72] and the Manning-Rosen potential [62]. It has the form

$$
\begin{aligned}
&G^{(V_2)}(\tau'',\tau',\vartheta'',\vartheta',\varphi'',\varphi';E) = (\sinh\tau'\sinh\tau'')^{-1}\\
&\quad\times\left(\begin{array}{c}\sum_{lh}\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac12)\,*}(\tilde\alpha',\tilde\beta')\Xi_{lm}^{(\pm k_1,\pm k_2,\pm\frac12)}(\tilde\alpha'',\tilde\beta'')\\ \sum_{lm}Y_l^{m\,*}(\vartheta',\varphi')Y_l^m(\vartheta'',\varphi'')\end{array}\right)\frac{M}{\hbar^2 R}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\quad\times\left(\frac{2}{\coth\tau'+1}\cdot\frac{2}{\coth\tau''+1}\right)^{(m_1+m_2+1)/2}\left(\frac{\coth\tau'-1}{\coth\tau'+1}\cdot\frac{\coth\tau''-1}{\coth\tau''+1}\right)^{(m_1-m_2)/2}\\
&\quad\times{}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1;\frac{\coth\tau_>-1}{\coth\tau_>+1}\right)\\
&\quad\times{}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1;\frac{2}{\coth\tau_<+1}\right)\ ,
\end{aligned}
\tag{3.129}
$$

where $L_E = \frac12(\sqrt{-(2MR^2E/\hbar^2+1)}/\hbar - 1)$, and $m_{1,2} = \lambda_2 \pm \sqrt{-2mR^2(2\alpha/R+E)-1}/\hbar$.

3.2.3 Elliptic-Parabolic 2 Coordinates.

In order to evaluate the path integral in elliptic-parabolic 2 coordinates, one first separates off the φ-path integration, and then performs a time transformation. This gives ($\lambda_1 = 2m \mp k_1 \mp k_2 + 1$)

$$
\begin{aligned}
&K^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';T) = R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]\\
&\quad\times\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\tanh a\tan\vartheta\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{(\cosh^2 a-\cos^2\vartheta)(\dot a^2+\dot\vartheta^2)+\sinh^2 a\sin^2\vartheta\dot\varphi^2}{\cosh^2 a\cos^2\vartheta}-\frac{\alpha}{R}\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\right.\right.\\
&\qquad\left.\left.+\frac{\hbar^2}{2MR^2}\coth^2 a\cot^2\vartheta\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\sinh^2 a\sin^2\vartheta}\right]dt\right\}
\end{aligned}
\tag{3.130}
$$

$$
\begin{aligned}
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}(\coth a'\coth a''\cot\vartheta'\cot\vartheta'')^{1/2}\sum_{m=0}^\infty\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\\
&\quad\times\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2R^2}\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}(\dot a^2+\dot\vartheta^2)-\frac{\alpha}{R}\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\right.\right.\\
&\qquad\left.\left.+\frac{\hbar^2\lambda_1^2}{2MR^2}\coth^2 a\cot^2\vartheta+\frac{\hbar^2}{8MR^2}\frac{\cosh^2 a+\cos^2\vartheta-1}{\cosh^2 a-\cos^2\vartheta}\right]dt\right\}
\end{aligned}
\tag{3.131}
$$

$$
\begin{aligned}
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}(\coth a'\coth a''\cot\vartheta'\cot\vartheta'')^{1/2}\sum_{m=0}^\infty\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\\
&\quad\times\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty ds''\int\limits_{a(0)=a'}^{a(s'')=a''}\mathcal{D}a(s)\int\limits_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''}\mathcal{D}\vartheta(s)
\end{aligned}
$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{M}{2}(\dot{a}^2+\dot{\vartheta}^2)-\frac{\hbar^2}{2M}\left(\frac{\lambda_1^2-\frac14}{\sin^2\vartheta}+\frac{\beta^2-\frac14}{\cos^2\vartheta}+\frac{\lambda_1^2-\frac14}{\sinh^2 a}-\frac{\nu^2-\frac14}{\cosh^2 a}\right)\right]ds\right\} , \qquad (3.132)$$

where $\beta^2=\frac14-2MER^2/\hbar^2, \nu^2=\frac14+2MR^2(2\alpha/R-E)/\hbar^2$. The analysis of this path integral is rather involved and requires the same Green function analysis as the corresponding two-dimensional case [75, 82], which will be not repeated here in all details.

Let us note first that in the case of the pure Coulomb problem we have to replace $\lambda_1=|j|$, $j\in\mathbb{Z}$, and the wave-functions in φ are circular waves, i.e., $\phi_j(\varphi)=\mathrm{e}^{\mathrm{i}j\varphi}/\sqrt{2\pi}$, $\varphi\in[0,2\pi)$. Everything else remains the same.

To analyze the general case we proceed exactly in an analogous way as in [82]. For the discrete spectrum we expand the ϑ-path integration into Pöschl-Teller potential wave functions $\Phi_{n_1}^{(\lambda_1,\beta)}(\vartheta)$, and the a-path integration into the bound state contribution of the modified Pöschl-Teller potential wave functions $\psi_{n_2}^{(\lambda_1,\nu)}(a)$ of (A.29). The emerging Green function representation $G_{disc.}^{(V_2)}(E)$ of $K_{disc.}^{(V_2)}(T)$ has poles which are determined by the equation

$$(2n_1+\lambda_1+\beta+1)=-(2n_2+\lambda_1-\nu+1) \ . \qquad (3.133)$$

Solving this equation for $E_{n_1n_2}$ yields exactly the energy-spectrum (3.118), with the principal quantum number $N=n_1+n_2+\lambda_1+1=1,\dots,N_{Max}$ as before. Taking the residuum gives the bound state wave-functions which are therefore given by

$$\Psi_{mn_1n_2}^{(V_2)}(a,\vartheta,\varphi;R)=(\coth a\cot\vartheta)^{1/2}\sqrt{\frac{\sigma_N^2-\tilde{N}^2}{2R^3\tilde{N}}}\,\psi_{n_1}^{(\lambda_1,\nu)}(a)\phi_{n_2}^{(\lambda_1,\beta)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi) \ , \qquad (3.134)$$

where ($\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ as in (3.73)),

$$(3.135)$$

$$\psi_{n_1}^{(\lambda_1,\nu)}(a)=\frac{1}{\Gamma(1+\lambda_1)}\left[\frac{2(\nu-\lambda_1-2n_1-1)\Gamma(n_1+1+\lambda_1)\Gamma(\nu-n_1)}{n_1!\Gamma(\nu\mp k_2-n_1)}\right]^{1/2}$$
$$\times(\sinh a)^{1/2+\lambda_1}(\cosh a)^{2n_1+1/2-\nu}{}_2F_2(-n_1,\nu-n_1;1+\lambda_1;\tanh^2 a) \qquad (3.136)$$

$$\phi_{n_2}^{(\lambda_1,\beta)}(\vartheta)=\left[2(\beta+\lambda_1+2n_2+1)\frac{n_2!\Gamma(\beta+\lambda_1+n_2+1)}{\Gamma(n_2+\lambda_1+1)\Gamma(n_2+\beta+1)}\right]^{1/2}$$
$$\times(\sin\vartheta)^{1/2+\lambda_1}(\cos\vartheta)^{\beta+1/2}P_{n_2}^{(\lambda_1,\beta)}(\cos 2\vartheta) \ . \qquad (3.137)$$

For the analysis of the continuous spectrum we must insert the entire Green functions of the Pöschl-Teller (A.28) and modified Pöschl-Teller potential (A.34). We then find the Green function $G^{(V_2)}(E)$ in elliptic-parabolic coordinates by considering the ds''-integration in (3.132) with the solutions of the Pöschl-Teller and modified Pöschl-Teller potential, respectively, and the result can be put in the following form (c.f. also [75] for some more details concerning the proper Green function analysis)

$$G^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';E)$$
$$=(R^2\tanh a'\tanh a''\tan\vartheta'\tan\vartheta'')^{-1/2}\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')$$
$$\times\left\{\frac12\sum_{n_2}\psi_{n_2}^{(\lambda_1,\nu)}(a'')\psi_{n_2}^{(\lambda_1,\nu)}(a')G_{PT}^{(\lambda_1,\beta)}(\vartheta'',\vartheta';E')\Big|_{E'=\hbar^2(2n_1+\lambda_1+\beta+1)^2/2MR^2}\right.$$
$$+\frac12\int_0^\infty dk\,\psi_k^{(\lambda_1,\nu)}(a'')\psi_k^{(\lambda_1,\nu)*}(a')G_{PT}^{(\lambda_1,\beta)}(\vartheta'',\vartheta';E')\Big|_{E'=-\hbar^2k^2/2MR^2}$$
$$\left.+\Big[\text{appropriate term with } a \text{ and } \vartheta \text{ interchanged}\Big]\right\} , \qquad (3.138)$$

in the notation of (A.25, A.28, A.29) and (A.34), respectively. Equation (3.138) also represents the Green function corresponding to the path integral (3.130). Analyzing the cuts gives the continuous states which have the form ($\tilde p^2=-\frac14-2MR^2(2\alpha/R-E)/\hbar^2$)

$$\Psi_{pkm}^{(V_2)}(a,\vartheta,\varphi;R)=(R^3\tanh a\tan\vartheta)^{-1/2}\psi_k^{(\lambda_1,\tilde p)}(a)\phi_k^{(\lambda_1,p)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}(\varphi)\ , \tag{3.139}$$

where

$$\begin{aligned}\psi_k^{(\lambda_1,\tilde p)}(a)&=\frac{\Gamma[\frac12(1+\lambda_1+\mathrm{i}\tilde p+\mathrm{i}k)]\Gamma[\frac12(1+\lambda_1+\mathrm{i}\tilde p-\mathrm{i}k)]}{\Gamma(1+\lambda_1)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}(\tanh a)^{\lambda_1-1/2}\\&\quad\times(\cosh a)^{\mathrm{i}k}{}_2F_1\bigg(\frac{1+\lambda_1+\mathrm{i}\tilde p+\mathrm{i}k}{2},\frac{1+\lambda_1-\mathrm{i}\tilde p+\mathrm{i}k}{2};1+\lambda_1;\tanh^2 a\bigg)\ ,\end{aligned}\tag{3.140}$$

$$\begin{aligned}\phi_k^{(\lambda_1,p)}(\vartheta)&=\frac{\Gamma[\frac12(1+\lambda_1+\mathrm{i}p+\mathrm{i}k)]\Gamma[\frac12(1+\lambda_1+\mathrm{i}p-\mathrm{i}k)]}{\Gamma(1+\lambda_1)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}(\tan\vartheta)^{\lambda_1-1/2}\\&\quad\times(\cos\vartheta)^{\mathrm{i}p+1+\lambda_1}{}_2F_1\bigg(\frac{1+\lambda_1+\mathrm{i}p+\mathrm{i}k}{2},\frac{1+\lambda_1-\mathrm{i}p+\mathrm{i}k}{2};1+\lambda_1;-\sin^2\vartheta\bigg)\ .\end{aligned}\tag{3.141}$$

The energy-spectra are as in (3.118,3.28), respectively. Putting both results together we obtain the path integral solution of the Coulomb problem on $\Lambda^{(3)}$ in elliptic parabolic 2 coordinates in the following form

$$\begin{aligned}&K^{(V_2)}(a'',a',\vartheta'',\vartheta',\varphi'',\varphi';T)\\&=\sum_{m=0}^\infty\bigg\{\sum_{n_1,n_2}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{mn_1n_2}^{(V_2)}(a'',\vartheta'',\varphi'';R)\Psi_{mn_1n_2}^{(V_2)}(a',\vartheta',\varphi';R)\\&\qquad+\int_0^\infty dk\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pkm}^{(V_2)}(a'',\vartheta'',\varphi'';R)\Psi_{pkm}^{(V_2)\,*}(a',\vartheta',\varphi';R)\bigg\}\ .\end{aligned}\tag{3.142}$$

3.3 A Radial Scattering Potential.

We consider the potential in its five separating coordinate systems ($k_{1,2,3}>0$)

$$V_3(\mathrm{u})=\frac{\hbar^2}{2MR^2}\Bigg[-\frac{k_0^2-\frac14}{u_0^2}+\frac{1}{\sqrt{u_1^2+u_2^2}}\bigg(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\bigg)+\frac{k_3^2-\frac14}{u_3^2}\Bigg]\tag{3.143}$$

Spherical ($\tau>0,\vartheta\in(0,\pi/2),\varphi\in(0,\pi)$):

$$=\frac{\hbar^2}{2MR^2}\Bigg[-\frac{k_0^2-\frac14}{\cosh^2\tau}+\frac{1}{\sinh^2\tau}\bigg(\frac{1}{4\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\bigg)\tag{3.144}$$

Equidistant-Cylindrical ($\tau_{1,2}>0,\varphi\in(0,\pi)$):

$$=\frac{\hbar^2}{2MR^2}\bigg(\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{\cosh^2\tau_1}\bigg(-\frac{k_0^2-\frac14}{\cosh^2\tau_2}+\frac{1}{4\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)\bigg)\bigg)\tag{3.145}$$

Prolate Elliptic ($\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi)$):

$$=-\frac{\hbar^2}{2MR^2}\bigg(-\frac{k_0^2-\frac14}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}+\frac{1}{4\,\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg)\tag{3.146}$$

Oblate Elliptic ($\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),\varphi\in(0,\pi)$):

$$=-\frac{\hbar^2}{2MR^2}\bigg(-\frac{k_0^2-\frac14}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}+\frac{1}{4\,\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\bigg)\tag{3.147}$$

Hyperbolic-Cylindrical 2 ($\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K'),\varphi\in(0,\pi)$):

$$=-\frac{\hbar^2}{2MR^2}\bigg(-\frac{k_0^2-\frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}+\frac{1}{4\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\bigg)\ .\tag{3.148}$$

For the five observables we find

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_3(\mathrm{u})\enspace,\\
I_2 &= \frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)\enspace,\\
I_3 &= \frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\bigg(\frac{1}{4\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\bigg)\enspace,\\
I_4 &= \frac{1}{2M}(K_1^2+K_2^2-L_3^2)+\frac{\hbar^2}{8MR^2\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)\enspace,\\
I_5 &= \frac{1}{2M}\Big[K_3^2+L_3^2-k^2(L_1^2+L_2^2)\Big]\\
&\qquad -\frac{\hbar^2}{2MR^2}\bigg(\frac{1}{4\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)+\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\bigg)\enspace.
\end{aligned}\right\}\tag{3.149}$$

Only in the first two coordinate systems an explicit solution is possible.

Spherical Coordinates. In the first separating coordinate system we have the following path integral representation together with its solution ($\lambda_1=m+\frac12(1\mp k_1\mp k_2),\lambda_2=2l\mp k_3+\lambda_1+1$)

$$\begin{aligned}
&K^{(V_3)}(\mathrm{u}'',\mathrm{u}';T)\\
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)-\frac{\hbar^2}{2MR^2}\bigg(-\frac{k_0^2-\frac14}{\cosh^2\tau}\right.\\
&\qquad\qquad\left.+\frac{1}{\sinh^2\tau}\bigg(\frac{1}{4\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}-1\bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}-\frac14\bigg)\bigg)\bigg]dt\right\}
\end{aligned}\tag{3.150}$$

$$=\sum_{m=0}^\infty\sum_{l=0}^\infty\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_p/\hbar}\Psi_{plm}^{(V_3)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_3)\,*}(\tau',\vartheta',\varphi';R)\enspace.\tag{3.151}$$

The path integral evaluation is successively performed by applying the path integral solution of the Pöschl-Teller potential in φ and ϑ, and for the $1/\sinh^2 r$-potential in τ. The spectrum is purely continuous and the wave-functions have the form

$$\Psi_{plm}^{(V_3)}(\tau,\vartheta,\varphi;R)=S_p^{(\lambda_2,k_0)}(\tau;R)\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)\phi_m^{(\pm k_2,\pm k_1)}\Big(\frac{\varphi}{2}\Big)\enspace,\tag{3.152}$$

and E_p as in (3.28). Here the wave-functions $S_p^{(\lambda_2,k_0)}$ are the usual continuous modified Pöschl-Teller wave-functions analogous to (3.77) with the parameters (λ_2,k_0), the $\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)$ are the same as in (3.91) and the wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ are the same as in (3.73) with $\varphi\to\varphi/2$ and an additional factor $1/\sqrt2$. The Green function is given by [92] ($E'=E+\hbar^2/2MR^2$, for simplicity we have set $k_0^2=1/4$)

$$\begin{aligned}
G^{(V_3)}(\mathrm{u}'',\mathrm{u}';E)&=\frac{2M}{\hbar^2R}(\sinh\tau'\sinh\tau'')^{-1/2}\\
&\quad\times\sum_{m=0}^\infty\sum_{l=0}^\infty\phi_l^{(\lambda_1,\pm k_3)}(\vartheta'')\phi_l^{(\lambda_1,\pm k_3)}(\vartheta')\phi_m^{(\pm k_2,\pm k_1)}\Big(\frac{\varphi''}{2}\Big)\phi_m^{(\pm k_2,\pm k_1)}\Big(\frac{\varphi'}{2}\Big)\\
&\quad\times\mathrm{e}^{-\mathrm{i}\pi\lambda_2}\mathcal{P}^{-\lambda_2}_{-1/2-\sqrt{-2MR^2E'}/\hbar}(\cosh\tau_>)\mathcal{Q}^{\lambda_2}_{-1/2+\sqrt{-2MR^2E'}/\hbar}(\cosh\tau_<)\enspace.
\end{aligned}\tag{3.153}$$

The functions $\mathcal{P}_\nu^\mu(z)$, $\mathcal{Q}_\nu^\mu(z)$ denote Legendre functions with argument $|z|>1$ [57, p.999].

Equidistant-Cylindrical Coordinates. The solution in the second coordinate system has the form ($\lambda_1 = 2m \mp k_1 \mp k_2 + 1$)

$$K^{(V_3)}(\mathrm{u}'', \mathrm{u}'; T)$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3} \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \cosh^2\tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \sinh\tau_2 \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\Big(\dot\tau_1^2 + \cosh^2\tau_1(\dot\tau_2^2 + \sinh^2\tau_2\dot\varphi^2)\Big) - \frac{\hbar^2}{2MR^2}\Bigg(\frac{k_3^2-\frac14}{\sinh^2\tau_1}$$

$$+\frac{1}{\cosh^2\tau_1}\Bigg(-\frac{k_0^2-\frac14}{\cosh^2\tau_2} + \frac{1}{4\sinh^2\tau_2}\Bigg(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}} - 1\Bigg) + 1\Bigg)\Bigg)\Bigg]dt\Bigg\} \tag{3.154}$$

$$= \sum_{m=0}^{\infty}\int_0^\infty dk \int_0^\infty dp\, \mathrm{e}^{-\mathrm{i}E_p/\hbar}\Psi_{pkm}^{(V_3)}(\tau_1'', \tau_2'', \varphi''; R)\Psi_{pkm}^{(V_3)\,*}(\tau_1', \tau_2', \varphi'; R)\ . \tag{3.155}$$

The path integral evaluation is successively performed by means of the path integral solution for the Pöschl-Teller potential in φ, for the $1/\sinh^2 r$-potential in τ_2 and for the modified Pöschl-Teller potential in τ_1. The continuous wave-functions have the form (λ_1 as before)

$$\Psi_{pkm}^{(V_3)}(\tau_1, \tau_2, \varphi; R) = S_p^{(\pm k_3, \mathrm{i}k)}(\tau_1; R)\psi_k^{(\lambda_1, \pm k_0)}(\tau_2)\phi_m^{(\pm k_2, \pm k_1)}\left(\frac{\varphi}{2}\right)\ . \tag{3.156}$$

with $\phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ as before, and the $S_p^{(\pm k_3,\mathrm{i}k)}(\tau_1; R)$ the same as in (3.97), and the $\psi_p^{(\lambda_1,k_0)}$ the same as in (3.152) with $\lambda_2 \to \lambda_1$, $\tau \to \tau_2$ and $R = 1$. The Green function $G^{(V_3)}(E)$ in these coordinates is constructed in a similar way as the previous case, where for the explicit expression in the radial variable τ_1 we must take $G_{mPT}^{(\pm k_3,\mathrm{i}k)}(E + \hbar^2/2MR^2)$.

3.4 A Stark-Effect Potential.

We consider the potential ($k_{1,2} > 0$)

$$V_4(\mathrm{u}) = \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right) + k_3 u_3\ . \tag{3.157}$$

In its four separating coordinate systems it has the form

Equidistant-Elliptic II ($\alpha \in (\mathrm{i}K', \mathrm{i}K'+K), \beta \in (0, K'), \tau > 0$):

$$V_4(\mathrm{u}) = \frac{1}{\cosh^2\tau}\frac{\hbar^2}{4MR^2}\Bigg[\frac{k_1^2+k_2^2-\frac12}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha} - \frac{1}{\mathrm{sn}^2\beta}\right)$$

$$+(k_1^2-k_2^2)\frac{k'}{k}\frac{k^2\,\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha + k'\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\Bigg] + k_3 R\sinh\tau \tag{3.158}$$

Equidistant-Semi-Hyperbolic ($\tau \in \mathbb{R}, \mu_{1,2} > 0$):

$$= \frac{\hbar^2}{4MR^2\cosh^2\tau}\frac{1}{\mu_1+\mu_2}\Bigg[(k_1^2+k_2^2-\tfrac12)\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)$$

$$+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1^2}}{\mu_1} - \frac{\sqrt{1+\mu_2^2}}{\mu_2}\right)\Bigg] + k_3 R\sinh\tau \tag{3.159}$$

Equidistant-Elliptic-Parabolic ($\tau \in \mathbb{R}, a > 0, \vartheta \in (0, \pi/2)$):

$$= \frac{\hbar^2}{2MR^2\cosh^2\tau}\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\left(\frac{k_1^2-\frac14}{\sin^2\vartheta} + \frac{k_2^2-\frac14}{\sinh^2 a}\right) + k_3 R\sinh\tau \tag{3.160}$$

Equidistant-Cylindrical ($\tau_1 \in \mathbb{R}, \tau_2 > 0, \varphi \in (0, \pi)$):

$$= \frac{\hbar^2}{8MR^2\cosh^2\tau_1\sinh^2\tau_2}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)} + \frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right) + k_3 R\sinh\tau_1\ . \tag{3.161}$$

The four constants of motion have the form

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_4(\mathrm{u})\ ,\\
I_2 &= \frac{1}{2M}(K_1^2+K_2^2-L_3^2)+\frac{\hbar^2}{8M\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\bigg)\ ,\\
I_3 &= \frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\bigg(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\bigg)\ ,\\
I_4 &= \frac{1}{2M}[(K_2-L_3)^2+K_1^2]+\frac{\hbar^2}{2M}\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\sin^2\vartheta}+\frac{k_2^2-\frac14}{\sinh^2 a}\bigg)\ .
\end{aligned}\right\}\tag{3.162}$$

Because there are only four observables, the potential V_4 is only minimally and not maximally super-integrable on the hyperboloid. However, its flat space analogue $V_5(\mathbf{x})$ is maximally super-integrable, and we have included V_4 in this section, though.

Unfortunately, the path integrals in all coordinate systems are not solvable. For instance, in the equidistant-cylindrical case it leads in the variable τ_1 to a spheroidal potential problem [153], i.e. ($\lambda_1=2m\mp k_1\mp k_2+1$),

$$\begin{aligned}
&K^{(V_4)}(\mathrm{u}'',\mathrm{u}';T)\\
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^2\tau_1\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\sinh\tau_2\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg[\frac{M}{2}R^2\Big(\dot\tau_1^2+\cosh^2\tau_1(\dot\tau_2^2+\sinh^2\tau_2\dot\varphi^2)\Big)-k_3R\sinh\tau_1\\
&\qquad-\frac{\hbar^2}{8MR^2}\bigg(\frac{1}{\cosh^2\tau_1}\bigg(\frac{1}{\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}-1\bigg)+\frac14\bigg)-\frac{1}{\sinh^2\tau_1}\bigg)\bigg]dt\Bigg\}\\
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R}(\cosh^2\tau_1'\cosh^2\tau_1''\sinh\tau_2'\sinh\tau_2'')^{-1/2}\sum_{m\in\mathbb{N}_0}\phi_m^{(\pm k_2,\pm k_1)}\Big(\frac{\varphi'}{2}\Big)\phi_m^{(\pm k_2,\pm k_1)}\Big(\frac{\varphi''}{2}\Big)\\
&\quad\times\int_0^\infty dk\frac{k\sinh\pi k}{\pi R}\Gamma(\tfrac12+\mathrm{i}k+\lambda_1)\mathcal{P}_{\mathrm{i}k-1/2}^{-\lambda_1\,*}(\cosh\tau_2')\mathcal{P}_{\mathrm{i}k-1/2}^{-\lambda_1}(\cosh\tau_2'')\\
&\quad\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg[\frac{M}{2}R^2\dot\tau_1^2-\frac{\hbar^2}{2M}\bigg(\frac{k^2+\frac14}{\cosh^2\tau_1}-\frac{1}{4\sinh^2\tau_1}\bigg)-k_3R\sinh\tau_1\bigg]dt\Bigg\}\ .
\end{aligned}\tag{3.163}$$

The other cases are similar, and we omit the statements of the other corresponding path integral representations.

4 Path Integral Formulation of the Minimally Superintegrable Potentials on $\Lambda^{(3)}$

In this Section we list our findings of the minimally super-integrable potentials on $\Lambda^{(3)}$. They include the following:

1. The class of potentials which are the analogues of the minimally super-integrable potentials in $\mathbb{R}^3$ [80]. For instance, the potentials V_5 and V_6 correspond to the double-ring shaped oscillator and the Hartmann potential in $\mathbb{R}^3$, respectively. The four found potentials are discussed in some detail.

2. The class of potentials which correspond to the group reduction $SO(3,1) \supset E(2)$, i.e., which are super-integrable in $\mathbb{R}^2$ [80]. Here the results of [80] will be used, and the problem of self-adjoint extensions for Hamiltonians unbounded from below is briefly mentioned.

3. The class of potentials which correspond to the group reduction $SO(3,1) \supset SO(3)$, i.e., which are super-integrable on $S^{(2)}$ [81]. In our list we have chosen for convenience a dependence according to $1/u_0^2$, but any function $F = F(u_0)$ admits separation of variables. Here the results of [81] and of Appendix B are used.

4. The class of potentials which correspond to the group reduction $SO(3,1) \supset SO(2,1)$, i.e., which are super-integrable on $\Lambda^{(2)}$ [82]. In our list we have chosen for convenience a dependence according to $1/u_3^2$, but any function $F = F(u_3)$ admits separation of variables. Because the features are repeating themselves, all those potentials can be treated simultaneously.

4.1 Analogues of the Minimally Superintegrable Potentials of $\mathbb{R}^3$.

4.1.1 Double Ring-Shaped Oscillator.

We consider the minimally super-integrable double ring-shaped potential V_5 on $\Lambda^{(3)}$ $(k_3 > 0)$

$$V_5(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{u_3^2} + \frac{F(u_2/u_1)}{u_1^2+u_2^2}\right) \ , \tag{4.1}$$

which in the five separating coordinate systems has the form (φ with appropriate range)

Spherical $(\tau > 0, \vartheta \in (0,\pi/2))$:

$$V_5(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \tanh^2\tau + \frac{\hbar^2}{2MR^2\sinh^2\tau}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta} + \frac{F(\tan\varphi)}{\sin^2\vartheta}\right) \tag{4.2}$$

Equidistant-Cylindrical $(\tau_{1,2} > 0)$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right) + \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1} + \frac{F(\tan\varphi)}{\cosh^2\tau_1\sinh^2\tau_2}\right) \tag{4.3}$$

Prolate Elliptic $(\alpha \in (\mathrm{i}K', \mathrm{i}K'+K), \beta \in (0,K'))$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right) - \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta} + \frac{F(\tan\varphi)}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right) \tag{4.4}$$

Oblate Elliptic $(\alpha \in (\mathrm{i}K', \mathrm{i}K'+K), \beta \in (0,K'))$:

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right) - \frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta} + \frac{F(\tan\varphi)}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\right) \tag{4.5}$$

Table 4.1: Minimally Superintegrable Potentials on $\Lambda^{(3)}$: Analoguous of Three-Dimensional Flat Space.

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_5(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2+u_3^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac14}{u_3^2} + \frac{F(u_2/u_1)}{u_1^2+u_2^2}\right)$	Spherical Equidistant-Cylindrical Prolate Elliptic Oblate Elliptic Hyperbolic Cylindrical 2	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2) + V_5(\mathrm{u}), \qquad I_2 = \frac{1}{2M}L_3^2 + F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac14}{\cos^2\vartheta} + \frac{F(\tan\varphi)}{\sin^2\vartheta}\right)$ $I_4 = \frac{1}{2M}(K_1^2+K_2^2-L_3)^2 - \frac{M}{2}\frac{\omega^2}{\cosh^2\tau_2} + \frac{\hbar^2}{2M}\frac{F(\tan\varphi)}{\tau_1\sinh^2\tau_2}$
$V_6(\mathrm{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2+u_3^2}} - 1\right)$ $+\frac{\hbar^2}{2M(u_1^2+u_2^2)}\left(\frac{\beta u_3}{\sqrt{u_1^2+u_2^2+u_3^2}} + F\left(\frac{u_2}{u_1}\right)\right)$ $\sinh^2 f = k'^2/k^2$	Spherical Prolate Elliptic II Semi-Hyperbolic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2) + V_6(\mathrm{u}), \qquad I_2 = \frac{1}{2M}L_3^2 + F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\frac{F(\tan\varphi)+\beta\cos\vartheta}{\sin^2\vartheta}$ $I_4 = \frac{1}{2M}[(\cosh 2f\mathbf{L}^2 - \frac12\sinh 2f(\{K_2,L_1\}-\{K_1,L_2\})] - \alpha R\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha - k'\,\mathrm{cn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}}$ $+\frac{\hbar^2}{4M}\left(\frac{F(\tan(\varphi)+1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha} - \frac{1}{\mathrm{sn}^2\beta}\right) - \beta\frac{k'}{k}\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha + k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\right)$
$V_7(\mathrm{u}) = F(u_1^2+u_2^2+u_3^2)$ $+\frac{\hbar^2}{2M}\left(-\frac{k_0^2-\frac14}{u_0^2} + \frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2} + \frac{k_3^2-\frac14}{u_3^2}\right)$	Sphero-Elliptic Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2) + V_7(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right)$ $I_4 = \frac{1}{4M}(L_1^2 + k'^2L_2^2) - \frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)}\left((k_1^2-\frac14)\left(\frac{1}{\mathrm{sn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right)\right.$ $\left.+(k_2^2-\frac14)\left(\frac{k'^2}{\mathrm{cn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right) - (k_3^2-\frac14)\left(\frac{k'^2}{\mathrm{dn}^2\tilde\alpha} - \frac{1}{\mathrm{sn}^2\tilde\beta}\right)\right)$
$V_8(\mathrm{u}) = \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}$ $+\frac{M}{2}\left(\frac{\alpha}{(u_0-u_3)^2} + \omega^2\frac{R^2+4u_1^2+u_2^2}{(u_0-u_3)^4} - \frac{\lambda u_1}{(u_0-u_3)^3}\right)$ $P_{x_i} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i},\ i=1,2$	Horicyclic Semi-Circular-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2) + V_8(\mathrm{u}), \qquad I_2 = \frac{1}{2M}P_{x_1}^2 + 2M\omega^2x_1^2 - \lambda x_1$ $I_3 = \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}$ $I_4 = \frac{1}{2M}(\{L_3, K_1+L_2\} + \{K_3, K_2-L_1\})$ $+\frac{\xi^2\eta^2}{2}\frac{2\alpha(\xi^2+\eta^2) + \lambda(\xi^4-\eta^4) + M\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2}$

Hyperbolic-Cylindrical 2 $(\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K'))$:

$$= \frac{M}{2}\omega^2R^2\bigg(1-\frac{1}{k^2\,\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\bigg)-\frac{\hbar^2}{2MR^2}\bigg(\frac{k_3^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}+\frac{F(\tan\varphi)}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\bigg)\ . \tag{4.6}$$

The observables have the form

$$\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_5(\mathrm{u})\ ,\\
I_2&=\frac{1}{2M}L_3^2+F(\tan\varphi)\ ,\\
I_3&=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\bigg(\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{F(\tan\varphi)}{\sin^2\vartheta}\bigg)\ ,\\
I_4&=\frac{1}{2M}(K_1^2+K_2^2-L_3^2)-\frac{M}{2}\frac{\omega^2}{\cosh^2\tau_2}+\frac{\hbar^2}{2M}\frac{F(\tan\varphi)}{\tau_1\sinh^2\tau_2}\ .
\end{aligned}\right\} \tag{4.7}$$

An explicit solution is available in two coordinate systems, and we have the following path integral representations together with their solutions:

Spherical Coordinates. In spherical coordinates the solution is not very different from the solution of the generalized oscillator on $\Lambda^{(3)}$, the only difference being the φ-dependence. Hence we obtain $(\lambda_2=2l\pm k_3+\lambda_F+1,\nu^2=M^2\omega^2R^4/\hbar^2+1/4,n=0,\dots,N_n<(\nu-\lambda_2-1)/2)$

$$\begin{aligned}
&K^{(V_5)}(\mathrm{u}'',\mathrm{u}';T)\\
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\omega^2\tanh^2\tau\Big)\\
&\qquad\qquad-\frac{\hbar^2}{2MR^2\sinh^2\tau}\bigg(\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{F(\tan\varphi)-\frac14}{\sin^2\vartheta}-\frac14\bigg)\bigg]dt\Bigg\}
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&=\int dE_\lambda\sum_{l=0}^{\infty}\Bigg\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{\lambda ln}^{(V_5)}(\tau'',\vartheta'',\varphi'';R)\Psi_{\lambda ln}^{(V_5)}(\tau',\vartheta',\varphi';R)\\
&\qquad+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{pl\lambda}^{(V_5)}(\tau'',\vartheta'',\varphi'';R)\Psi_{pl\lambda}^{(V_5)*}(\tau',\vartheta',\varphi';R)\Bigg\}\ .
\end{aligned} \tag{4.9}$$

The bound state wave-functions and the energy-spectrum are given by $(N=l+n)$

$$\Psi_{\lambda ln}^{(V_5)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_n^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)\phi_\lambda^{(F)}(\varphi)\ , \tag{4.10}$$

$$E_N=-\frac{\hbar^2}{2MR^2}\Big[\Big(2(N+1)\pm k_3+\lambda_F-\nu\Big)^2-1\Big]+\frac{M}{2}\omega^2R^2\ . \tag{4.11}$$

The scattering states and continuous spectrum have the form

$$\Psi_{pl\lambda}^{(V_5)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_p^{(\lambda_2,\nu)}(\tau;R)\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)\phi_\lambda^{(F)}(\varphi)\ , \tag{4.12}$$

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)+\frac{M}{2}\omega^2R^2\ . \tag{4.13}$$

Here the wave-functions $\phi_\lambda^{(F)}(\varphi)$ are the eigenfunctions corresponding to the potential term $F(\tan\varphi)$ with eigenvalues $E_\lambda=\hbar^2\lambda^2/2M$, and the $\phi_l^{(\lambda_F,\pm k_3)}(\vartheta)$ are the same as in (3.91) with $\lambda_1\to\lambda_F$, and the wave-functions $S_n^{(\lambda_2,\nu)}(\tau;R)$ and $S_p^{(\lambda_2,\nu)}(\tau;R)$ are the same as in (3.85) and (3.87), respectively.

Equidistant-Cylindrical Coordinates. $(\lambda_1 = 2l + \lambda_F - \nu + 1, \nu^2 = M^2\omega^2 R^4/\hbar^2 + 1/4)$

$$K^{(V_5)}(\mathrm{u}'', \mathrm{u}'; T) = R^{-3} \exp\left[-\frac{\mathrm{i}}{\hbar} T\left(\frac{\hbar^2}{2MR^2} + \frac{M}{2} R^2 \omega^2\right)\right]$$

$$\times \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \cosh^2 \tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \sinh \tau_2 \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2 + \cosh^2\tau_1(\dot\tau_2^2 + \sinh^2\tau_1\dot\varphi^2) + \frac{\omega^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1}\left(\frac{F(\tan\varphi) - \frac14}{\sinh^2\tau_2} + \frac14\right) + \frac{k_3^2 - \frac14}{\sinh^2\tau_1}\right)\right]dt\right\} \quad (4.14)$$

$$= \int dE_\lambda \left\{\sum_{l=0}^{N_l}\left[\sum_{n=0}^{N_n} \mathrm{e}^{-\mathrm{i}\hbar E_N T/\hbar} \Psi_{\lambda ln}^{(V_5)}(\tau_1'', \tau_2'', \varphi''; R)\Psi_{\lambda ln}^{(V_5)}(\tau_1', \tau_2', \varphi'; R)\right.\right.$$

$$\left. + \int_0^\infty dp\, \mathrm{e}^{-\mathrm{i}E_p T/\hbar} \Psi_{pl\lambda}^{(V_5)}(\tau_1'', \tau_2'', \varphi''; R)\Psi_{pl\lambda}^{(V_5)\,*}(\tau_1', \tau_2', \varphi'; R)\right]$$

$$\left. + \int_0^\infty dk \int_0^\infty dp\, \mathrm{e}^{-\mathrm{i}E_p T/\hbar} \Psi_{pk\lambda}^{(V_5)}(\tau_1'', \tau_2'', \varphi''; R)\Psi_{pk\lambda}^{(V_5)\,*}(\tau_1', \tau_2', \varphi'; R)\right\} \ . \quad (4.15)$$

Again we have similar features as for the general oscillator, hence we have one set of bound state wave-functions, and two sets of scattering states. They are given by $(n = 0, \dots, N_n < (\lambda_2 - k_3 - 1)/2, l = 0, \dots, N_l < (\nu - \lambda_F - 1)/2)$

$$\Psi_{\lambda ln}^{(V_5)}(\tau_1, \tau_2, \varphi; R) = (\cosh^2\tau_1 \sinh\tau_2)^{-1/2} S_n^{(\pm k_3, \lambda_2)}(\tau_1; R)\psi_l^{(\lambda_F, \nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi) \ , \quad (4.16)$$

$$\Psi_{pk\lambda}^{(V_5)}(\tau_1, \tau_2, \varphi; R) = (\cosh^2\tau_1 \sinh\tau_2)^{-1/2} S_p^{(\pm k_3, \lambda_2)}(\tau_1; R)\psi_l^{(\lambda_F, \nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi) \ , \quad (4.17)$$

$$\Psi_{pl\lambda}^{(V_5)}(\tau_1, \tau_2, \varphi; R) = (\cosh^2\tau_1 \sinh\tau_2)^{-1/2} S_p^{(\pm k_3, \mathrm{i}k)}(\tau_1; R)\psi_k^{(\lambda_F, \nu)}(\tau_2)\phi_\lambda^{(F)}(\varphi) \ . \quad (4.18)$$

The wave-functions $\psi_{l,k}^{(\lambda_F, \nu)}(\tau_2)$ are the same as in (3.72) and (3.81) with $\pm k_1 \to \lambda_F$ and $m \to l$, respectively. The energy-spectra E_N and E_p are the same as in the previous paragraph.

The Green function corresponding to the potential V_5 in the spherical and equidistant cylindrical coordinate systems are easy to construct. After separating off the first two path integrations, the Green function in the radial variable is found by the appropriate insertion of the parameters (κ, λ) into $G_{mPT}^{(\kappa,\lambda)}(E)$ from the corresponding radial wave-functions $S^{(\kappa,\lambda)}$. This is very similar as the discussion for the Green functions in the general oscillator case, and this is not repeated once again.

4.1.2 Hartmann Potential.

The next potential represents the analogue of the Hartmann potential V_6 in $\mathbb{R}^3$ [80]. We consider

$$V_6(\mathrm{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2 + u_2^2 + u_3^2}} - 1\right) + \frac{\hbar^2}{2M(u_1^2 + u_2^2)}\left(\frac{\beta u_3}{\sqrt{u_1^2 + u_2^2 + u_3^2}} + F\left(\frac{u_2}{u_1}\right)\right) \ , \quad (4.19)$$

which in the two separating coordinate systems has the form (φ with appropriate range)

Spherical $(\tau > 0, \vartheta \in (0, \pi/2))$:

$$V_6(\mathrm{u}) = -\frac{\alpha}{R}(\coth\tau - 1) + \frac{\hbar^2}{2MR^2\sinh^2\tau}\frac{F(\tan\varphi) + \beta\cos\vartheta}{\sin^2\vartheta} \quad (4.20)$$

Prolate Elliptic II ($\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K')$) :

$$-\frac{\alpha}{R}\left(\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha-k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}-1\right)$$
$$+\frac{\hbar^2}{4MR^2}\left(\frac{F(\tan\varphi)+1}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha}-\frac{1}{\mathrm{sn}^2\beta}\right)-\beta\frac{k'}{k}\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha+k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\right)\Bigg]\ . \tag{4.21}$$

For the observables we find

$$\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_6(\mathrm{u})\ ,\\
I_2&=\frac{1}{2M}L_3^2+F(\tan\varphi)\ ,\\
I_3&=\frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\frac{F(\tan\varphi)+\beta\cos\vartheta}{\sin^2\vartheta}\ ,
\end{aligned}\right\} \tag{4.22}$$

and the fourth observable is given by

$$I_4=\frac{1}{2M}\Big[\cosh 2f\mathbf{L}^2-\tfrac{1}{2}\sinh 2f(\{K_2,L_1\}-\{K_1,L_2\})\Big]-\alpha R\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\beta-k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}$$
$$+\frac{\hbar^2}{4M}\left(\frac{F(\tan\varphi)+1}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\left(\frac{k'^2}{\mathrm{dn}^2\alpha}-\frac{1}{\mathrm{sn}^2\beta}\right)-\beta\frac{k'}{k}\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha+k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\right)\ . \tag{4.23}$$

We treat only the spherical case with $F(\tan\varphi)=\gamma$. Then we obtain together with $\gamma\geq|\beta|$, $\lambda_\pm^2=n^2+\gamma\pm\beta$, $\lambda_2=m+(\lambda_++\lambda_-+1)/2$, $\varphi\in[0,2\pi)$:

$$K^{(V_6)}(\tau'',\tau',\vartheta'',\vartheta',\varphi'',\varphi';T)=R^{-3}\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(\frac{\hbar^2}{2MR^2}+\frac{\alpha}{R}\right)\right]$$
$$\times\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)+\frac{\alpha}{R}\coth\tau\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{8MR^2\sinh^2\tau}\left(\frac{\gamma+\beta-\frac{1}{4}}{\sin^2(\vartheta/2)}+\frac{\gamma-\beta-\frac{1}{4}}{\cos^2(\vartheta/2)}-\frac{1}{4}\right)\right]dt\right\} \tag{4.24}$$

$$=\sum_{m=0}^\infty\sum_{l=0}^\infty\left\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\Psi_{nlm}^{(V_6)}(\tau'',\vartheta'',\varphi'';R)\Psi_{nlm}^{(V_6)\,*}(\tau',\vartheta',\varphi';R)\right.$$
$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_{plm}^{(V_6)}(\tau'',\vartheta'',\varphi'';R)\Psi_{plm}^{(V_6)\,*}(\tau',\vartheta',\varphi';R)\right\}\ . \tag{4.25}$$

The path-integration in ϑ is of the Pöschl-Teller type, whereas the path integration in τ is essentially the same as for the Coulomb potential. Therefore the wave-functions for the bound and continuous spectrum are given by ($n=0,\dots,N_n<\sqrt{R/a}-\lambda_2-1/2$, $a=\hbar^2/M\alpha$ is the Bohr radius)

$$\Psi_{nlm}^{(V_6)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_N(\tau;R)\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\ , \tag{4.26}$$

$$\Psi_{plm}^{(V_6)}(\tau,\vartheta,\varphi;R)=(\sinh^2\tau\sin\vartheta)^{-1/2}S_p(\tau;R)\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\ , \tag{4.27}$$

$$\phi_l^{(\lambda_+,\lambda_-)}(\vartheta)=\left[(2l+\lambda_++\lambda_-+1)\frac{l!\Gamma(l+\lambda_++\lambda_-+1)}{\Gamma(l+\lambda_++1)\Gamma(l+\lambda_-+1)}\right]^{1/2}$$
$$\times\left(\sin\frac{\vartheta}{2}\right)^{1/2+\lambda_+}\left(\cos\frac{\vartheta}{2}\right)^{1/2+\lambda_-}P_l^{(\lambda_+,\lambda_-)}(\cos\vartheta)\ , \tag{4.28}$$

with the Coulomb wave-functions $S_N(\tau;R)$, $S_p(\tau;R)$ as in (3.117, 3.119) and the energy-spectra (3.118, 3.28), respectively. In the case when $R\to\infty$ the flat space limit is recovered [80, 122, 123]. The Green function corresponding to the potential V_6 is constructed in complete analogy to the Coulomb Green function in section (3.2).

4.1.3 Generalized Radial Potential.

We consider the potential ($k_{0,1,2,3}>0$)

$$V_7(\mathrm{u}) = F(u_1^2+u_2^2+u_3^2)+\frac{\hbar^2}{2M}\Bigg(-\frac{k_0^2-\frac14}{u_0^2}+\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\Bigg)\ , \tag{4.29}$$

which in the two separating coordinate systems has the form

Spherical ($\tau>0, \vartheta\in(0,\pi/2), \varphi\in(0,\pi/2)$):

$$V_7(\mathrm{u}) = F(\sinh^2\tau)+\frac{\hbar^2}{2MR^2}\Bigg[\frac{1}{\sinh^2\tau}\bigg(\frac{1}{\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\bigg)-\frac{k_0^2-\frac14}{\cosh^2\tau}\Bigg] \tag{4.30}$$

Sphero-Elliptic $\big(\tau>0, \tilde\alpha\in(0,K), \tilde\beta\in(0,K')\big)$:

$$= F(\sinh^2\tau)+\frac{\hbar^2}{2MR^2}\Bigg(\frac{1}{\sinh^2\tau}\bigg(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\bigg)-\frac{k_0^2-\frac14}{\cosh^2\tau}\Bigg)\ . \tag{4.31}$$

This potential is the analogue of the minimally super-integrable potential $V_1(\mathbf{x})$ in $\mathrm{I\!R}^3$ [80]. The corresponding observables have the form

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_7(\mathrm{u})\ ,\\
I_2 &= \frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)\ ,\\
I_3 &= \frac{1}{2M}\mathbf{L}^2+\frac{\hbar^2}{2M}\Bigg(\frac{1}{\sin^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\Bigg)\ ,\\
I_4 &= \frac{1}{2M}(L_1^2+k'^2L_2^2)-\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta)}\Bigg((k_1^2-\tfrac14)\bigg(\frac{1}{\mathrm{sn}^2\tilde\alpha}-\frac{k^2}{\mathrm{dn}^2\tilde\beta}\bigg)\\
&\qquad +(k_2^2-\tfrac14)\bigg(\frac{k'^2}{\mathrm{cn}^2\tilde\alpha}-\frac{k^2}{\mathrm{dn}^2\tilde\beta}\bigg)-(k_3^2-\tfrac14)\bigg(\frac{k'^2}{\mathrm{dn}^2\tilde\alpha}-\frac{1}{\mathrm{sn}^2\tilde\beta}\bigg)\Bigg)\ .
\end{aligned}\right\} \tag{4.32}$$

We have the following two path integral representations

$K^{(V_7)}(\mathrm{u}'',\mathrm{u}';T)$

<u>Sphero-Elliptic</u>, $\lambda_2=2(m+l)\pm k_3+\lambda_1+2$ (l,h as in subsection 3.1.2):

$$\begin{aligned}
&= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\tilde\alpha(t')=\tilde\alpha'}^{\tilde\alpha(t'')=\tilde\alpha''}\mathcal{D}\tilde\alpha(t)\int\limits_{\tilde\beta(t')=\tilde\beta'}^{\tilde\beta(t'')=\tilde\beta''}\mathcal{D}\tilde\beta(t)(k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta)\\
&\quad\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\Big(\dot\tau^2+\sinh^2\tau(k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta)(\dot{\tilde\alpha}^2+\dot{\tilde\beta}^2)\Big)-F(\tau)\\
&\qquad-\frac{\hbar^2}{2MR^2}\Bigg(-\frac{k_0^2-\frac14}{\cosh^2\tau}+\frac{1}{\sinh^2\tau}\bigg(\frac{k_1^2-\frac14}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\bigg)\Bigg)\Bigg]dt\Bigg\}
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
&= (R^2\sinh^2\tau'\sinh^2\tau''\,\mathrm{sn}\tilde\alpha'\,\mathrm{cn}\tilde\alpha'\,\mathrm{dn}\tilde\alpha'\,\mathrm{sn}\tilde\beta'\,\mathrm{cn}\tilde\beta'\,\mathrm{dn}\tilde\beta'\,\mathrm{sn}\tilde\alpha''\,\mathrm{cn}\tilde\alpha''\,\mathrm{dn}\tilde\alpha''\,\mathrm{sn}\tilde\beta''\,\mathrm{cn}\tilde\beta''\,\mathrm{dn}\tilde\beta'')^{-1/2}\\
&\quad\times\sum_{lm}\Xi_{lm}^{(\pm k_1,\pm k_2,\pm k_3)}(\tilde\alpha'',\tilde\beta')\Xi_{lm}^{(\pm k_1,\pm k_2,\pm k_3)\,*}(\tilde\alpha',\tilde\beta')K_{lm}^{(V_7)}(\tau'',\tau';T)\ ,
\end{aligned} \tag{4.34}$$

Spherical, $\lambda_1 = 2m \mp k_1 \mp k_2 + 1, \lambda_2 = 2l \mp k_3 + \lambda_1 + 1$:

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t) \sinh^2\tau \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\tau^2 + \sinh^2\tau(\dot\vartheta^2 + \sin^2\vartheta\dot\varphi^2)\Big) - F(\tau)\right.\right.$$

$$\left.\left. -\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi} - \frac14\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta} - \frac14\right) - \frac{k_0^2-\frac14}{\cosh^2\tau}\right)\right]dt\right\} \tag{4.35}$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R}(\sinh^2\tau'\sinh^2\tau''\sin\vartheta'\sin\vartheta'')^{-1/2}$$

$$\times \sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}(\varphi'')\phi_m^{(\pm k_2,\pm k_1)}(\varphi')\sum_{l=0}^{\infty}\phi_l^{(\lambda_1,\pm k_3)}(\vartheta'')\phi_l^{(\lambda_1,\pm k_3)}(\vartheta')K_{lm}^{(V_7)}(\tau'',\tau';T)\ , \tag{4.36}$$

with the remaining path integral $K_{lm}^{(V_7)}(T)$

$$K_{lm}^{(V_7)}(\tau'',\tau';T) = \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot\tau^2 - F(\tau) - \frac{\hbar^2}{2MR^2}\left(\frac{\lambda_2^2-\frac14}{\sinh^2\tau} - \frac{k_0^2-\frac14}{\cosh^2\tau}\right)\right]dt\right\}\ . \tag{4.37}$$

The wave-functions $\phi_m^{(\pm k_2,\pm k_1)}(\varphi)$ and $\phi_l^{(\lambda_1,\pm k_3)}(\vartheta)$ are the same as in (3.73) and (3.91), respectively. This path integral cannot be further specified until $F(\tau) \equiv F(\sinh^2\tau)$ is known. The special case $F \equiv 0$ is trivial. In this case the radial Green function is proportional to the Green function of the modified Pöschl-Teller potential $G_{mPT}^{(\lambda_2,\pm k_0)}(E + \hbar^2/2MR^2)$.

4.1.4 Analogue of the Holt-Potential.

The potential V_8 can be considered as an analogue of the minimally super-integrable Holt-potential $V_6(\mathbf{x})$ in $\mathbb{R}^3$ [80] $(\alpha,\lambda,\omega > 0)$

$$V_8(\mathrm{u}) = \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2} + \frac{\alpha}{(u_0-u_3)^2} + \frac{M}{2}\omega^2\frac{R^2+4u_1^2+u_2^2}{(u_0-u_3)^4} - \frac{\lambda u_1}{(u_0-u_3)^2}\ . \tag{4.38}$$

In the two separating coordinate systems it has the form

Horicyclic $(x_2, y > 0, x_1 \in \mathbb{R})$:

$$V_8(\mathrm{u}) = \frac{y^2}{R^2}\left[\alpha + \frac{M}{2}\omega^2(4x_1^2 + x_2^2 + y^2) - \lambda x_1\right] + y^2\frac{\hbar^2}{2MR^2}\frac{k_2^2-\frac14}{x_2^2} \tag{4.39}$$

Semi-Circular-Parabolic $(\xi,\eta,\varrho > 0)$:

$$= \frac{\xi^2\eta^2}{R^2}\left[\frac{\alpha(\xi^2+\eta^2) - \frac{\lambda}{2}(\eta^4-\xi^4) + \frac{M}{2}\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2} + \left(\frac{M}{2}\omega^2\varrho^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{\varrho^2}\right)\right]\ . \tag{4.40}$$

For the constants of motion we find $(P_{x_i} = -\mathrm{i}\hbar\partial_{x_i},\ i = 1,2)$

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_8(\mathrm{u})\ ,\\
I_2 &= \frac{1}{2M}P_{x_1}^2 + 2M\omega^2x_1^2 - \lambda x_1\ ,\\
I_3 &= \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}\ ,\\
I_4 &= \frac{1}{4M}\Big(\{L_3, K_1+L_2\} + \{K_3, K_2-L_1\}\Big)\\
&\quad + \frac{\xi^2\eta^2}{2}\frac{2\alpha(\xi^2+\eta^2) - \lambda(\eta^4-\xi^4) + M\omega^2(\xi^6+\eta^6)}{\xi^2+\eta^2}\ .
\end{aligned}\right\} \tag{4.41}$$

The effect of the x_2- and ϱ-path integration in both cases $(x_2,\varrho>0)$ is that in separating off the corresponding variable, the quantity α is shifted by the additional quantum numbers. The resulting path integrals in the variables (y,x_1) and (ξ,η) separate, however, only the former can be evaluated. Indeed, almost the same path integral problem we have already solved in [82]. The solution in horicyclic coordinates then has the following structure ($z=x_1-\lambda/4M\omega^2$)

$$K^{(V_8)}(\mathrm{u}'',\mathrm{u}';T)=\frac{1}{R^3}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^3}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot{\mathbf{x}}^2+\dot{y}^2}{y^2}-\frac{y^2}{R^2}\Big(\alpha+\frac{M}{2}\omega^2(4x_1^2+x_2^2+y^2)-\lambda x_1\Big)-\frac{y^2\hbar^2}{2MR^2}\frac{k_2^2-\frac14}{x_2^2}\right]dt\right\}\tag{4.42}$$

$$=\frac{2M\omega}{\hbar R}\sqrt{x_2'x_2''}\sum_{l\in\mathbb{N}_0}\frac{l!}{\Gamma(l\pm k_2+1)}\left(\frac{M\omega}{\hbar}x_2'x_2''\right)^{\pm k_2}$$

$$\times\exp\left(-\frac{M\omega}{2\hbar}({x_2'}^2+{x_2''}^2)\right)L_l^{(\pm k_2)}\left(\frac{M\omega}{\hbar}{x_2'}^2\right)L_n^{(\pm k_2)}\left(\frac{M\omega}{\hbar}{x_2''}^2\right)\ .$$

$$\times\sum_{m\in\mathbb{N}_0}\left(\frac{2M\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^m m!}H_m\left(\sqrt{\frac{2M\omega}{\hbar}}\,z'\right)H_m\left(\sqrt{\frac{2M\omega}{\hbar}}\,z''\right)\exp\left[-\frac{M\omega}{\hbar}({z'}^2+{z''}^2)\right]$$

$$\times\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\left[\frac{\mathrm{i}M}{2\hbar}\int_{t'}^{t''}\left(R^2\frac{\dot{x}^2+\dot{y}^2}{y^2}-\frac{y^2}{R^2}\Big(E_{\alpha,\omega,\lambda}+\omega^2y^2\Big)\right)dt\right]\ ,\tag{4.43}$$

with the quantity $E_{\alpha,\omega,\lambda}$ given by

$$E_{\alpha,\omega,\lambda}=\alpha+\hbar\omega(2m+2l\pm k_2+2)-\frac{\lambda^2}{8M\omega^2}\ .\tag{4.44}$$

A path integral like this was calculated in [66], and we must distinguish two case, first where $E_{\alpha,\omega,\lambda}>0$, and second $E_{\alpha,\omega,\lambda}<0$. In the first case only a continuous spectrum occurs, whereas in the second bound states can exist with the number of levels given by $n=0,1,\dots,N_n=[E_{\alpha,\omega,\lambda}/2\hbar\omega-1/2]$. According to [82] we obtain therefore the following path integral solution for $V_8(\mathrm{u})$ in horicyclic coordinates ($\nu=-\mathrm{i}\sqrt{2MR^2E/\hbar^2-1/4}\,$)

$$K^{(V_8)}(\mathrm{u}'',\mathrm{u}';T)=\frac{1}{R}\sum_{n=0}^{\infty}\psi_n(x_2')\psi_n(x_2'')\sum_{m=0}^{\infty}\psi_m(x_1')\psi_m(x_1'')\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}$$

$$\times\frac{\Gamma[\frac12(1+\nu+E_{\alpha,\omega,\lambda}/\hbar\omega)]}{\sqrt{y'y''}\,\hbar\omega\Gamma(1+\nu)}W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\nu/2}\left(\frac{M\omega}{\hbar}y_>^2\right)M_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\nu/2}\left(\frac{M\omega}{\hbar}y_<^2\right)\tag{4.45}$$

$$=\sum_{l,m=0}^{\infty}\left[\sum_{n=0}^{N_n}\Psi_{nlm}^{(V_8)}(x_1'',x_2'',y'';R)\Psi_{nlm}^{(V_8)}(x_1',x_2',y';R)\,\mathrm{e}^{-\mathrm{i}E_nT/\hbar}\right.$$

$$\left.+\int_0^\infty dp\Psi_{plm}^{(V_8)}(x_1'',x_2'',y'';R)\Psi_{plm}^{(V_8)\,*}(x_1',x_2',y';R)\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\right]\ .\tag{4.46}$$

The bound state wave-functions have the form

$$\Psi_{nmq}^{(V_8)}(x_1,x_2,y;R)=\psi_n(y;R)\psi_m(x_1)\psi_l(x_2)\tag{4.47}$$

where

$$\psi_n(y;R) = \sqrt{\frac{2n!(|E_{\alpha,\omega,\lambda}|/\hbar\omega - 2n - 1)y}{R^3\Gamma(|E_{\alpha,\omega,\lambda}|/\hbar\omega - n)}}\bigg(\frac{M\omega}{\hbar}y^2\bigg)^{|E_{\alpha,\omega,\lambda}|/2\hbar\omega - n - 1/2}$$
$$\times\exp\bigg(-\frac{M\omega}{2\hbar}y^2\bigg)L_n^{(|E_{\alpha,\omega,\lambda}|/\hbar\omega - 2n-1)}\bigg(\frac{M\omega}{\hbar}y^2\bigg) \ , \tag{4.48}$$
$$\psi_m(x_1) = \bigg(\frac{2M\omega}{\pi\hbar 2^{2m}(m!)^2}\bigg)^{1/4} H_m\Bigg(\sqrt{\frac{2M\omega}{\hbar}}\bigg(x_1 - \frac{\lambda}{8\omega^2}\bigg)\Bigg)\exp\Bigg(-\frac{M\omega}{\hbar}\bigg(x_1 - \frac{\lambda}{8\omega^2}\bigg)^2\Bigg) \ , \tag{4.49}$$
$$\psi_l(x_2) = \sqrt{\frac{2M\omega}{\hbar}\frac{l!}{\Gamma(l\pm k_2+1)}}x_2\bigg(\frac{M\omega}{\hbar}x_2^2\bigg)^{\pm k_2/2}\exp\bigg(-\frac{M\omega}{2\hbar}x_2^2\bigg)L_l^{(\pm k_2)}\bigg(\frac{M\omega}{\hbar}x_2^2\bigg) \ , \tag{4.50}$$

with the discrete energy-spectrum given by

$$E_n = \frac{\hbar^2}{8MR^2} - \frac{\hbar^2}{2MR^2}\bigg(\frac{|E_{\alpha,\omega,\lambda}|}{\hbar\omega} - 2n - 1\bigg)^2 \ . \tag{4.51}$$

The continuous wave-functions and the energy-spectrum have the form

$$\Psi_{plm}^{(V_8)}(x_1,x_2,y;R) = \psi_p(y;R)\psi_m(x_1)\psi_l(x_2) \tag{4.52}$$
$$\psi_p(y;R) = \sqrt{\frac{\hbar}{M\omega}\frac{p\sinh\pi p}{2\pi^2R^3y}}\,\Gamma\bigg[\frac{1}{2}\bigg(1 + \mathrm{i}p + \frac{E_{\alpha,\omega,\lambda}}{\hbar\omega}\bigg)\bigg]W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\mathrm{i}p/2}\bigg(\frac{M\omega}{\hbar}y^2\bigg) \ , \tag{4.53}$$
$$E_p = \frac{\hbar^2}{2MR^2}\bigg(p^2 + \frac{1}{4}\bigg) \ , \tag{4.54}$$

and the $\psi_m(x_1),\psi_l(x_2)$ as in (4.49,4.50). The Green function $G^{(V_8)}(E)$ of the potential V_8 is given by (4.45).

4.2 Minimally Superintegrable Potentials from the Group Chain $\mathrm{SO}(3,1)\supset E(2)$.

4.2.1 Subsystem of Oscillator.

The potential V_9 contains the super-integrable harmonic oscillator in $\mathbb{R}^2$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider ($k_{1,2} > 0$)

$$V_9(\mathrm{u}) = F(u_0 - u_3) + \frac{M}{2}\omega^2\frac{u_1^2 + u_2^2}{(u_0 - u_3)^4} + \frac{\hbar^2}{2M}\bigg(\frac{k_1^2 - \frac{1}{4}}{u_1^2} + \frac{k_2^2 - \frac{1}{4}}{u_2^2}\bigg) \ , \tag{4.55}$$

which in the three separating coordinate systems has the form

Horicyclic $(x_{1,2}, y > 0)$:
$$V_9(\mathrm{u}) = F\bigg(\frac{R}{y}\bigg) + \frac{y^2}{R^2}\bigg[\frac{M}{2}\omega^2(x_1^2 + x_2^2) + \frac{\hbar^2}{2M}\bigg(\frac{k_1^2 - \frac{1}{4}}{x_1^2} + \frac{k_2^2 - \frac{1}{4}}{x_2^2}\bigg)\bigg] \tag{4.56}$$
Horicyclic-Cylindrical $(y,\varrho > 0, \varphi\in(0,\pi/2))$:
$$= F\bigg(\frac{R}{y}\bigg) + \frac{y^2}{R^2}\bigg[\frac{M}{2}\omega^2\varrho^2 + \frac{\hbar^2}{2M}\frac{y^2}{\varrho^2}\bigg(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\bigg)\bigg] \tag{4.57}$$
Horicyclic-Elliptic $(y,\mu > 0, \varphi\in(0,\pi/2))$:
$$= F\bigg(\frac{R}{y}\bigg) + \frac{y^2}{R^2}\bigg[\frac{M}{2}\omega^2(\cosh^2\mu\cos^2\nu + \sinh^2\mu\sin^2\nu)$$
$$+ \frac{\hbar^2}{2M}\bigg(\frac{k_1^2 - \frac{1}{4}}{\cosh^2\mu\cos^2\nu} + \frac{k_2^2 - \frac{1}{4}}{\sinh^2\mu\sin^2\nu}\bigg)\bigg] \ . \tag{4.58}$$

Table 4.2: Minimally Superintegrable Potentials on $\Lambda^{(3)}$ from the Group Chain $\mathrm{SO}(3,1) \supset E(2)$.

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_9(\mathrm{u}) = F(u_0 - u_3) + \frac{M}{2}\omega^2 \frac{u_1^2 + u_2^2}{(u_0 - u_3)^4}$ $+ \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{u_1^2} + \frac{k_2^2 - \frac{1}{4}}{u_2^2}\right)$ $P_{x_i} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i}$, $i = 1, 2$	Horicyclic Horicyclic-Cylindrical Horicyclic-Elliptic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_9(\mathrm{u})$ $I_2 = \frac{1}{2M}P_{x_1}^2 + \frac{M}{2}\omega^2 x_1^2 + \frac{\hbar^2}{2M}\frac{k_1^2 - \frac{1}{4}}{x_1^2}$ $I_3 = \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2 x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{x_2^2}$ $I_4 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right)$
$V_{10}(\mathrm{u}) = F(u_0 - u_3) + \frac{M}{2}\omega^2 \frac{4u_1^2 + u_2^2}{(u_0 - u_3)^4}$ $+ \frac{k_1 u_1}{(u_0 - u_3)^3} + \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{u_2^2}$ $P_{x_i} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_i}$, $i = 1, 2$	Horicyclic Horicyclic-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_{10}(\mathrm{u})$ $I_2 = \frac{1}{2M}P_{x_1}^2 + 2M\omega^2 x_1^2 + k_1 x_1$ $I_3 = \frac{1}{2M}P_{x_2}^2 + \frac{M}{2}\omega^2 x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{x_2^2}$ $I_4 = \frac{1}{4M}\{L_3, K_1 + L_2\} + \frac{M\omega^2(\xi^6 + \eta^6) + 2k_1(\xi^4 - \eta^4) + \hbar^2(k_2^2 - \frac{1}{4})(1/\xi^2 + 1/\eta^2)/M}{2(\xi^2 + \eta^2)}$
$V_{11}(\mathrm{u}) = F(u_0 - u_3) - \frac{\alpha}{\sqrt{u_1^2 + u_2^2}}\frac{1}{u_0 - u_3}$ $+ \frac{R^2\hbar^2}{4M}\frac{1}{(u_0 - u_3)^2}\frac{1}{\sqrt{u_1^2 + u_2^2}}$ $\times\left(\frac{k_1^2 - \frac{1}{4}}{\sqrt{u_1^2 + u_2^2} + u_1} + \frac{k_2^2 - \frac{1}{4}}{\sqrt{u_1^2 + u_2^2} - u_1}\right)$	Horicyclic-Cylindrical Horicyclic-Elliptic II Horicyclic-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_{11}(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2(\varphi/2)} + \frac{k_2^2 - \frac{1}{4}}{\sin^2(\varphi/2)}\right)$ $I_3 = \frac{1}{2M}\Big[(K_1 + L_2)^2 + (K_2 - L_1)^2\Big] - \frac{\alpha}{\varrho} + \frac{\hbar^2}{8M\varrho^2}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2(\varphi/2)} + \frac{k_2^2 - \frac{1}{4}}{\sin^2(\varphi/2)}\right)$ $I_4 = \frac{1}{4M}\{L_3, P_2\} + \frac{1}{\xi\eta}\Big[-\alpha(\xi - \eta) + (\frac{1}{2} - 2k_1^2)\frac{\eta}{\xi} + (2k_2^2 - \frac{1}{2})\frac{\xi}{\eta}\Big]$
$V_{12}(\mathrm{u}) = F(u_0 - u_3) - \frac{\alpha}{\sqrt{u_1^2 + u_2^2}}\frac{1}{u_0 - u_3}$ $+ \frac{\beta_1\sqrt{\sqrt{u_1^2 + u_2^2} + u_1} + \beta_2\sqrt{\sqrt{u_1^2 + u_2^2} - u_1}}{2(u_0 - u_3)^{3/2}\sqrt{u_1^2 + u_2^2}}$ $P_y = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y}$	Horicyclic-Mutually-Parabolic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_{12}(\mathrm{u})$ $I_2 = \frac{1}{2M}P_y^2 + F(R/y)$ $I_3 = \frac{1}{4M}\{L_3, P_1\} - \frac{\alpha(\lambda - \mu) + \beta_1\mu\sqrt{\lambda} - \beta_2\lambda\sqrt{\mu}}{\lambda + \mu}$ $I_4 = \frac{1}{4M}\{L_3, P_2\} - \frac{\alpha(\xi - \eta) + (\beta_1 + \beta_2)\eta\sqrt{\xi/2} - (\beta_1 + \beta_2)\xi\sqrt{\eta/2}}{\xi + \eta}$

If we choose $\omega=0$ and $F\propto y^2$, the potential V_9 is additionally separable and solvable in the coordinate systems XXV. and XXVI, and separable in XXIII. and XXIV.

The constants of motion are ($P_{x_i}=-\mathrm{i}\hbar\partial_{x_i}$, $i=1,2$)

$$\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_9(\mathrm{u})\ ,\\
I_2&=\frac{1}{2M}P_{x_1}^2+\frac{M}{2}\omega^2x_1^2+\frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{x_1^2}\ ,\\
I_3&=\frac{1}{2M}P_{x_2}^2+\frac{M}{2}\omega^2x_2^2+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}\ ,\\
I_4&=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)\ .
\end{aligned}\right\}\tag{4.59}$$

We discuss the path integral representations in all coordinate systems for the potential V_9 simultaneously. We obtain ($N=n_1+n_2,\mathbf{x}\in\mathrm{I\!R}^2$)

$$\begin{aligned}
K^{(V_9)}(\mathrm{u}'',\mathrm{u}';T)&=\frac{\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}}{R^3}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^3}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\\
&\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot y^2+\dot{\mathbf{x}}^2}{y^2}-F\Big(\frac{R}{y}\Big)-\frac{y^2}{R^2}\bigg(\frac{M}{2}\omega^2\mathbf{x}^2+\frac{\hbar^2}{2M}\Big(\frac{k_1^2-\frac14}{x_1^2}+\frac{k_2^2-\frac14}{x_2^2}\Big)\bigg)\right]dt\right\}
\end{aligned}\tag{4.60}$$

$$\begin{aligned}
&=\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}\frac{y'y''}{R}\sum_{n_1,n_2=0}^\infty\Psi_{n_1n_2}^{(GHO)}(\mathbf{x}'')\Psi_{n_1n_2}^{(GHO)}(\mathbf{x}')\\
&\times\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot y^2}{y^2}-F\Big(\frac{R}{y}\Big)-\frac{y^2}{R^2}\hbar\omega(2N\pm k_1\pm k_2+2)\right]dt\right\}\ .
\end{aligned}\tag{4.61}$$

The wave-functions $\Psi_{n_1n_2}^{(GHO)}(\mathbf{x})$ are the wave-functions of the generalized harmonic oscillator (GHO) in $\mathrm{I\!R}^2$ in one of the three possible coordinate system representations [80]. The remaining y-path integral cannot be further evaluated until the potential $F(R/y)$ is specified. The case of $F\propto y^4$ was discussed in section (4.1.4). The case of $F\propto y^3$ can be treated by the methods of [76].

Let us consider the following variation of V_9

$$V_9'(\mathrm{u})=\frac{M}{2}\frac{\omega^2}{(u_0-u_3)^2}+\frac{\hbar^2}{2M}\bigg(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\bigg)\ ,\tag{4.62}$$

which is separable in four more coordinate systems, and the corresponding cases have the form

Elliptic-Parabolic 1: $(a,\varrho>0,\vartheta\in(0,\pi/2))$:

$$V_9'(\mathrm{u})=\frac{M}{2}\frac{\omega^2}{R^2}\frac{\cosh^2a\cos^2\vartheta}{(1+\varrho^2)^2}+\frac{\hbar^2}{2MR^2}\bigg(\cosh^2a\cos^2\vartheta\frac{k_1^2-\frac14}{\varrho^2}+\frac{k_2^2-\frac14}{\tanh^2a\tan^2\vartheta}\bigg)\tag{4.63}$$

Hyperbolic-Parabolic 1: $(b,\varrho>0,\vartheta\in(0,\pi/2))$:

$$=\frac{M}{2}\frac{\omega^2}{R^2}\frac{\sinh^2b\sin^2\vartheta}{(1+\varrho^2)^2}+\frac{\hbar^2}{2MR^2}\bigg(\sinh^2b\sin^2\vartheta\frac{k_1^2-\frac14}{\varrho^2}+\frac{k_2^2-\frac14}{\coth^2b\cot^2\vartheta}\bigg)\ .\tag{4.64}$$

Elliptic-Parabolic 2: $(a>0,\vartheta\in(0,\pi/2),\varphi\in(0,\pi/2))$:

$$=\frac{M}{2}\frac{\omega^2}{R^2}\cosh^2a\cos^2\vartheta+\frac{\hbar^2}{2MR^2}\coth^2a\cot^2\vartheta\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)\tag{4.65}$$

Hyperbolic-Parabolic 2: $(b>0,\vartheta\in(0,\pi/2),\varphi\in(0,\pi/2))$:

$$=\frac{M}{2}\frac{\omega^2}{R^2}\sinh^2b\sin^2\vartheta+\frac{\hbar^2}{2MR^2}\tanh^2b\tan^2\vartheta\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)\ .\tag{4.66}$$

For the first two cases we have the following two path integral representations

$$K^{(V_9')}(\mathrm{u}'',\mathrm{u}';T)$$

Elliptic-Parabolic 1:

$$= \frac{\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}}{R^3}\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^3 a\cos^3\vartheta}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)$$

$$\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\frac{(\cosh^2 a-\cos^2\vartheta)(\dot a^2+\dot\vartheta^2)+\dot\varrho^2}{\cosh^2 a\cos^2\vartheta}$$
$$-\frac{\cosh^2 a\cos^2\vartheta}{R^2}\Bigg(\frac{M}{2}\frac{\omega^2}{(1+\varrho^2)^2}+\frac{\hbar^2}{2M}\Bigg(\frac{k_1^2-\frac14}{\varrho^2}+\frac{k_2^2-\frac14}{\sinh^2 a\sin^2\vartheta}\Bigg)\Bigg)\Bigg]dt\Bigg\}\ , \tag{4.67}$$

Hyperbolic-Parabolic 1:

$$= \frac{\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}}{R^3}\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\frac{\sinh^2 b+\sin^2\vartheta}{\sinh^3 b\sin^3\vartheta}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)$$

$$\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\frac{(\sinh^2 b+\sin^2\vartheta)(\dot b^2+\dot\vartheta^2)+\dot\varrho^2}{\sinh^2 b\sin^2\vartheta}$$
$$-\frac{\sinh^2 b\sin^2\vartheta}{R^2}\Bigg(\frac{M}{2}\frac{\omega^2}{(1+\varrho^2)^2}+\frac{\hbar^2}{2M}\Bigg(\frac{k_1^2-\frac14}{\varrho^2}+\frac{k_2^2-\frac14}{\cosh^2 b\cos^2\vartheta}\Bigg)\Bigg)\Bigg]dt\Bigg\}\ . \tag{4.68}$$

Neither of the two path integrals can be evaluated. In the other two path integral representations we are lead after the φ-separation to spheroidal path integral problems, c.f. [75, 82] for an heuristic approach for the solution of such problems.

4.2.2 Subsystem of Holt Potential.

The potential V_{10} contains the super-integrable Holt potential [101] in $\mathbb{R}^2$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider ($k_1\in\mathbb{R}, k_2>0$)

$$V_{10}(\mathrm{u}) = F(u_0-u_3)+\frac{M}{2}\omega^2\frac{4u_1^2+u_2^2}{(u_0-u_3)^4}+\frac{k_1u_1}{(u_0-u_3)^3}+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2} \tag{4.69}$$

which in the two separating coordinate systems has the form

Horicyclic ($x_2, y>0, x_1\in\mathbb{R}$) :

$$V_{10}(\mathrm{u}) = F\bigg(\frac{R}{y}\bigg)+\frac{y^2}{R^2}\bigg[\frac{M}{2}\omega^2(4x_1^2+x_2^2)+k_1x_1+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}\bigg] \tag{4.70}$$

Horicyclic-Parabolic ($\xi,\eta,y>0$) :

$$= F\bigg(\frac{R}{y}\bigg)+\frac{y^2}{R^2}\bigg[\frac{M}{2}\omega^2\Big((\xi^2-\eta^2)^2+\xi^2\eta^2\Big)+\frac{k_1}{2}(\xi^2-\eta^2)+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{\xi^2\eta^2}\bigg]\ . \tag{4.71}$$

For the constants of motion we find ($P_{x_i}=-\mathrm{i}\hbar\partial_{x_i}$, $i=1,2$)

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{10}(\mathrm{u})\ ,\\
I_2 &= \frac{1}{2M}P_{x_1}^2+2M\omega^2x_1^2+k_1x_1\ ,\\
I_3 &= \frac{1}{2M}P_{x_2}^2+\frac{M}{2}\omega^2x_2^2+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}\ ,\\
I_4 &= \frac{1}{4M}\{L_3,K_1+L_2\}+\frac{M\omega^2(\xi^6+\eta^6)+2k_1(\xi^4-\eta^4)+\hbar^2(k_2^2-\frac14)(1/\xi^2+1/\eta^2)/M}{2(\xi^2+\eta^2)}\ .
\end{aligned}\right\} \tag{4.72}$$

Similarly as in the previous case we treat both coordinate space representations simultaneously, and we have ($\mathbf{x}\in\mathbb{R}^2$)

$$K^{(V_{10})}(\mathrm{u}'',\mathrm{u}';T)=\frac{\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}}{R^3}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^3}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot y^2+\dot{\mathbf{x}}^2}{y^2}-F\left(\frac{R}{y}\right)-\frac{y^2}{R^2}\left(\frac{M}{2}\omega^2(4x_1^2+x_2^2)+k_1x_1+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{x_2^2}\right)\right]dt\right\} \tag{4.73}$$

$$=\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}\frac{y'y''}{R}\sum_{n_1,n_2=0}^{\infty}\Psi_{n_1n_2}^{(Holt)}(\mathbf{x}'')\Psi_{n_1n_2}^{(Holt)}(\mathbf{x}')$$

$$\times\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot y^2}{y^2}-F\left(\frac{R}{y}\right)-\frac{y^2}{R^2}\hbar\omega\left(n_1+2n_2\pm k_2+\frac32\right)\right]dt\right\}\ . \tag{4.74}$$

The wave-functions $\Psi_{n_1n_2}^{(Holt)}(\mathbf{x})$ are the wave-functions of the Holt potential [101] in $\mathbb{R}^2$ in one of the two possible coordinate system representations [80]. The remaining y-path integral again cannot be further evaluated until the potential $F(R/y)$ is specified.

4.2.3 Subsystem of Coulomb Potential.

The potential V_{11} contains the super-integrable Coulomb (C) potential in $\mathbb{R}^2$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider ($k_{1,2}>0$)

$$V_{11}(\mathrm{u})=F(u_0-u_3)-\frac{\alpha}{\sqrt{u_1^2+u_2^2}}\frac{1}{u_0-u_3}$$
$$+\frac{R^2}{(u_0-u_3)^2}\frac{\hbar^2}{4M}\frac{1}{\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)\ , \tag{4.75}$$

which in the three separating coordinate systems has the form

Horicyclic-Cylindrical $(y,\varrho>0,\varphi\in(0,\pi))$:

$$V_{11}(\mathrm{u})=F\left(\frac{R}{y}\right)+\frac{y^2}{R^2}\left[-\frac{\alpha}{\varrho}+\frac{\hbar^2}{8M\varrho^2}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)\right] \tag{4.76}$$

Horicyclic-Elliptic II $(y,\xi>0,\eta\in(0,\pi/2))$:

$$=F\left(\frac{R}{y}\right)+\frac{y^2}{R^2}\left[-\frac{\alpha}{d(\cosh\xi+\cos\eta)}+\frac{\hbar^2}{4Md(\cosh^2\xi-\cos^2\eta)}\left(\frac{(k_1^2+k_2^2-\frac12)+(k_2^2-k_1^2)\cos\eta}{\sin^2\eta}\right.\right.$$
$$\left.\left.+\frac{(k_1^2+k_2^2-\frac12)+(k_2^2-k_1^2)\cosh\xi}{\sinh^2\xi}\right)\right] \tag{4.77}$$

Horicyclic-Parabolic $(\xi,\eta,y>0)$:

$$=F\left(\frac{R}{y}\right)+\frac{y^2}{R^2}\left[-\frac{2\alpha}{\xi^2+\eta^2}+\frac{\hbar^2}{2M(\xi^2+\eta^2)}\left(\frac{k_1^2-\frac14}{\xi^2}+\frac{k_2^2-\frac14}{\eta^2}\right)\right]\ . \tag{4.78}$$

The observables have the form ($P_i=-\mathrm{i}\hbar\partial_{x_i}$ in $\mathbb{R}^2$, $i=1,2$)

$$\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{11}(\mathrm{u})\ ,\\
I_2&=\frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)\ ,\\
I_3&=\frac{1}{2M}\Big[(K_1+L_2)^2+(K_2-L_1)^2\Big]-\frac{\alpha}{\varrho}+\frac{\hbar^2}{8M\varrho^2}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)\ ,\\
I_4&=\frac{1}{4M}\{L_3,P_2\}+\frac{1}{\xi\eta}\bigg[-\alpha(\xi-\eta)+(\tfrac12-2k_1^2)\frac{\eta}{\xi}+(2k_2^2-\tfrac12)\frac{\xi}{\eta}\bigg]\ .
\end{aligned}\right\}\tag{4.79}$$

We have for instance in the horicyclic-cylindrical system the path integral representation ($\lambda=m+(1\pm k_1\pm k_2)/2,\mathbf{x}\in\mathbb{R}^2$)

$$\begin{aligned}
K^{(V_{11})}(\mathrm{u}'',\mathrm{u}';T)&=\frac{\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}}{R^3}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^3}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\varrho\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\frac{\dot y^2+\dot\varrho^2+\varrho^2\dot\varphi^2}{y^2}-F\Big(\frac{R}{y}\Big)\\
&\qquad-\frac{y^2}{R^2}\bigg(\frac{\hbar^2}{8M\varrho^2}\bigg(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\bigg)-\frac{\alpha}{\varrho}\bigg)\Bigg]dt\Bigg\}
\end{aligned}\tag{4.80}$$

$$\begin{aligned}
&=\sum_{m=0}^\infty\Bigg[\sum_{n=0}^\infty\Psi_{nm}^{(C)\,*}(\mathbf{x}')\Psi_{nm}^{(C)}(\mathbf{x}'')+\int_{\mathbb{R}}dp\,\Psi_{pm}^{(C)\,*}(\mathbf{x}')\Psi_{pm}^{(C)}(\mathbf{x}'')\Bigg]\\
&\quad\times\frac{1}{R}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\frac{\dot y^2}{y^2}-F\Big(\frac{R}{y}\Big)-y^2\frac{\hbar^2}{2MR^2}\lambda_{b,c}^2\Bigg]dt-\frac{3\mathrm{i}\hbar T}{8MR^2}\Bigg\}\ .
\end{aligned}\tag{4.81}$$

Here denote the wave-functions $\Psi_{nm}^{(C)}(\mathbf{x})$ and $\Psi_{pm}^{(C)}(\mathbf{x})$ the discrete and continuous wave-functions of the Coulomb problem in $\mathbb{R}^2$ in one of the three coordinate space representations [80]. In the discrete case we have $\lambda_b^2=2MR^2E_n/\hbar^2$ with ($\lambda_1=m+(1\pm k_1\pm k_2)/2$)

$$E_n=-\frac{M\alpha^2}{2\hbar^2(n+\lambda_1+\frac12)^2}\ ,\tag{4.82}$$

and in the continuous case just $\lambda_c=k>0$. The remaining y-path integral again cannot be further evaluated until the potential $F(R/y)$ is specified. However, let us consider the case $F\equiv0$. With $\lambda_c=k$ we obtain for the y-path integral the path integral representation for Liouville quantum mechanics [87], i.e., we get

$$\begin{aligned}
&\frac{1}{R}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{M}{2}R^2\frac{\dot y^2}{y^2}-y^2k^2\frac{\hbar^2}{2MR^2}\Bigg]dt-\frac{3\mathrm{i}\hbar T}{8MR^2}\Bigg\}\\
&=\frac{2}{\pi^2R^3}\int_0^\infty dp\,p\sinh\pi pK_{\mathrm{i}p}(ky')K_{\mathrm{i}p}(ky'')\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\ ,
\end{aligned}\tag{4.83}$$

and $E_p=\hbar^2(p^2+1)/2MR^2$.

In order to take into account the negative bound states of the two-dimensional Coulomb subsystem we first observe that $\lambda_b^2\equiv-\kappa^2<0$ ($\kappa>0$). This system therefore requires a proper self-adjoint extension because it is unbounded from below. Actually, this problem emerges in the consideration of the quantum motion on the SU(1,1) group manifold, respectively on the

O(2, 2) hyperboloid. The spectrum consists of two parts, an infinite negative discrete and a positive continuous spectrum.

This difficulty is resolved by means of harmonic analysis on the SU(1, 1) group manifold, respectively the SO(1, 2) or O(2, 2) hyperboloid and the corresponding wave-functions are the matrix element expansions of the Titchmarch transformation ([13, 32, 73, 75], [147, p.140], [154, p.54], [185, p.93–95]. The corresponding path integral representation has been discussed in [75] with the result of the following path integral identity, which is also the solution of the remaining y-path integration in the present case:

$$\begin{aligned}
&\frac{1}{R}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{M}{2}R^2\frac{\dot y^2}{y^2}+y^2\frac{\hbar^2\kappa^2}{2MR^2}\right)dt-\frac{3\mathrm{i}\hbar T}{8MR^2}\right]\\
&=\sum_{n\in\mathbb{N}}\mathrm{e}^{-\mathrm{i}E_n^{(\alpha)}T/\hbar}\psi_n^{(\alpha)}(y')\psi_n^{(\alpha)}(y'')+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\psi_p^{(\alpha)*}(y')\psi_p^{(\alpha)}(y'')\ . \qquad (4.84)
\end{aligned}$$

The bound states are given by

$$\psi_n^{(\alpha)}(y) = \sqrt{\frac{2(2n+\alpha)}{R}}\,J_{2n+\alpha}(\kappa y)\ , \qquad (4.85)$$

$$E_n^{(\alpha)} = -\frac{\hbar^2}{2MR^2}\Big[(2n+\alpha)^2-1\Big]\ , \qquad (4.86)$$

and the continuous states are

$$\psi_p^{(\alpha)}(y) = \sqrt{\frac{p}{2R\sinh\pi p}}\Big[J_{\mathrm{i}p}(\kappa y)+J_{-\mathrm{i}p}(\kappa y)\Big]\ , \qquad (4.87)$$

$$E_p = \frac{\hbar^2}{2MR^2}(p^2+1)\ , \qquad (4.88)$$

and α is the parameter of the self-adjoint extension.

4.2.4 Subsystem of Modified Coulomb Potential.

The potential V_{12} contains the super-integrable modified Coulomb potential in $\mathbb{R}^2$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider ($\beta_{1,2}\in\mathbb{R}$)

$$\begin{aligned}
V_{12}(\mathrm{u}) &= F(u_0-u_3)-\frac{\alpha}{\sqrt{u_1^2+u_2^2}}\frac{1}{u_0-u_3}\\
&\quad+\frac{1}{(u_0-u_3)^{3/2}}\frac{\beta_1\sqrt{\sqrt{u_1^2+u_2^2}+u_1}+\beta_2\sqrt{\sqrt{u_1^2+u_2^2}-u_1}}{2\sqrt{u_1^2+u_2^2}}\ , \qquad (4.89)
\end{aligned}$$

which has in mutually parabolic coordinate systems the form ($\xi,\eta,y>0$)

$$V_{12}(\mathrm{u}) = F\left(\frac{R}{y}\right)-\frac{y^2}{R^2}\left[\frac{\alpha-(\beta_1\xi+\beta_2\eta)}{\xi^2+\eta^2}\right]\ . \qquad (4.90)$$

The constants of motion are (λ,μ is a parabolic system mutually orthogonal to ξ,η) ($P_i=-\mathrm{i}\hbar\partial_{x_i}$ in $\mathbb{R}^2$, $i=1,2$, $P_y=-\mathrm{i}\hbar\partial_y$)

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{12}(\mathrm{u})\ ,\\
I_2 &= \frac{1}{2M}\Big[(K_1+L_2)^2+(K_2-L_1)^2\Big]+F\left(\frac{R}{y}\right)\ ,\\
I_3 &= \frac{1}{4M}\{L_3,P_1\}-\frac{\alpha(\lambda-\mu)+\beta_1\mu\sqrt{\lambda}-\beta_2\lambda\sqrt{\mu}}{\lambda+\mu}\ ,\\
I_4 &= \frac{1}{4M}\{L_3,P_2\}-\frac{\alpha(\xi-\eta)+(\beta_1+\beta_2)\eta\sqrt{\xi/2}-(\beta_1+\beta_2)\xi\sqrt{\eta/2}}{\xi+\eta}\ .
\end{aligned}\right\} \qquad (4.91)$$

We state the corresponding path integral formulation which have the form

$$
\begin{aligned}
&K^{(V_{12})}(\mathrm{u}'',\mathrm{u}';T)\\
&=\frac{\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}}{R^3}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^3}\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\xi^2+\eta^2)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot y^2+(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)}{y^2}-F\left(\frac{R}{y}\right)+\frac{y^2}{R^2}\left(\frac{\alpha-(\beta_1\xi+\beta_2\eta)}{\xi^2+\eta^2}\right)\right]dt\right\} \qquad (4.92)\\
&=\left\{\sum_{n_1,n_2=0}^{\infty}\Psi_{n_1,n_2}(\xi',\eta')\Psi_{n_1,n_2}(\xi'',\eta'')+\sum_{e,o}\int_{\mathbb{R}}d\zeta\int_{\mathbb{R}}dp\Psi_{p,\zeta}^{(e,o)*}(\xi',\eta')\Psi_{p,\zeta}^{(e,o)}(\xi'',\eta'')\right\}\\
&\qquad\times\mathrm{e}^{-3\mathrm{i}\hbar T/8MR^2}\frac{y'y''}{R}\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\dot y^2}{y^2}-F\left(\frac{R}{y}\right)-y^2\frac{\hbar^2}{2MR^2}\lambda_{b,c}^2\right]dt\right\} \qquad (4.93)
\end{aligned}
$$

Here denote the wave-functions $\Psi_{n_1,n_2}(\xi,\eta)$ and $\Psi_{p,\zeta}^{(e,o)}(\xi,\eta)$ the discrete and continuous wave-functions of the corresponding problem in $\mathbb{R}^2$ [80]. In the continuous case we have for the parameter $\lambda_c=p$, whereas in the discrete case we have $\lambda_c^2=2ME_{n_1n_2}/\hbar^2$ with $(N=n_1+n_2, n_{1,2}\in\mathbb{N}_0)$

$$
\begin{aligned}
E_{n_1,n_2} &= -\frac{M}{2}\omega_N^2\ , \qquad \omega_N=y_N+\frac{2\alpha}{3\hbar N}\ , \qquad y_N=u_1+u_2\ , &(4.94)\\
u_{1,2} &= \sqrt[3]{\left(\frac{2\alpha}{3\hbar N}\right)^3+\frac{\beta_1^2+\beta_2^2}{MN\hbar}\mp\sqrt{\left(\frac{\beta_1^2+\beta_2^2}{MN\hbar}\right)^2+2\frac{\beta_1^2+\beta_2^2}{MN\hbar}\left(\frac{2\alpha}{3\hbar N}\right)^3}}\ . &(4.95)
\end{aligned}
$$

The remaining y-path integral again cannot be further evaluated until the potential $F(R/y)$ is specified. Of course, for $\lambda_b^2<0$ the same feature as for the potential V_{11} must be taken into account, and again we obtain for the case $F\equiv 0$ the solutions (4.85, 4.87). The Green functions of the potentials $V_{9,10,11,12}$ can be constructed by using the result of (4.45), which is left to the reader.

4.3 Minimally Superintegrable Potentials from the Group Chain SO(3,1) ⊃ SO(3).

In the following two potentials we can consider two choices of the functions $F(u_0)$, i.e.,

$$F_1(u_0)=-\frac{\hbar^2}{2M}\frac{k_0^2-\frac14}{u_0^2}\ , \qquad F_2(u_0)=\alpha R\frac{u_0}{\sqrt{u_0^2-1}}\ . \qquad (4.96)$$

F_1 leads the usual modified Pöschl-Teller path integral problem, whereas F_2 gives a Kepler problem on the hyperboloid. The latter has already been discussed, and therefore in the following we make the choice of F_1 just for convenience.

4.3.1 Subsystem of Oscillator.

At first sight, the potential V_7', c.f. table 4.3, contains the super-integrable Higgs-oscillator on $S^{(2)}$ due to its sub-group construction, therefore it is minimally super-integrable on $\Lambda^{(3)}$. However, as seen, e.g., in the spherical system it is by an appropriate redefinition of the parameters equivalent to the potential V_7, c.f. subsection 4.1.3. We have mentioned this potential nevertheless to emphasize the SO(3) subgroup property of the corresponding analogue of the flat space case.

Table 4.3: Minimally Superintegrable Potentials on $\Lambda^{(3)}$ from the Group Chain $\mathrm{SO}(3,1) \supset \mathrm{SO}(3)$.

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_7'(\mathrm{u}) = F(u_0) + \frac{M}{2}\frac{u_1^2+u_2^2}{u_1^2+u_2^2+u_3^2}\frac{\omega^2}{u_3^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2} + \frac{k_2^2-\frac{1}{4}}{u_2^2}\right)$ $\lambda^2 = \frac{M^2\omega^2}{\hbar^2}R^4 + \frac{1}{4}$	Sphero-Elliptic Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_7'(\mathrm{u})$ $I_2 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right) + \frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}\right)$ $I_3 = \frac{1}{2M}L_3^2 + \left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$ $I_4 = \frac{1}{2M}(L_1^2 + k'^2L_2^2) - \frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)}\Big[(k_1^2-\tfrac{1}{4})\left(\frac{1}{\mathrm{sn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right)$ $+(k_2^2-\tfrac{1}{4})\left(\frac{k'^2}{\mathrm{cn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right) - (\lambda^2-\tfrac{1}{4})\left(\frac{k'^2}{\mathrm{dn}^2\tilde\alpha} - \frac{1}{\mathrm{sn}^2\tilde\beta}\right)\Big]$
$V_{13}(\mathrm{u}) = F(u_0) - \frac{\alpha}{u_1^2+u_2^2+u_3^2}\frac{u_3}{\sqrt{u_1^2+u_2^2}}$ $+\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Sphero-Elliptic* Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_{13}(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2(\varphi/2)} + \frac{k_2^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 - \alpha\cot\vartheta + \frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2(\varphi/2)} + \frac{k_2^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right)$ $I_4 = \frac{1}{2M}(\tfrac{1}{2}\sin 2f\{L_1, L_3\} - \cos 2f L_3^2)$ $-\alpha\frac{k^2k'\,\mathrm{sn}^2\tilde\alpha\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta - (k^2 + k'^2\,\mathrm{cn}^2\tilde\beta)k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}$ $+\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)}$ $\times\Bigg(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha}$ $+\frac{(k_1^2+k_2^2-\frac{1}{2})k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Bigg)$

* after appropriate rotation, $\sin^2 f = k^2$.

4.3.2 Subsystem of Coulomb Potential.

The potential V_{13} contains the super-integrable Coulomb potential on $S^{(2)}$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider $(k_{1,2}>0)$

$$V_{13}(\mathrm{u})=-\frac{\alpha}{u_1^2+u_2^2+u_3^2}\frac{u_3}{\sqrt{u_1^2+u_2^2}}-\frac{\hbar^2}{2M}\frac{k_0^2-\frac14}{u_0^2}+\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)\ , \tag{4.97}$$

which in the two separating coordinate systems has the form

Spherical $(\tau>0,\vartheta\in(0,\pi),\varphi\in(0,\pi))$:

$$V_{13}(\mathrm{u})=\frac{1}{R^2\sinh^2\tau}\left[-\alpha\cot\vartheta+\frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)\right]-\frac{\hbar^2}{2MR^2}\frac{k_0^2-\frac14}{\cosh^2\tau} \tag{4.98}$$

Sphero-Elliptic II $(\tau>0,\tilde\alpha\in(0,K),\tilde\beta\in(0,K'))$:

$$\begin{aligned}&=\frac{1}{R^2\sinh^2\tau}\Bigg[-\tilde\alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta-k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta}\\&\quad+\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta)}\Bigg(\frac{(k_1^2+k_2^2-\frac12)k'^2+(k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha}\\&\quad+\frac{(k_1^2+k_2^2-\frac12)k^2+(k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Bigg)\Bigg]-\frac{\hbar^2}{2MR^2}\frac{k_0^2-\frac14}{\cosh^2\tau}\ .\end{aligned} \tag{4.99}$$

The corresponding observables are

$$\left.\begin{aligned}I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{13}(\mathrm{u})\ ,\\I_2&=\frac{1}{2M}L_3^2+\frac{1}{8M}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)\ ,\\I_3&=\frac{1}{2M}\mathbf{L}^2-\alpha\cot\vartheta+\frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)\ ,\\I_4&=\frac{1}{2M}(\tfrac12\sin 2f\{L_1,L_3\}-\cos 2fL_3^2)-\alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta-k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta}\\&\quad+\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta)}\Bigg(\frac{(k_1^2+k_2^2-\frac12)k'^2+(k_2^2-k_1^2)k'\ \mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha}\\&\quad+\frac{(k_1^2+k_2^2-\frac12)k^2+(k_2^2-k_1^2)k\ \mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Bigg)\ .\end{aligned}\right\} \tag{4.100}$$

Here denote

$$k^2=\frac{a_2-a_1}{a_3-a_1}=\sin^2 f\ ,\qquad k'^2=\frac{a_3-a_2}{a_3-a_1}=\cos^2 f\ ,\qquad k^2+k'^2=1\ . \tag{4.101}$$

and $2f$ is the interfocus distance on the upper semisphere of the ellipses on the sphere [81]. Because the potential V_{13} contains the Coulomb potential on the two-dimensional sphere as the subsystem, we can separate the subsystem off and obtain ($\mathbf{\Omega}=(\Omega_1,\Omega_2)\in S^{(2)}$ are the corresponding coordinates on $S^{(2)}$, $\lambda_1=m+(1\mp k_1\mp k_2)/2$)

$$K^{(V_{13})}(\mathrm{u}'',\mathrm{u}';T)=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\mathbf{\Omega}(t')=\mathbf{\Omega}'}^{\mathbf{\Omega}(t'')=\mathbf{\Omega}''}\mathcal{D}\mathbf{\Omega}(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2+\sinh^2\tau\dot{\mathbf{\Omega}}^2)+\frac{\hbar^2}{2MR^2}\frac{k_0^2-\frac14}{\cosh^2\tau}\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2MR^2}\frac{1}{\sinh^2\tau}\left(\frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{k_2^2-\frac14}{\sqrt{s_1^2+s_2^2}-s_1}\right)-\frac14\right)\right]dt\right\} \tag{4.102}$$

$$=\sum_{m=0}^\infty\sum_{l=0}^\infty\left\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_nT\hbar}\Psi_{mln}^{(V_{13})\,*}(\tau',\mathbf{\Omega}';R)\Psi_{mln}^{(V_{13})}(\tau'',\mathbf{\Omega}'';R)\right.$$
$$\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT\hbar}\Psi_{plm}^{(V_{13})\,*}(\tau',\mathbf{\Omega}';R)\Psi_{plm}^{(V_{13})}(\tau'',\mathbf{\Omega}'';R)\right\}\ . \tag{4.103}$$

The bound state wave-functions and the energy-spectrum are given by

$$\Psi_{lmn}^{(V_{13})}(\tau,\mathbf{\Omega};R)=(\sinh\tau)^{-1}S_n^{(\lambda_2,\pm k_3)}(\tau;R)\Psi_{lm}^{(\alpha)}(\mathbf{\Omega})\ , \tag{4.104}$$
$$E_n=-\frac{\hbar^2}{2MR^2}(2n+\lambda_2\mp k_0+1)^2\ , \tag{4.105}$$

and the continuous states are

$$\Psi_{plm}^{(V_{13})}(\tau,\mathbf{\Omega};R)=(\sinh\tau)^{-1}S_p^{(\lambda_2,\pm k_3)}(\tau;R)\Psi_{lm}^{(\alpha)}(\mathbf{\Omega})\ , \tag{4.106}$$

and E_p as in (3.28). The wave-functions $\Psi_{lm}^{(\alpha)}(\mathbf{\Omega})$ are the wave-functions of the Coulomb problem on $S^{(2)}$ [81]. The quantum number λ_2 is defined by the energy-spectrum of the Coulomb potential on $S^{(2)}$ which can be positive or negative, i.e., $\lambda_2^2=2MR^2E_l/\hbar^2$ with [81] $E_l=\hbar^2(\tilde l^2-\frac14)/2MR^2-M\alpha^2/2\hbar^2\tilde l^2$. The wave-functions $S_n^{(\lambda_2,\pm k_3)}(\tau;R)$, $S_p^{(\lambda_2,\pm k_3)}(\tau;R)$ are the same as in (3.71,3.77) with $(\lambda_1,\lambda_2)\to(\lambda_2,\pm k_3)$. For E_l negative the same problem of a proper self-adjoint extension arises as for the potential V_{11}, however we do not know this extension, and we are not able to write down the corresponding path integral representation either.

The radial Green functions of the potential V_{13} is of the form

$$G^{(V_{13})}(\tau'',\tau';E)=(R\sinh\tau''\sinh\tau')^{-1}G_{mPT}^{(\lambda_2,\pm k_3)}\left(\tau'',\tau';E+\frac{\hbar^2}{2MR^2}\right)\ . \tag{4.107}$$

4.4 Minimally Superintegrable Potentials from the Group Chain $\mathrm{SO}(3,1)\supset\mathrm{SO}(2,1)$.

4.4.1 Construction of the Potentials V_{14}–V_{18}.

In the following potentials we can consider two choices of the functions $F(u_3)$, i.e.

$$F_1(u_3)=\frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{u_3^2}\ ,\qquad F_2(u_3)=\alpha R\frac{u_3}{\sqrt{1+u_3^2}}\ . \tag{4.108}$$

F_1 leads the usual modified Pöschl-Teller path integral problem, whereas F_2 yields a Rosen-Morse potential on the hyperboloid. The latter gives the following solution in terms of the equidistant variable τ_1 [66, 132, 82]

$$K^{(F_2)}(\tau_1'',\tau_1';T)$$
$$=\frac1R\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{M}{2}R^2\dot\tau_1^2-\alpha R\tanh\tau_1-\frac{\hbar^2}{2MR^2}\frac{k^2+\frac14}{\cosh^2\tau_1}\right)dt\right] \tag{4.109}$$
$$=\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}\frac{M}{\hbar^2}\frac{\Gamma(m_1-L_k)\Gamma(L_k+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$

Table 4.4: Minimally Superintegrable Potentials on $\Lambda^{(3)}$ from the Group Chain $SO(3,1) \supset SO(2,1)$.

Potential $V(\mathrm{u})$	Coordinate Systems	Observables
$V_{14}(\mathrm{u}) = F(u_3) + \frac{M}{2}\frac{\omega^2}{u_0^2-u_1^2-u_2^2}\frac{u_1^2+u_2^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\right)$	Equidistant-Elliptic Equidistant-Hyperbolic Equidistant-Cylindrical Equidistant	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{14}(\mathrm{u}), \quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_4 = \frac{1}{2M}K_2^2-\frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_3}+\frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{\sinh^2\tau_3}$
$V_{15}(\mathrm{u}) = F(u_3) - \frac{\alpha}{u_0^2-u_1^2-u_2^2}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2}}-1\right) + \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Equidistant-Elliptic* Equidistant-Semi-Hyperbolic Equidistant-Elliptic-Parabolic Equidistant-Cylindrical	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{15}(\mathrm{u}), \quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)$ $I_4 = \frac{1}{4M}\{K_1,L_3\}-\alpha R\frac{\sqrt{1+\mu_1}+\sqrt{1+\mu_2}}{\mu_1+\mu_2}+\frac{\hbar^2}{4M}\frac{1}{\mu_1+\mu_2}$ $\times\left[(k_1^2+k_2^2-\frac12)\left(\frac{\mu_1}{\mu_2}+\frac{\mu_2}{\mu_1}\right)+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1}}{\mu_1}-\frac{\sqrt{1+\mu_2}}{\mu_2}\right)\right]$
$V_{16}(\mathrm{u}) = F(u_3) + \frac{\alpha}{(u_0-u_1)^2} + \frac{M}{2}\omega^2\frac{u_0^2-u_1^2+3u_2^2}{(u_0-u_1)^4} - \lambda\frac{u_2}{(u_0-u_1)^3}$	Equidistant- Semi-Circular-Parabolic Equidistant-Horicyclic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{16}(\mathrm{u}), \quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}(K_1-L_3)^2+\alpha+2M\omega^2x^2-\lambda x$ $I_4 = \frac{1}{4M}(\{K_1,K_2\}-\{K_2,L_3\})$ $+\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left[\alpha(\xi^2+\eta^2)+\frac{\lambda}{2}(\xi^2-\eta^4)+\frac{M}{2}\omega^2(\xi^6+\eta^6)\right]$
$V_{17}(\mathrm{u}) = F(u_3) + \frac{M}{2}\frac{\omega^2}{(u_0-u_1)^2} + \frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{u_2^2}$	Equidistant-Elliptic-Parabolic Equidistant-Hyperbolic-Parabolic Equidistant- Semi-Circular-Parabolic Equidistant Equidistant-Horicyclic	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{17}(\mathrm{u}), \quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{2M}(K_1-L_3)^2+\frac{\hbar^2}{2M}\frac{\kappa^2-\frac14}{x^2}$ $I_4 = \frac{1}{2M}K_2^2+\frac{M}{2}\omega^2e^{2\tau_3}$
$V_{18}(\mathrm{u}) = F(u_3) + \frac{\alpha}{u_0^2-u_1^2-u_2^2}\frac{u_2}{\sqrt{u_0^2-u_1^2}}$	Equidistant- Semi-Circular-Parabolic Equidistant	$I_1 = \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{18}(\mathrm{u}), \quad I_2 = K_1^2+K_2^2-L_3^2$ $I_3 = \frac{1}{4M}(\{K_1,K_2\}-\{K_2,L_3\})+\alpha R\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{1}{\xi^2}-\frac{1}{\eta^2}\right)$ $I_4 = K_2^2$

* after appropriate rotation, $\sin^2 f = k^2$.

$$\times\left(\frac{1-\tanh\tau_1'}{2}\cdot\frac{1-\tanh\tau_1''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\tanh\tau_1'}{2}\cdot\frac{1+\tanh\tau_1''}{2}\right)^{(m_1+m_2)/2}$$

$$\times{}_2F_1\left(-L_k+m_1,L_k+m_1+1;m_1+m_2+1;\frac{1+\tanh\tau_{1,>}}{2}\right)$$

$$\times{}_2F_1\left(-L_k+m_1,L_k+m_1+1;m_1-m_2+1;\frac{1-\tanh\tau_{1,<}}{2}\right) \qquad (4.110)$$

$$=\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi_p^{(F_2)}(\tau_1'')\Psi_p^{(F_2)*}(\tau_1')\ . \qquad (4.111)$$

Here denote $L_k=-2\mathrm{i}k-\frac{1}{2}$, $m_{1,2}=\sqrt{M/2}(\sqrt{-\alpha R-E}\pm\sqrt{\alpha R-E})/\hbar$, and (4.110) is the Green function corresponding to the path integral (4.109). The wave-functions and the energy-spectrum of the continuous states are (where $\pm$ distinguishes between incoming and outgoing scattering states, respectively)

$$\Psi_p^{(F_2)}(\tau_1)=\frac{1}{R\Gamma(1+m_1\pm m_2)}\frac{\sqrt{M\sinh(\pi|m_1\pm m_2|)/2}}{\hbar|\sin\pi(m_1+L_k)|}$$

$$\times\left(\frac{1+\tanh\tau_1}{2}\right)^{(m_1+m_2)/2}\left(\frac{1-\tanh\tau_1}{2}\right)^{(m_1-m_2)/2}$$

$$\times{}_2F_1\left(m_1+L_k+1,m_1-L_k;1+m_1\pm m_2;\frac{1\pm\tanh\tau_1}{2}\right)\ , \qquad (4.112)$$

$$E_p=\frac{\hbar^2}{2MR^2}(p^2+1)-\alpha R\ . \qquad (4.113)$$

In the following we make the choice of F_1 just for convenience.

Subsystem of Oscillator on $\Lambda^{(2)}$. The potential V_{14} contains the super-integrable Higgs-oscillator on $\Lambda^{(2)}$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider ($k_{1,2,3}>0$)

$$V_{14}(\mathbf{u})=\frac{M}{2}\frac{\omega^2}{u_0^2-u_1^2-u_2^2}\frac{u_1^2+u_2^2}{u_0^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2}+\frac{k_2^2-\frac{1}{4}}{u_2^2}+\frac{k_3^2-\frac{1}{4}}{u_3^2}\right)\ , \qquad (4.114)$$

which in the four separating coordinate systems has the form

Equidistant-Elliptic ($\tau>0,\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K')$) :

$$V_{14}=\frac{1}{R^2\cosh^2\tau}\left[\frac{M}{2}\omega^2\left(1-\frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right)\right.$$
$$\left.-\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}+\frac{k_2^2-\frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right)\right]+\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau} \qquad (4.115)$$

Equidistant-Hyperbolic ($\tau>0,\mu\in(\mathrm{i}K',\mathrm{i}K'+K),\eta\in(0,K')$) :

$$=\frac{1}{R^2\cosh^2\tau}\left[\frac{M}{2}\omega^2\left(1-\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right)\right.$$
$$\left.-\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{k_2^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right]+\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau} \qquad (4.116)$$

Equidistant-Cylindrical ($\tau_{1,2}>0,\varphi\in(0,\pi/2)$) :

$$=\frac{1}{R^2\cosh_1^2\tau}\left[\frac{M}{2}\omega^2\tanh^2\tau_2+\frac{\hbar^2}{2M}\frac{1}{\sinh^2\tau_2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)\right]+\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac{1}{4}}{\sinh^2\tau_1} \qquad (4.117)$$

Equidistant ($\tau_{1,2,3}>0$) :

$$=\frac{1}{R^2\cosh^2\tau_1}\left[\frac{M}{2}\omega^2\left(1-\frac{1}{\cosh^2\tau_2\cosh^2\tau_3}\right)\right.$$

$$+\frac{\hbar^2}{2M}\bigg(\frac{k_2^2-\frac14}{\sinh^2\tau_2}+\frac{k_1^2-\frac14}{\cosh^2\tau_2\sinh^2\tau_3}\bigg)\bigg]+\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau_1}\ . \tag{4.118}$$

The observables are

$$\left.\begin{aligned}
I_1&=\frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{14}(\mathrm{u})\ ,\\
I_2&=K_1^2+K_2^2-L_3^2\ ,\\
I_3&=\frac{1}{2M}L_3^2+\frac{\hbar^2}{2M}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)\ ,\\
I_4&=\frac{1}{2M}K_2^2-\frac{M}{2}\frac{\omega^2R^4}{\cosh^2\tau_3}+\frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{\sinh^2\tau_3}\ .
\end{aligned}\right\} \tag{4.119}$$

Subsystem of Coulomb Potential on $\Lambda^{(2)}$. The potential V_{15} contains the super-integrable Coulomb potential on $\Lambda^{(2)}$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider $(k_{1,2,3}>0)$

$$\begin{aligned}
V_{15}(\mathrm{u})=&-\frac{\alpha}{u_0^2-u_1^2-u_2^2}\bigg(\frac{u_0}{\sqrt{u_1^2+u_2^2}}-1\bigg)+\frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{u_3^2}\\
&+\frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\bigg(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\bigg)\ ,
\end{aligned} \tag{4.120}$$

which in the four separating coordinate systems has the form

Equidistant-Elliptic Rotated $(\tau>0,\alpha\in(\mathrm{i}K',\mathrm{i}K'+K),\beta\in(0,K'),)$:

$$\begin{aligned}
V_{15}(\mathrm{u})=&\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau}+\frac{1}{R^2\cosh^2\tau}\Bigg\{-\frac{\alpha}{R}\bigg(\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\beta-k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}-1\bigg)\\
&+\frac{\hbar^2}{4MR^2}\bigg[\frac{k_1^2+k_2^2-\frac12}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\bigg(\frac{k'^2}{\mathrm{dn}^2\alpha}-\frac{1}{\mathrm{sn}^2\beta}\bigg)+(k_1^2-k_2^2)\frac{k'}{k}\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha+k'\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta}\bigg]\Bigg\}
\end{aligned} \tag{4.121}$$

Equidistant-Semi-Hyperbolic $(\tau,\mu_{1,2}>0)$:

$$\begin{aligned}
=&\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau}+\frac{1}{R^2\cosh^2\tau}\Bigg\{-\alpha\bigg(\frac{\sqrt{1+\mu_1^2}+\sqrt{1+\mu_2}}{\mu_1+\mu_2}-1\bigg)+\frac{\hbar^2}{4MR^2}\frac{1}{\mu_1+\mu_2}\\
&\times\bigg[(k_1^2+k_2^2-\tfrac12)\bigg(\frac{1}{\mu_1}+\frac{1}{\mu_2}\bigg)+(k_1^2-k_2^2)\bigg(\frac{\sqrt{1+\mu_1^2}}{\mu_1}-\frac{\sqrt{1+\mu_2^2}}{\mu_2}\bigg)\bigg]\Bigg\}
\end{aligned} \tag{4.122}$$

Equidistant-Elliptic-Parabolic $(a,\tau>0,\vartheta\in(0,\pi/2))$:

$$\begin{aligned}
=&\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau}+\frac{1}{R^2\cosh^2\tau}\bigg[-\alpha\bigg(\frac{\cosh^2a+\cos^2\vartheta}{\cosh^2a-\cos^2\vartheta}-1\bigg)\\
&+\frac{\hbar^2}{2M}\frac{\cosh^2a\cos^2\vartheta}{\cosh^2a-\cos^2\vartheta}\bigg(\frac{k_1^2-\frac14}{\sin^2\vartheta}+\frac{k_2^2-\frac14}{\sinh^2a}\bigg)\bigg]
\end{aligned} \tag{4.123}$$

Equidistant-Cylindrical $(\tau_{1,2}>0,\varphi\in(0,\pi))$:

$$=\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{R^2\cosh^2\tau_1}\bigg[-\alpha(\coth\tau_2-1)+\frac{\hbar^2}{8M\sinh^2\tau_2}\bigg(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\bigg)\bigg]\ . \tag{4.124}$$

The observables are

$$
\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{15}(\mathrm{u})\ ,\\
I_2 &= K_1^2+K_2^2-L_3^2\ ,\\
I_3 &= \frac{1}{2M}L_3^2+\frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\sin^2(\varphi/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi/2)}\right)\ ,\\
I_4 &= \frac{1}{4M}\{K_1,L_3\}-\alpha R\frac{\sqrt{1+\mu_1^2}+\sqrt{1+\mu_2^2}}{\mu_1+\mu_2}\\
&\quad+\frac{\hbar^2}{4M}\left[(k_1^2+k_2^2-\tfrac12)\left(\frac{\mu_1/\mu_2+\mu_2/\mu_1}{\mu_1+\mu_2}\right)+(k_1^2-k_2^2)\frac{\mu_2\sqrt{1+\mu_1^2}-\mu_1\sqrt{1+\mu_2^2}}{\mu_1\mu_2(\mu_1+\mu_2)}\right]\ .
\end{aligned}\right\} \tag{4.125}
$$

The Potential V_{16}. We consider the potential V_{16} which in the two separating coordinate systems has the following form

$$
V_{16}(\mathrm{u})=\frac{\alpha}{(u_0-u_1)^2}+\frac{M}{2}\omega^2\frac{u_0^2-u_1^2+3u_2^2}{(u_0-u_1)^4}-\lambda\frac{u_2}{(u_0-u_1)^3}+\frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{u_3^2} \tag{4.126}
$$

Equidistant-Horicyclic $(\tau,y>0,x\in\mathbb{R})$:

$$
=\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau}+\frac{y^2}{R^2\cosh^2\tau}\left[\alpha+\frac{M}{2}\omega^2(4x^2+y^2)-\lambda x\right] \tag{4.127}
$$

Equidistant-Semi-Circular Parabolic $(\tau,\xi,\eta>0)$:

$$
=\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau_1}+\frac{1}{R^2\cosh^2\tau}\frac{\xi^2\eta^2}{\xi^2+\eta^2}\Big[\alpha(\xi^2+\eta^2)-\tfrac{\lambda}{2}(\eta^4-\xi^4)+\tfrac{M}{2}\omega^2(\xi^6+\eta^6)\Big]\ . \tag{4.128}
$$

The constants of motion have the form

$$
\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2-\mathbf{L}^2)+V_{16}(\mathrm{u})\ ,\\
I_2 &= K_1^2+K_2^2-L_3^2\ ,\\
I_3 &= \frac{1}{2M}(K_1-L_3)^2+\alpha+2M\omega^2x^2-\lambda x\ ,\\
I_4 &= \frac{1}{4M}(\{K_1,K_2\}-\{K_2,L_3\})\\
&\qquad+\frac{\xi^2\eta^2}{\xi^2+\eta^2}\Big[\alpha(\xi^2+\eta^2)+\tfrac{\lambda}{2}(\xi^2-\eta^4)+\tfrac{M}{2}\omega^2(\xi^6+\eta^6)\Big]\ .
\end{aligned}\right\} \tag{4.129}
$$

The Potential V_{17}. The potential V_{17} contains the super-integrable potential V_3 on $\Lambda^{(2)}$, therefore it is minimally super-integrable on $\Lambda^{(3)}$. We consider $(k_{1,3}>0)$

$$
V_{17}(\mathrm{u})=\frac{M}{2}\frac{\omega^2}{(u_0-u_1)^2}+\frac{\hbar^2}{2M}\left(\frac{k_2^2-\frac14}{u_2^2}+\frac{k_3^2-\frac14}{u_3^2}\right)\ , \tag{4.130}
$$

which in the five separating coordinate systems has the form

Equidistant-Elliptic-Parabolic $(a,\tau>0,\varphi\in(0,\pi/2))$:

$$
V_{17}(\mathrm{u})=\frac{1}{R^2\cosh^2\tau}\left(\frac{M}{2}\omega^2\cosh^2 a\cos^2\vartheta+\frac{\hbar^2}{2M}(k_2^2-\tfrac14)\coth^2 a\cot^2\vartheta\right)+\frac{\hbar^2}{2MR^2}\frac{k_3^2-\frac14}{\sinh^2\tau} \tag{4.131}
$$

Equidistant-Hyperbolic-Parabolic $(b, \tau > 0, \varphi \in (0, \pi/2))$:

$$= \frac{1}{R^2\cosh^2\tau}\left(\frac{M}{2}\omega^2\sinh^2 b\sin^2\vartheta + \frac{\hbar^2}{2M}(k_2^2 - \tfrac{1}{4})\tanh^2 b\tan^2\vartheta\right) + \frac{\hbar^2}{2MR^2}\frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau} \tag{4.132}$$

Equidistant-Semi-Circular-Parabolic (separable only for $|k_2| = 1/2,\ \tau, \xi, \eta > 0$):

$$= \frac{1}{R^2\cosh^2\tau}\frac{M}{2}\omega^2\xi^2\eta^2 + \frac{\hbar^2}{2MR^2}\frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau} \tag{4.133}$$

Equidistant $(\tau_{1,2} > 0, \tau_3 \in \mathbb{R})$:

$$= \frac{1}{R^2\cosh^2\tau_1}\left(\frac{M}{2}\frac{\omega^2 e^{2\tau_3}}{\cosh^2\tau_2} + \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{\cosh^2\tau_2}\right) + \frac{\hbar^2}{2MR^2}\frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau_1} \tag{4.134}$$

Equidistant-Horicyclic $(\tau, x, y > 0)$:

$$= \frac{y^2}{R^2\cosh^2\tau}\left(\frac{M}{2}\omega^2 + \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{x^2}\right) + \frac{\hbar^2}{2MR^2}\frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau} \ . \tag{4.135}$$

For the observables we find

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_{17}(\mathrm{u}) \ , \\
I_2 &= K_1^2 + K_2^2 - L_3^2 \ , \\
I_3 &= \frac{1}{2M}(K_1 - L_3)^2 + \frac{\hbar^2}{2M}\frac{\kappa^2 - \frac{1}{4}}{x^2} \ , \\
I_4 &= \frac{1}{2M}K_2^2 + \frac{M}{2}\omega^2 e^{2\tau_3} \ .
\end{aligned}\right\} \tag{4.136}$$

The Potential V_{18}. We consider the potential $V_{18}(\mathrm{u})$ in its two separating coordinate systems

$$V_{18}(\mathrm{u}) = \frac{\alpha}{u_0^2 - u_1^2 - u_2^2}\frac{u_2}{\sqrt{u_0^2 - u_1^2}} + \frac{\hbar^2}{2M}\frac{k_3^2 - \frac{1}{4}}{u_3^2} \tag{4.137}$$

Equidistant $(\tau_{1,2} > 0, \tau_3 \in \mathbb{R})$:

$$= \alpha\frac{\tanh\tau_2}{R^2\cosh^2\tau_1} + \frac{\hbar^2}{2MR^2}\frac{k_3^2 - \frac{1}{4}}{\cosh^2\tau} \tag{4.138}$$

Semi-Circular Parabolic $(\tau, \xi, \eta > 0)$:

$$= \frac{\alpha}{R^2\cosh^2\tau}\frac{\xi^2\eta^2}{\xi^2 + \eta^2}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right) + \frac{\hbar^2}{2MR^2}\frac{k_3^2 - \frac{1}{4}}{\sinh^2\tau} \ , \tag{4.139}$$

and the observables have the form

$$\left.\begin{aligned}
I_1 &= \frac{1}{2MR^2}(\mathbf{K}^2 - \mathbf{L}^2) + V_{18}(\mathrm{u}) \ , \\
I_2 &= K_1^2 + K_2^2 - L_3^2 \ , \\
I_3 &= \frac{1}{4M}(\{K_1, K_2\} - \{K_2, L_3\}) + \alpha R\frac{\xi^2\eta^2}{\xi^2 + \eta^2}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right) \ , \\
I_4 &= K_2^2 \ .
\end{aligned}\right\} \tag{4.140}$$

4.4.2 Path Integral Discussion of the Potentials V_{14}–V_{18}.

The common structure of the potentials allows us to treat them simultaneously. Let us denote by $\tilde V_j(\mathrm{v})$ $(j=1,\dots,5)$ the five super-integrable potentials on $\Lambda^{(2)}$ with $\mathrm{v}\in\Lambda^{(2)}$. We then obtain for each of the $\tilde V_i$ in the corresponding separating coordinate systems $(i=14,\dots,18, k_3>0$, and identify $\tau\equiv\tau_1$ appropriately)

$$\begin{aligned}
&K^{(V_i)}(\mathrm{u}'',\mathrm{u}';T)\\
&=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/2MR^2}}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\cosh^2\tau\int\limits_{\mathrm{v}(t')=\mathrm{v}'}^{\mathrm{v}(t'')=\mathrm{v}''}\frac{\mathcal{D}\mathrm{v}(t)}{v_0}\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2-\cosh^2\tau\dot{\mathrm{v}}^2)-\frac{\hbar^2}{2MR^2}\left(\frac{k_3^2-\frac14}{\sinh^2\tau}+\frac{\tilde V(\mathrm{v})+\frac14}{\cosh^2\tau}\right)\right]dt\right\} && (4.141)\\
&=\int d\varrho_1\int d\varrho_2\left\{\sum_{n=0}^{N_n}\mathrm{e}^{-\mathrm{i}E_nT\hbar}\Psi^{(V_i)}_{\varrho_1\varrho_2n}(\tau',\mathrm{v}';R)\Psi^{(V_i)}_{\varrho_1\varrho_2n}(\tau'',\mathrm{v}'';R)\right.\\
&\qquad\qquad\left.+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT\hbar}\Psi^{(V_i)}_{\varrho_1\varrho_2p}(\tau',\mathrm{v}';R)\Psi^{(V_i)}_{\varrho_1\varrho_2p}(\tau'',\mathrm{v}'';R)\right\}\ . && (4.142)
\end{aligned}$$

The bound state spectrum has the form

$$\Psi^{(V_i)}_{lmn}(\tau,\mathrm{v};R)=(\cosh^2\tau)^{-1/2}S_n^{(\lambda_1,\pm k_3)}(\tau;R)\Psi^{(\tilde V_j)}_{\varrho_1\varrho_2}(\mathrm{v})\ , \qquad (4.143)$$

$$E_n=-\frac{\hbar^2}{2MR^2}(2n+\lambda_1\mp k_3+1)^2\ , \qquad (4.144)$$

and the continuous spectrum is given by

$$\Psi^{(V_i)}_{\varrho_1\varrho_2p}(\tau,\mathrm{v};R)=(\cosh^2\tau)^{-1/2}S_p^{(\lambda_1,\pm k_3)}(\tau;R)\Psi^{(\tilde V_j)}_{\varrho_1\varrho_2}(\mathrm{v})\ , \qquad (4.145)$$

and E_p as in (3.28). Here denote the wave-functions $\Psi^{(\tilde V_j)}_{\varrho_1\varrho_2}(\mathrm{v})$ the corresponding wave-functions of the two-dimensional subsystem on $\Lambda^{(2)}$, and the wave-functions $S_n^{(\pm k_3,\lambda_1)}(\tau;R)$, $S_p^{(\pm k_3,\lambda_1)}(\tau;R)$ are the same as in (3.71,3.77) with $(\lambda_1,\lambda_2)\to(\lambda_1,\pm k_3)$. The spectral expansions $\int d\varrho_i$ denote Lebesque-Stieltjes integrals in order to contain both discrete and continuous quantum numbers. The subspectra of the corresponding subsystems may be positive or negative, i.e., $\lambda_1^2=2MR^2E_{\tilde V_j}/\hbar^2$ may be positive or negative, respectively. If λ_1^2 is positive, $N_n=0$ and only continuous states exist for the corresponding V_i. If λ_1^2 is negative, bound states can exist with $N_n=\frac12(|\lambda_1|-k_3-1)$. If the $1/\sinh^2\tau$-term vanishes, the analytic continuation to an attractive potential is trivial, and we obtain in the radial variable a symmetric Rosen-Morse potential defined in $\tau\in\mathbb{R}$. The radial bound state wave-functions in this case are (3.56) and the scattering states (3.60).

The radial Green functions of the potentials V_i is of the form $(i=14,15,16,17,18)$

$$G^{(V_i)}(\tau'',\tau';E)=(R\cosh\tau''\cosh\tau')^{-1}G_{mPT}^{(\pm k_3,\lambda_1)}\left(\tau'',\tau';E+\frac{\hbar^2}{2MR^2}\right)\ . \qquad (4.146)$$

For $|k_3|=1/2$ the corresponding radial Green function is given by $(E'=E+\hbar^2/2MR^2)$

$$\begin{aligned}
&G^{(V_i)}(\tau'',\tau';E)\\
&=\frac{M}{\hbar^2R}(\cosh\tau''\cosh\tau')^{-1}\Gamma\left(\frac1\hbar\sqrt{-2MR^2E'}-\lambda_1+\frac12\right)\Gamma\left(\frac1\hbar\sqrt{-2MR^2E'}+\lambda_1+\frac12\right)\\
&\qquad\times P_{\lambda_1-1/2}^{-\sqrt{-2MR^2E'}/\hbar}(\tanh\tau_<)P_{\lambda_1-1/2}^{-\sqrt{-2MR^2E'}/\hbar}(-\tanh\tau_>)\ . && (4.147)
\end{aligned}$$

5 Summary and Discussion

The purpose of this paper has been to present a comprehensive discussion of super-integrable potentials on the three-dimensional hyperboloid $\Lambda^{(3)}$. It has included an enumeration of the coordinate systems on $\Lambda^{(3)}$ as known from the literature, a systematic search of maximally and minimally super-integrable potentials by appropriate generalizations from the Euclidean space, the statement of the constants of motions, respectively operators, and in the soluble cases the evaluation of the corresponding path integral representation in order to find the quantum mechanical propagators, the Green functions, the discrete and continuous wave-functions, and the energy spectra, respectively.

In the enumeration of the 34 coordinate systems in section 2 we have followed [116, 162], however, supplemented by the corresponding Hamiltonian and the form of a corresponding separable potential, several rotated coordinate systems, i.e., the sphero-elliptic rotated (2.31), the equidistant-elliptic rotated (2.44), and the prolate-elliptic rotated (2.139). These rotated systems correspond in their respective flat space limit to sphero-conical II, cylindrical elliptic II, and prolate-spheroidal II coordinate systems, which in turn contain as additional degenerate systems the respective parabolic systems. However, for the complicated two-parametric systems XXIX.–XXIV. hardly any statement and usage could have been made.

In section 3 we have presented our results concerning the maximally super-integrable potentials on $\Lambda^{(3)}$. These have been the (generalized) Higgs-oscillator $V_1(\mathrm{u})$, the (generalized) Coulomb potential $V_2(\mathrm{u})$, and a specific scattering potential $V_3(\mathrm{u})$. The potential $V_4(\mathrm{u})$, which is only minimally super-integrable on $\Lambda^{(3)}$, has been included in this section due to the fact that its flat space analogue in $\mathbb{R}^3$ is maximally super-integrable.

The Higgs-oscillator and the Coulomb potential have been discussed in some detail, first for the pure oscillator and Coulomb case, second with the incorporation of additional centrifugal terms which do not spoil the property of maximally super-integrability, similarly as the corresponding cases in $\mathbb{R}^3$ and on $S^{(3)}$. The energy spectrum and degeneracy of levels was also discussed.

In section 4 we have discussed the minimally super-integrable potentials on $\Lambda^{(3)}$. We have found the four analogues of the flat space case, in particular the ring-shaped oscillator, the Hartmann potential, a radial potential, and a Holt potential. The remaining minimally super-integrable potentials have emerged from the subgroup structure of $\mathrm{SO}(3,1)$, i.e., we have had to take into account the group chains $\mathrm{SO}(3,1)\supset E(2)$, $\mathrm{SO}(3,1)\supset \mathrm{SO}(3)$, $\mathrm{SO}(3,1)\supset \mathrm{SO}(2,1)$, which have given rise to four, one and five new minimally super-integrable potentials, respectively. In total, we have found 15 minimally super-integrable potentials on $\Lambda^{(3)}$. Whereas we have treated the ring-shaped oscillator and the Hartmann potential in some detail, the discussion for the other potentials has been mostly rather sketchy because the underlying super-integrable two-dimensional systems have been already solved in previous publications.

We have therefore continued the study of super-integrable systems in spaces of constant curvature as started by Bonatsos et al, Higgs, Granovsky et al., Hietarinta, Kalnins et al., Kibler et al., Izmest'ev et al., Katayama, Mardoyan et al., Otchik et al., Vinitsky et al., and others. At the same time, we have extended these investigations by taking into account generalizations of already known potentials, and a wider range of potentials.

Furthermore we would like to draw the attention to the following observations:

1. Let us consider the potential V_{19} in semi-hyperbolic coordinates

$$\begin{aligned} V_{19}(\mathrm{u}) &= \frac{M}{2}\omega^2\bigg(\frac{4u_0^2u_3^2}{R^2}+u_1^2+u_2^2\bigg)+2k_3u_0u_3+\frac{\hbar^2}{2M}\bigg(\frac{k_1^2-\frac14}{u_1^2}+\frac{k_2^2-\frac14}{u_2^2}\bigg)\\ &=\frac{R^2}{\mu_1+\mu_2}\bigg[\frac{M}{2}\omega^2(\mu_1^3+\mu_2^3)+k_3(\mu_1^2-\mu_2^2)\bigg]+\frac{\hbar^2}{2MR^2}\frac{1}{\mu_1\mu_2}\bigg(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\bigg)\ , \end{aligned} \tag{5.1}$$

and the potential is separable in this system. It has two flat-space limits in its full range of parameters, i.e., without the restrictions we made in section 2.3.22, namely circular polar and parabolic coordinates. Whereas it was possible to just simply state the potential $V_4(\mathrm{u})$, no explicit solution could have been found. This Stark-effect like potential could be of some interest, in particular in comparison with the potential $V_{19}(\mathrm{u})$. The potential in this limit corresponds to the second maximally super-integrable potential $V_2(\mathbf{x})$ of [80], c.f. table D.2, and the limiting case separates in these two coordinate systems, and in addition in the cartesian and circular elliptic system.

2. Let us consider the potential V_{20} in semi-hyperbolic coordinates

$$\begin{aligned} V_{20}(\mathrm{u}) &= \frac{M}{2}\omega^2\left(\frac{4u_0^2u_3^2}{R^2}+u_1^2+u_2^2\right)+\frac{\hbar^2}{2M}\frac{F(u_2/u_1)}{u_1^2+u_2^2} \ . \\ &= \frac{R^2}{\mu_1+\mu_2}\frac{M}{2}\omega^2(\mu_1^3+\mu_2^3)+\frac{\hbar^2}{2MR^2}\frac{F(\tan\varphi)}{\mu_1\mu_2} \ , \end{aligned} \tag{5.2}$$

and the potential is separable in this system. The potential in the flat space limit corresponds to the sixth minimally super-integrable potential $V_6(\mathbf{x})$ of [80], c.f. table D.3, and the limiting case separates in circular polar and parabolic coordinates.

3. The expressions for $V_{19}(\mathrm{u})$ and $V_{20}(\mathrm{u})$ suggest analogous expressions on the two- and three-dimensional sphere. Let us consider on the two-dimensional sphere the following two potentials ($\mathbf{s}=(s_1,s_2,s_3)\in S^{(2)}$)

$$\begin{aligned} V_3^{(S^{(2)})}(\mathbf{s}) &= \frac{M}{2}\omega^2\left(\frac{4s_1^2s_3^2}{R^2}+s_2^2\right)+2k_1s_1s_3+\frac{\hbar^2}{2M}\frac{k_2^2-\frac14}{s_2^2} \\ &= \frac{R^2}{\mathrm{cn}^2\alpha+\mathrm{cn}^2\beta}\left[\frac{M}{2}\omega^2(\mathrm{cn}^6\alpha+\mathrm{cn}^6\beta)+k_1(\mathrm{cn}^4\alpha-\mathrm{cn}^4\beta)\right]+\frac{\hbar^2}{2MR^2}\frac{k_2^2-\frac14}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta} \ , \end{aligned} \tag{5.3}$$

$$\begin{aligned} V_4^{(S^{(2)})}(\mathbf{s}) &= -\frac{\alpha}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}}+\frac{\beta_1\sqrt{\sqrt{s_1^2s_3^2+R^2s_2^2}+s_1s_3}+\beta_2\sqrt{\sqrt{s_1^2s_3^2+R^2s_2^2}-s_1s_3}}{2R\sqrt{s_1^2s_3^2+R^2s_2^2}} \\ &= -\sqrt{2}\frac{\alpha}{R}\frac{\mathrm{sn}\alpha\,\mathrm{dn}\alpha-\mathrm{sn}\beta\,\mathrm{dn}\beta}{\mathrm{cn}^2\alpha+\mathrm{cn}^2\beta}+\frac{\beta_1\,\mathrm{cn}\alpha+\beta_2\,\mathrm{cn}\beta}{\mathrm{cn}^2\alpha+\mathrm{cn}^2\beta} \ , \end{aligned} \tag{5.4}$$

where we have inserted the elliptic II system on $S^{(2)}$, c.f. Appendix C and [81], and we have put for the moduli $k=k'=1/\sqrt{2}$. According to [110] this coordinate system yields in the flat space limit parabolic coordinates in $\mathbb{R}^2$. We have thus found the analogues of the Holt-potential and of the modified Coulomb potential on $S^{(2)}$, which are however only integrable but not super-integrable. Therefore, we have found on the two-dimensional sphere and on the two-dimensional hyperboloid, c.f. [81, 82], all the analogues of the super-integrable potentials of two-dimensional Euclidean space.

4. Considering in an analogous way the $k=k'=1/\sqrt{2}$ version of the prolate elliptic system on $S^{(3)}$ [81] ($\mathbf{s}=(s_1,s_2,s_3,s_4)\in S^{(3)}$)

$$\left.\begin{aligned} s_1 &= R\ \mathrm{cn}\alpha\,\mathrm{cn}\beta\cos\varphi \ , \quad s_3=\frac{R}{\sqrt{2}}(\mathrm{sn}\alpha\,\mathrm{dn}\beta+\mathrm{dn}\alpha\,\mathrm{sn}\beta) \ , \\ s_2 &= R\ \mathrm{cn}\alpha\,\mathrm{cn}\beta\sin\varphi \ , \quad s_4=\frac{R}{\sqrt{2}}(\mathrm{dn}\alpha\,\mathrm{sn}\beta-\mathrm{sn}\alpha\,\mathrm{dn}\beta) \ , \end{aligned}\right\} \tag{5.5}$$

we can construct two more potentials on $S^{(3)}$ which correspond to the maximally super-integrable potential $V_2(\mathbf{x})$ and $V_5(\mathbf{x})$, c.f. tables D.5, D.6. They have the form:

$$V_8^{(S^{(3)})}(\mathbf{s}) = \frac{M}{2}\omega^2\left(\frac{4s_3^2s_4^2}{R^2}+s_1^2+s_2^2\right)+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{s_1^2}+\frac{k_2^2-\frac14}{s_2^2}\right)$$

$$
\begin{aligned}
&= \frac{\mathrm{cn}^6\alpha + \mathrm{cn}^6\beta}{\mathrm{cn}^2\alpha + \mathrm{cn}^2\beta}\frac{M}{2}R^2\omega^2 + \frac{\hbar^2}{2MR^2\,\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta}\bigg(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\bigg) \ , \qquad (5.6)\\
V_9^{(S^{(3)})}(\mathbf{s}) &= 2k_3 s_3 s_4 + \frac{\hbar^2 R^2}{2M\sqrt{s_3^2 s_4^2 + R^2(s_1^2+s_2^2)}}\\
&\quad\times\bigg(\frac{\beta_1^2-\frac{1}{4}}{\sqrt{s_3^2s_4^2+R^2(s_1^2+s_2^2)}+s_3s_4} + \frac{\beta_1^2-\frac{1}{4}}{\sqrt{s_3^2s_4^2+R^2(s_1^2+s_2^2)}-s_3s_4}\bigg)\\
&= k_3R^2\frac{\mathrm{cn}^4\alpha - \mathrm{cn}^4\beta}{\mathrm{cn}^2\alpha + \mathrm{cn}^2\beta} + \frac{\hbar^2}{2MR^2}\frac{2}{\mathrm{cn}^2\alpha + \mathrm{cn}^2\beta}\bigg(\frac{\beta_1^2-\frac{1}{4}}{\mathrm{cn}^2\alpha} + \frac{\beta_1^2-\frac{1}{4}}{\mathrm{cn}^2\beta}\bigg) \ . \qquad (5.7)
\end{aligned}
$$

However, both potentials are only integrable but not super-integrable on $S^{(3)}$.

5. Following along these lines we are able to construct on $S^{(3)}$ the analogue of the minimally super-integrable potential $V_6(\mathbf{x})$ in $\mathrm{I\!R}^3$, c.f. table D.3, i.e.,

$$
\begin{aligned}
V_{10}^{(S^{(3)})}(\mathbf{s}) &= \frac{M}{2}\omega^2\bigg(\frac{4s_3^2s_4^2}{R^2} + s_1^2 + s_2^2\bigg) + \frac{\hbar^2}{2M}\frac{F(s_2/s_1)}{s_1^2+s_2^2}\\
&= \frac{M}{2}R^2\omega^2\frac{\mathrm{cn}^6\alpha + \mathrm{cn}^6\beta}{\mathrm{cn}^2\alpha + \mathrm{cn}^2\beta} + \frac{\hbar^2}{2MR^2}\frac{F(\tan\varphi)}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta} \ . \qquad (5.8)
\end{aligned}
$$

Therefore we are able to construct suitable *parabolic* coordinates on the the two- and three-dimensional sphere, together with the corresponding separable potentials which are the analogues of two- and three-dimensional flat space, respectively. However, we have not found a possibility to construct the analogues of the minimally super-integrable potentials in $\mathrm{I\!R}^3$ which correspond to the group chain $E(3) \supset E(2)$. A corresponding group chain for $S^{(3)}$ does not exist, and therefore it is not surprising that it is not possible to construct the corresponding *circular* coordinate systems nor the corresponding potentials.

6. The previous observations allow the following statement: We have found all five analogues of the maximally super-integrable potentials in $\mathrm{I\!R}^3$, where we have the following identification (where the enumeration of the potentials in $\mathrm{I\!R}^3$ is according to [80], and the enumeration of the potentials on $S^{(3)}$ according to [81], c.f. tables D.2, D.5):

Table 5.1: Correspondence of Maximally Super-Integrable Potentials in Three Dimensions

$V_{\Lambda^{(3)}}(\mathrm{u})$	#Systems	$V_{\mathrm{I\!R}^3}(\mathbf{x})$	#Systems	$V_{S^{(3)}}(\mathbf{s})$	#Systems
$V_1(\mathrm{u})$	14(8)	$V_1(\mathbf{x})$	8	$V_1(\mathbf{s})$	6(8)
$V_2(\mathrm{u})$	5(4)	$V_3(\mathbf{x})$	4	$V_2(\mathbf{s})$	3(4)
$V_3(\mathrm{u})$	5(4)	$V_4(\mathbf{x})$	4	$V_3(\mathbf{s})$	2(2) [3(4)]
$V_4(\mathrm{u})$	4(4)	$V_5(\mathbf{x})$	4	$V_9(\mathbf{s})$	1(1)
$V_{19}(\mathrm{u})$	1(2)	$V_2(\mathbf{x})$	4	$V_8(\mathbf{s})$	1(1)

In parenthesis we have indicated the number of limiting coordinate systems, as $R \to \infty$. Note that for $V_3(\mathbf{s})$ we have two separating coordinate systems. In $V_3(\mathbf{s})$ we have also indicated the additional coordinate system which emerge, and causes an additional observable, if $k_3^2 - 1/4 = 0$, c.f. table D.5. From the rotated sphero-elliptic system on $S^{(3)}$ two coordinate systems on $\mathrm{I\!R}^3$ can be obtained by means of contraction, as $R \to \infty$, the cylindrical elliptic II and the cylindrical parabolic. Note also that $V_4(\mathrm{u})$ is only minimally super-integrable, but not maximally super-integrable. Furthermore, in [81] several potential systems were overlooked. However, the additional systems turn out to be only integrable but not super-integrable.

Table 5.2: Correspondence of Minimally Super-Integrable Potentials in Three Dimensions

$V_{\Lambda^{(3)}}(\mathrm{u})$	#Systems	$V_{\mathbb{R}^3}(\mathbf{x})$	#Systems	$V_{S^{(3)}}(\mathbf{s})$	#Systems
		Analogues of flat space			
$V_5(\mathrm{u})$	5(3)	$V_5(\mathbf{x})$	4	$V_4(\mathbf{s})$	4(4)
$V_6(\mathrm{u})$	3(4)	$V_7(\mathbf{x})$	3	$V_5(\mathbf{s})$	2(3)
$V_7(\mathrm{u})$	2(2)	$V_1(\mathbf{x})$	2	$V_6(\mathbf{s})$	2(2)
$V_8(\mathrm{u})$	2(1)	$V_3(\mathbf{x})$	2	$V_{10}(\mathbf{s})$	1(1)
$V_{20}(\mathrm{u})$	1(2)	$V_6(\mathbf{x})$	2	$V_9(\mathbf{s})$	1(2)
		Potentials emerging from $\mathrm{SO}(3,1) \supset E(2)$			
$V_9(\mathrm{u})$	3(3)	$V_2(\mathbf{x})$	3	–	–
V_9'	7(3)	$\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)+F(z)$	3	–	–
$V_{10}(\mathrm{u})$	2(2)	$V_3(\mathbf{x})$	2	–	–
$V_{11}(\mathrm{u})$	3(3)	$V_4(\mathbf{x})$	3	–	–
$V_{12}(\mathrm{u})$	2(2)	$V_8(\mathbf{x})$	2	–	–
		Potentials emerging from $\mathrm{SO}(3,1) \supset \mathrm{SO}(3)$			
$V_7'(\mathrm{u})$	2(2)	$V_1(\mathbf{x})$	2	$V_6(\mathbf{s})$	2(2)
$V_{13}(\mathrm{u})$	2(2)	$V_9(\mathbf{x})$	2	$V_7(\mathbf{s})$	2(2)
		Potentials emerging from $\mathrm{SO}(3,1) \supset \mathrm{SO}(2,1)$			
$V_{14}(\mathrm{u})$	4(3)	$V_2(\mathbf{x})$	3	–	–
$V_{15}(\mathrm{u})$	4(3)	$V_4(\mathbf{x})$	3	–	–
$V_{16}(\mathrm{u})$	2(1)	$V_3(\mathbf{x})$	2	–	–
$V_{17}(\mathrm{u})$	5(2)	$\frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{x^2}+F(z)$	2	–	–
$V_{18}(\mathrm{u})$	2(1)	$\alpha x + F(z)$	2	–	–

7. Furthermore, we have therefore found all analogues of the minimally super-integrable potential in $\mathbb{R}^3$, which is summarized in table 5.2, where the enumeration of the potentials in $\mathbb{R}^3$ is according to [80], and the enumeration of the potentials on $S^{(3)}$ according to [81], c.f. tables D.3, D.6. (In parenthesis we have indicated the maximal number of limiting coordinate systems, as $R \to \infty$). For three minimally super-integrable potentials in $\mathbb{R}^3$ we have found two, respectively three generalizations on the three-dimensional hyperboloid, a feature which is due to the rich subgroup structure of $\mathrm{SO}(3,1)$.

8. The linear potentials on the sphere and on the hyperboloid seem to have a structure according to $s_3 s_4$, respectively $u_0 u_3$, which turns out to be separable in an appropriately chosen *parabolic coordinate system*. However, on spaces of constant (non-vanishing) curvature, there seems to be no analogue of an *cartesian coordinate system*, which separates these kinds of potentials as well.

9. We also have found two more minimally super-integrable potentials in $\mathbb{R}^3$ and on $S^{(3)}$, respectively. They correspond to the subgroup chain $E(3) \supset \mathrm{SO}(3)$ and $\mathrm{SO}(4) \supset \mathrm{SO}(3)$. These cases have been overlooked in [44] and [81]. Whereas in both cases the subgroup Higgs oscillator on $S^{(2)}$ is easy to solve, we must distinguish for the subgroup Kepler problem on $S^{(2)}$ two cases. First, where the interaction on $S^{(2)}$ is repulsive, second where it is attractive. The repulsive case causes no problems. However, the attractive case yields a discrete negative subspectrum due to strongly attractive potentials in the variable r in

$\mathbb{R}^3$ and χ on $S^{(3)}$. It would seem very odd to discard these cases because it *seems* ill-defined. Rather we must find an analytic continuation for these kinds of problems. The case of $\mathbb{R}^3$ can be treated in the context of the free motion on pseudo-Euclidean space, c.f.e.g. [75, 154], and the solution is known. The situation is unfortunately very different for the strongly attractive singular potential on $S^{(3)}$. Here, no solution seems to be known yet (however, c.f. [32] for the corresponding hyperbolic case).

10. The coordinate system XXX. separates a radial potential according to $V(u_2, u_3) \propto \alpha/u_2^2 + \beta/u_3^2$, XXXIV. a potential according to $V(u_3) \propto \beta/u_3^2$, which are, however, trivial and not very interesting. The systems XXXI.–XXXIII. separate a potential according to

$$V_{21}(\mathrm{u}) = \frac{\alpha}{(u_0 + u_1)^2} + \frac{\beta}{u_2^2} + \frac{\gamma}{u_3^2} \ , \tag{5.9}$$

where we do not know any application, and no explicit solution.

11. Let us finally note another application of the prolate elliptic coordinate system. It has the property that it separates the two-center Coulomb problem on the hyperboloid, similarly as the prolate elliptic system on the sphere separates the two-center Coulomb problem on $S^{(3)}$ [11, 164]. Let us consider two point charges located at $u_{1,2} = (1, 0, 0, \pm k')/k$ on the hyperboloid. Then it is not difficult to show by means of the prolate elliptic coordinate system that one has in algebraic form ($Z_\pm = Z_1 \pm Z_2$)

$$\begin{aligned} V(u_1, u_2, u) &= -Z_1 \frac{u_1 \cdot u}{\sqrt{(u_1 \cdot u)^2 - 1}} - Z_2 \frac{u_2 \cdot u}{\sqrt{(u_2 \cdot u)^2 - 1}} \\ &= -\frac{Z_+ \sqrt{(\varrho_1 - a_2)(\varrho_1 - a_3)} - Z_- \sqrt{(\varrho_2 - a_2)(\varrho_2 - a_3)}}{\varrho_1 - \varrho_2} \ . \end{aligned} \tag{5.10}$$

A detailed investigation of this problem will be presented elsewhere [85].

We cannot say for sure if we really have found all possible super-integrable potentials on the hyperboloid. For a systematic search one must solve differential equations which emerge from the general form of a potential separable in a particular coordinate system, and changing variables. Because there are 34 coordinate systems on the hyperboloid which separate the Schrödinger equation, there are $33! \simeq 8.7 \cdot 10^{36}$ of such differential equations. This is not tractable, and one has to look for alternative procedures, in particular physical arguments. In this respect, we have found the relevant potentials which matters from a physical point of view, and which are the analogues of the flat space limit $\mathbb{R}^3$, including the corresponding coordinate systems.

Summarizing, we have achieved an enumeration and classification of super-integrable systems in spaces of constant (positive, zero, or negative) curvature. Further studies along these lines could include the investigation of the corresponding interbasis expansions, the contraction of the wave-functions in the curved spaces with respect to their Euclidean flat space limit, c.f. [110], their pseudo-Euclidean flat space limit, and the solution of the various superintegrable potentials in the generic, respectively parametric coordinate systems [86]. Among the latter, the most important cases are the Coulomb problems, for instance the Coulomb problem in $\Lambda^{(2)}$ or $\Lambda^{(3)}$ in semi-hyperbolic coordinates, and the investigation of the Stark-effect in spaces of constant curvature which includes the solution of the corresponding Schrödinger equations, or an analysis of the anisotropic Kepler problem [17, 94, 97, 177] in a space of (non-zero) constant curvature. We hope to return to these issues in the future.

Acknowledgements

The authors gratefully acknowledge financial support from the Heisenberg-Landau program. C.Grosche would like to thank the members of the Joint Institut for Nuclear Research, Dubna, for their kind hospitality. G.S.Pogosyan would like to thank the members of the II.Institut für Theoretische Physik, Hamburg University, for their kind hospitality.

A Elementary Path Integral Techniques

In order to set up our notation for path integrals on curved manifolds we proceed in the canonical way (see, e.g., DeWitt [38], D'Olivio and Torres [39], Feynman [45], Feynman and Hibbs [46], Gervais and Jevicki [54], [61, 88], McLaughlin and Schulman [146], Mayes and Dowker [152], Mizrahi [155], and Omote [163]). In the following $\mathbf{x}$ denotes D-dimensional cartesian coordinates, $\mathbf{q}$ D-dimensional arbitrary coordinates, and x, y, z etc. are one-dimensional coordinates.

A.1 Defining the Path Integral.

We start by considering the classical Lagrangian corresponding to the line element $ds^2 = g_{ab}dq^a dq^b$ of the classical motion in some D-dimensional Riemannian space

$$\mathcal{L}_{Cl}(\mathbf{q},\dot{\mathbf{q}}) = \frac{M}{2}\left(\frac{ds}{dt}\right)^2 - V(\mathbf{q}) = \frac{M}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) \ . \tag{A.1}$$

The quantum Hamiltonian is *constructed* by means of the Laplace-Beltrami operator

$$H = -\frac{\hbar^2}{2M}\Delta_{LB} + V(\mathbf{q}) = -\frac{\hbar^2}{2M}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}g^{ab}\sqrt{g}\frac{\partial}{\partial q^b} + V(\mathbf{q}) \tag{A.2}$$

as a *definition* of the quantum theory on a curved space. Here are $g = \det(g_{ab})$, $(g^{ab}) = (g_{ab})^{-1}$, and $\Delta_{LB} = g^{-1/2}\partial_a g^{ab} g^{1/2}\partial_b$. The scalar product for wave-functions on the manifold reads $(f,g) = \int d\mathbf{q}\sqrt{g}f^*(\mathbf{q})g(\mathbf{q})$, and the momentum operators which are hermitian with respect to this scalar product are given by

$$p_a = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2}\right) \ , \qquad \Gamma_a = \frac{\partial \ln\sqrt{g}}{\partial q^a} \ . \tag{A.3}$$

In terms of the momentum operators (A.3) we can rewrite H by using an ordering prescription called product according [61], where we assume $g_{ab} = h_{ac}h_{cb}$ (other lattice formulations like the important midpoint prescription (MP) which is corresponding to the Weyl ordering in the Hamiltonian, we do not discuss). Then we obtain for the Hamiltonian (A.2)

$$H = -\frac{\hbar^2}{2M}\Delta_{LB} + V(\mathbf{q}) = \frac{1}{2M}h^{ac}p_a p_b h^{cb} + \Delta V(\mathbf{q}) + V(\mathbf{q}) \ , \tag{A.4}$$

and for the path integral we obtain

$$\begin{aligned} K(\mathbf{q}'',\mathbf{q}';T) &= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathcal{D}\mathbf{q}(t)\sqrt{g(\mathbf{q})}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V(\mathbf{q})\right]dt\right\} \\ &\equiv \lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int d\mathbf{q}_k\sqrt{g(\mathbf{q}_k)} \\ &\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{bc}(\mathbf{q}_j)h_{ac}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b - \epsilon V(\mathbf{q}_j) - \epsilon\Delta V(\mathbf{q}_j)\right]\right\} \ . \end{aligned} \tag{A.5}$$

ΔV_{PF} denotes the well-defined quantum potential

$$\Delta V_{PF}(\mathbf{q}) = \frac{\hbar^2}{8M}\Big[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_b)_{,b} + g^{ab}{}_{,ab} + 2h^{ac}h^{bc}{}_{,ab} - h^{ac}{}_{,a}h^{bc}{}_{,b} - h^{ac}{}_{,b}h^{bc}{}_{,a}\Big] \ . \tag{A.6}$$

Here we have used the abbreviations $\epsilon = (t''-t')/N \equiv T/N$, $\Delta\mathbf{q}_j = \mathbf{q}_j - \mathbf{q}_{j-1}$, $\bar{q}_j = \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_{j-1})$ for $\mathbf{q}_j = \mathbf{q}(t' + j\epsilon)$ $(t_j = t' + \epsilon j, j = 0, \ldots, N)$ and we interpret the limit $N \to \infty$ as equivalent to $\epsilon \to 0$, T fixed. The lattice representation can be obtained by exploiting the composition law of the time-evolution operator $U = \exp(-\mathrm{i}HT/\hbar)$, respectively its semi-group property, and the the discretized path integral emerges in a natural way. The classical Lagrangian is modified into an effective Lagrangian via $\mathcal{L}_{eff} = \mathcal{L}_{Cl} - \Delta V$. In cartesian coordinates ordering problems do not appear in the Hamiltonian, and the path integral takes on the simple form (with obvious lattice discretization)

$$K(\mathbf{x}'', \mathbf{x}'; T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right]dt\right\} \ . \tag{A.7}$$

If the metric tensor is diagonal, i.e., $g_{ab} = f_a^2\delta_{ab}$, the quantum potential simplifies into

$$\Delta V_{PF}(\mathbf{q}) = \frac{\hbar^2}{8M}\frac{1}{f_a^2}\left[\left(\frac{f_{b,a}}{f_b}\right)^2 - 4\frac{f_{a,aa}}{f_a} + 4\frac{f_{a,a}}{f_a}\left(2\frac{f_{a,a}}{f_a} - \frac{f_{b,a}}{f_b}\right) + 2\left(\frac{f_{b,a}}{f_b}\right)_{,a}\right] \ . \tag{A.8}$$

Let us assume that g_{ab} is proportional to the unit tensor, i.e., $g_{ab} = f^2\delta_{ab}$. Then ΔV_{PF} simplifies into

$$\Delta V_{PF} = \hbar^2\frac{D-2}{8M}\frac{(4-D)f_{,a}^2 + 2f\cdot f_{,aa}}{f^4} \ . \tag{A.9}$$

This implies, that if the dimension of the space is $D = 2$, the quantum correction ΔV_{PF} vanishes.

Let us consider the special case that the metric is of the form ($\mathbf{q} = (a, b, z)$ are some three-dimensional coordinates)

$$ds^2(\mathbf{q}) = h^{-2}(da^2 + db^2) + u^2dz^2 \ , \tag{A.10}$$

with $h = h(a, b)$, $u = u(a, b)$. Then the quantum potential is of the form

$$\Delta V_{PF} = \frac{\hbar^2}{8M}\frac{h^2}{u^2}\Big[2u(u_{,aa} + u_{,bb}) - (u_{,a}^2 + u_{,b}^2)\Big] \ . \tag{A.11}$$

A.2 Transformation Techniques.

Point Canonical Transformations.

Indispensable tools in path integral techniques are transformation rules. In order to avoid cumbersome notation, we restrict ourselves to the one-dimensional case. For the general case we refer to DeWitt [38], Duru and Kleinert [42], Fischer, Leschke and Müller [47, 48], Gervais and Jevicki [54], Refs. [68, 88, 90, 91], Inomata [107], Kleinert [130, 131], and Storchak [183], and references therein. Implementing a transformation $x = F(q)$, one has to keep all terms of $O(\epsilon)$ in (A.7). Expanding about midpoints, the result is

$$\begin{aligned}K(F(q''), F(q'); T) &= \Big[F'(q'')F'(q')\Big]^{-1/2}\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{1/2}\prod_{k=1}^{N-1}\int dq_k \cdot \prod_{l=1}^{N} F'(\bar{q}_l)\\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}F'^2(\bar{q}_j)(\Delta q_j)^2 - \epsilon V(F(\bar{q}_j)) - \frac{\epsilon\hbar^2}{8M}\frac{F''^2(\bar{q}_j)}{F'^4(\bar{q}_j)}\right]\right\} &\text{(A.12)}\\ &\equiv \Big[F'(q'')F'(q')\Big]^{-\frac{1}{2}}\int\limits_{q(t')=q'}^{q(t'')=q''}\mathcal{D}q(t)F'\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}F'^2\dot{q}^2 - V(F(q)) - \frac{\hbar^2}{2M}\frac{F''^2}{F'^4}\right]dt\right\} \ . &\text{(A.13)}\end{aligned}$$

Space-Time Transformations.

It is obvious that the path integral representation (A.13) is not completely satisfactory. Whereas the transformed potential $V(F(q))$ may have a convenient form when expressed in the new coordinate q, the kinetic term $\frac{M}{2}F'^2\dot{q}^2$ is in general nasty. Here the so-called "time transformation" comes into play which leads in combination with the "space transformation" already carried out to general "space-time transformations" in path integrals. The time transformation is implemented [42, 88, 107, 131, 166, 179, 183] by introducing a new "pseudo-time" s''. In order to do this, one first makes use of the operator identity

$$\frac{1}{H-E} = f_r(x)\frac{1}{f_l(x)(H-E)f_r(x)}f_l(x) \ , \tag{A.14}$$

where H is the Hamiltonian corresponding to the path integral $K(T)$, and $f_{l,r}(x)$ are functions in q and t, multiplying from the left or from the right, respectively, onto the operator $(H-E)$. Secondly, one introduces a new pseudo-time s'' and assumes that the constraint

$$\int_0^{s''} ds f_l(F(q(s)))f_r(F(q(s))) = T = t''-t' \tag{A.15}$$

has for all admissible paths a unique solution $s''>0$ given by

$$s'' = \int_{t'}^{t''}\frac{dt}{f_l(x)f_r(x)} = \int_{t'}^{t''}\frac{ds}{F'^2(q(s))} \ . \tag{A.16}$$

Here one has made the choice $f_l(F(q(s))) = f_r(F(q(s))) = F'(q(s)))$ in order that in the final result the metric coefficient in the kinetic energy term is equal to one. A convenient way to derive the corresponding transformation formulæ uses the energy dependent Green function $G(E)$ of the kernel $K(T)$ defined by

$$G(q'',q';E) = \left\langle q''\left|\frac{1}{H-E-\mathrm{i}\epsilon}\right|q'\right\rangle = \frac{\mathrm{i}}{\hbar}\int_0^\infty dT \mathrm{e}^{\mathrm{i}(E+\mathrm{i}\epsilon)T/\hbar}K(q'',q';T) \ , \tag{A.17}$$

where a small positive imaginary part ($\epsilon>0$) has been added to the energy E. (Very often we shall not explicitly write the $\mathrm{i}\epsilon$, but will tacitly assume that the various expressions are regularized according to this rule). For the path integral (A.13) one obtains the following transformation formula

$$K(x'',x';T) = \int_{\mathrm{I\!R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G(q'',q';E) \ , \tag{A.18}$$

$$G(q'',q';E) = \frac{\mathrm{i}}{\hbar}\Big[F'(q'')F'(q')\Big]^{1/2}\int_0^\infty ds''\hat{K}(q'',q';s'') \ , \tag{A.19}$$

with the transformed path integral $\hat{K}$ given by

$$\begin{aligned}\hat{K}(q'',q';s'') &= \lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\epsilon\hbar}\right)^{1/2}\prod_{k=1}^{N-1}\int dq_k \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\epsilon}(\Delta q_j)^2 - \epsilon F'^2(\bar{q}_j)\Big(V(F(\bar{q}_j))-E\Big) - \epsilon\Delta V(\bar{q}_j)\right]\right\} \qquad \text{(A.20)} \\ &\equiv \int\limits_{q(0)=q'}^{q(s'')=q''}\mathcal{D}q(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{M}{2}\dot{q}^2 - F'^2(q)\Big(V(F(q)-E\Big) - \Delta V(q)\right]ds\right\} \qquad \text{(A.21)}\end{aligned}$$

with the quantum potential ΔV given by

$$\Delta V(q)=\frac{\hbar^2}{8M}\left(3\frac{F''^2}{F'^2}-2\frac{F'''}{F'}\right)\ . \tag{A.22}$$

Note that ΔV has the form of a Schwarz derivative of F. A rigorous lattice derivation is far from being trivial and has been discussed by some authors. Recent attempts to put it on a sound footing can be found in Castrigiano and Stärk [23], Fischer et al. [47, 48] and Young and DeWitt-Morette [195].

A.3 Separation of Variables.

Separation Formula for the Path Integral.

By the same technique also the separation of variables in path integrals can be stated, c.f. [66]. Let us consider a $D=d+d'$ dimensional system, where $\mathbf{x}$ represents the d-dimensional coordinate and $\mathbf{z}$ the d'-dimensional coordinate. For simplicity we consider the special case where the metric tensor for the $\mathbf{x}$ coordinates is equal to $f^2(\mathbf{z})\mathbb{1}$, and the metric tensor for the $\mathbf{z}$ coordinates is diagonal and denoted by $\mathbf{g}=\mathbf{g}(\mathbf{z})$ with elements $g_i=g_{ii}(\mathbf{z})$, $i=1,\dots,d'$. Furthermore, we incorporate a potential of the special form $\hat W(\mathbf{x},\mathbf{z})=W(\mathbf{z})+V(\mathbf{x})/f^2(\mathbf{z})$ which include all quantum potentials arising from metric terms. Then ($g=\prod g_i^2$)

$$\begin{aligned}
&\int\limits_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''}\mathcal{D}\mathbf{z}(t)f^d(\mathbf{z})\sqrt{g}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\Big((\mathbf{g}\cdot\dot{\mathbf{z}})^2+f^2(\mathbf{z})\dot{\mathbf{x}}^2\Big)-\left(\frac{V(\mathbf{x})}{f^2(\mathbf{z})}+W(\mathbf{z})\right)\right]dt\right\}\\
&=[f(\mathbf{z}')f(\mathbf{z}'')]^{-d/2}\int dE_\lambda\Psi_\lambda^*(\mathbf{x}')\Psi_\lambda(\mathbf{x}'')\int\limits_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''}\mathcal{D}\mathbf{z}(t)\sqrt{g}\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}(\mathbf{g}\cdot\dot{\mathbf{z}})^2-W(\mathbf{z})-\frac{E_\lambda}{f^2(\mathbf{z})}\right]dt\right\}\ .
\end{aligned}\tag{A.23}$$

Here we assume that the d-dimensional $\mathbf{x}$-path integration has the special representation

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{\mathbf{x}}^2-V(\mathbf{x})\right]dt\right\}=\int dE_\lambda\Psi_\lambda^*(\mathbf{x}')\Psi_\lambda(\mathbf{x}'')\mathrm{e}^{-\mathrm{i}E_\lambda T/\hbar}\ . \tag{A.24}$$

We make frequently use of (A.23).

A.4 Path Integral Identity for the Pöschl-Teller Potential.

As we shall see, we encounter particularly in the case of the Higgs oscillator, the Pöschl-Teller and the modified Pöschl-Teller potential in our path integral problems. The path integral solution of the Pöschl-Teller potential reads as follows (Böhm and Junker [13], Duru [40], [75, 91, 92], Fischer et al. [47], Inomata et al. [108], Kleinert and Mustapic [132], $0<x<\pi/2$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{x}^2-\frac{\hbar^2}{2M}\left(\frac{\alpha^2-\frac14}{\sin^2 x}+\frac{\beta^2-\frac14}{\cos^2 x}\right)\right]dt\right\}$$

$$= \sum_{n\in\mathbb{N}_0} \mathrm{e}^{-\mathrm{i}E_nT/\hbar}\phi_n^{(\alpha,\beta)}(x')\phi_n^{(\alpha,\beta)}(x'') = \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G_{PT}^{(\alpha,\beta)}(x'',x';E) \ . \tag{A.25}$$

The bound state wave-functions and the energy spectrum are given by

$$\phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2n+1)\frac{n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right]^{1/2} \times(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x) \ , \tag{A.26}$$

$$E_n = \frac{\hbar^2}{2M}(2n+\alpha+\beta+1)^2 \ . \tag{A.27}$$

The $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. The Pöschl-Teller wave-functions $\phi^{(\alpha,\beta)}(x)$ are normalized to unity with respect to the scalar product $\int_0^{\pi/2} |\phi_n^{(\alpha,\beta)}(x)|^2 dx = 1$. The Green function $G_{PT}^{(\alpha,\beta)}(E)$ has the form

$$\begin{aligned}
G_{PT}^{(\alpha,\beta)}(x'',x';E) &= \frac{M}{2\hbar^2}\sqrt{\sin x' \sin x''}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\times\left(\frac{1-\cos 2x'}{2}\cdot\frac{1-\cos 2x''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos 2x'}{2}\cdot\frac{1+\cos 2x''}{2}\right)^{(m_1+m_2)/2}\\
&\times {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1; \frac{1-\cos 2x_<}{2}\right)\\
&\times {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1; \frac{1+\cos 2x_>}{2}\right) \ ,
\end{aligned} \tag{A.28}$$

where $m_{1,2} = \frac{1}{2}(\beta \pm \alpha)$, $L_E = -\frac{1}{2} + \frac{1}{2}\sqrt{2ME}/\hbar$, ${}_2F_1(a,b;c;z)$ is the hypergeometric function, and $x_>, x_<$ denotes the larger, respectively smaller of x', x''.

A.5 Path Integral Identity for the Modified Pöschl-Teller Potential.

The case of the modified Pöschl-Teller potential is given by [13, 48, 75, 91, 92, 108, 132]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{r}^2 - \frac{\hbar^2}{2M}\left(\frac{\kappa^2-\frac{1}{4}}{\sinh^2 r} - \frac{\lambda^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\}$$

$$= \sum_{n=0}^{N_{max}} \mathrm{e}^{-\mathrm{i}E_nT/\hbar}\psi_n^{(\kappa,\lambda)*}(r')\psi_n^{(\kappa,\lambda)}(r'') + \int_0^\infty dp\, \mathrm{e}^{-\mathrm{i}E_pT/\hbar}\psi_p^{(\kappa,\lambda)*}(r')\psi_p^{(\kappa,\lambda)}(r'') \ , \tag{A.29}$$

$$= \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G_{mPT}^{(\kappa,\lambda)}(r'',r';E) \ . \tag{A.30}$$

The bound states have the form

$$\psi_n^{(\kappa,\lambda)}(r) = N_n^{(\kappa,\lambda)}(\sinh r)^{\kappa+1/2}(\cosh r)^{n-\lambda+1/2}{}_2F_1(-n,\lambda-n;1+\kappa;\tanh^2 r) \ , \tag{A.31}$$

$$N_n^{(\kappa,\lambda)} = \frac{1}{\Gamma(1+\kappa)}\left[\frac{2(\lambda-\kappa-2n-1)\Gamma(n+1+\kappa)\Gamma(\lambda-n)}{\Gamma(\lambda-\kappa-n)n!}\right]^{1/2}$$

$$E_n = -\frac{\hbar^2}{2M}(2n+\kappa-\lambda+1)^2 \ . \tag{A.32}$$

Here denote $n = 0, 1, \ldots, N_{max} = [\frac{1}{2}(\lambda-\kappa-1)] \geq 0$, and only a finite number of bound states can exist depending on the strength of the attractive potential trough and the repulsive centrifugal term as well. Here $[x]$ denotes the integer part of the real number x. The continuous states are

$$\psi_p^{(\kappa,\lambda)}(r) = N_p^{(\kappa,\lambda)}(\cosh r)^{\mathrm{i}p}(\tanh r)^{\kappa+1/2}{}_2F_1\bigg(\frac{\lambda+\kappa+1-\mathrm{i}p}{2},\frac{\kappa-\lambda+1-\mathrm{i}p}{2};1+\kappa;\tanh^2 r\bigg) \tag{A.33}$$

$$N_p^{(\kappa,\lambda)} = \frac{1}{\Gamma(1+\kappa)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\bigg(\frac{\lambda+\kappa+1-\mathrm{i}p}{2}\bigg)\Gamma\bigg(\frac{\kappa-\lambda+1-\mathrm{i}p}{2}\bigg)\quad .$$

The Green function $G_{mPT}^{(\kappa,\lambda)}(E)$ has the form

$$\begin{aligned}
G_{mPT}^{(\kappa,\lambda)}(r'',r';E) &= \frac{M}{2\hbar^2}\frac{\Gamma(m_1-L_\lambda)\Gamma(L_\lambda+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\times(\cosh r'\cosh r'')^{-(m_1-m_2)}(\tanh r'\tanh r'')^{m_1+m_2+1/2}\\
&\times{}_2F_1\bigg(-L_\lambda+m_1,L_\lambda+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 r_<}\bigg)\\
&\times{}_2F_1\bigg(-L_\lambda+m_1,L_\lambda+m_1+1;m_1+m_2+1;\tanh^2 r_>\bigg)\quad ,
\end{aligned} \tag{A.34}$$

where we have set $m_{1,2}=\frac{1}{2}(\kappa\pm\sqrt{-2ME}/\hbar)$, $L_\lambda=\frac{1}{2}(\lambda-1)$. We make extensively use of the solutions of the Pöschl-Teller and the modified Pöschl-Teller potential, respectively.

A.6 Path Integral Identity for the Rosen-Morse Potential.

For completeness we cite the path integral solution of the Rosen-Morse potential according to [132]

$$\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\bigg[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg(\frac{M}{2}\dot{x}^2-A\tanh\frac{x}{R}+\frac{B}{\cosh^2\frac{x}{R}}\bigg)dt\bigg]\\
&=\frac{MR}{\hbar^2}\frac{\Gamma(m_1-L_B)\Gamma(L_B+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\quad\times\bigg(\frac{1-\tanh\frac{x'}{R}}{2}\cdot\frac{1-\tanh\frac{x''}{R}}{2}\bigg)^{\frac{m_1-m_2}{2}}\bigg(\frac{1+\tanh\frac{x'}{R}}{2}\cdot\frac{1+\tanh\frac{x''}{R}}{2}\bigg)^{\frac{m_1+m_2}{2}}\\
&\quad\times{}_2F_1\bigg(-L_B+m_1,L_B+m_1+1;m_1+m_2+1;\frac{1+\tanh\frac{x_>}{R}}{2}\bigg)\\
&\quad\times{}_2F_1\bigg(-L_B+m_1,L_B+m_1+1;m_1-m_2+1;\frac{1-\tanh\frac{x_<}{R}}{2}\bigg)
\end{aligned} \tag{A.35}$$

$$=\sum_{n=0}^{N_n}\frac{\Psi_n^*(x')\Psi_n(x'')}{E_n-E}+\sum_\pm\int_0^\infty dp\frac{\Psi_p^{(\pm)*}(x')\Psi_p^{(\pm)}(x'')}{A+\hbar^2p^2/2MR^2-E}\quad . \tag{A.36}$$

Here denote $L_B=-\frac{1}{2}+\frac{1}{2}\sqrt{8MBR^2/\hbar^2+1}$, $m_{1,2}=\sqrt{M/2}\,R(\sqrt{-A-E}\pm\sqrt{A-E})/\hbar$. The wave functions and the energy-spectrum are given by $\Big[s\equiv\sqrt{1+8MBR^2/\hbar^2},\ 0,\ldots,N_n<\frac{1}{2}(s-1)-\sqrt{M|A|R^2/2}/\hbar,\ k_1=\frac{1}{2}(1+s),\ k_2=\frac{1}{2}[1+\frac{1}{2}(s-2n-1)-\frac{2MAR^2}{\hbar(s-2n-1)}]$, note $k_2-\frac{1}{2}>0\Big]$:

$$\begin{aligned}
\Psi_n &= \bigg[\bigg(\frac{1}{R}-\frac{4MAR}{\hbar^2(s-2n-1)^2}\bigg)\frac{(s-2k_2-2n)n!\,\Gamma(s-n)}{\Gamma(s+1-n-2k_2)\Gamma(2k_2+n)}\bigg]^{1/2}\\
&\quad\times 2^{n+(1-s)/2}\bigg(1-\tanh\frac{x}{R}\bigg)^{\frac{1}{2}s-k_2-n}
\end{aligned}$$

$$\times\left(1+\tanh\frac{x}{R}\right)^{k_2-\frac{1}{2}} P_n^{(s-2k_2-2n,2k_2-1)}\left(\tanh\frac{x}{R}\right) \ , \tag{A.37}$$

$$E_n^{(RM)} = -\left[\frac{\hbar^2(s-2n-1)^2}{8MR^2} + \frac{2MA^2R^2}{\hbar^2(s-2n-1)^2}\right] \ . \tag{A.38}$$

The wave functions and the energy-spectrum of the continuous states are given by

$$\begin{aligned}\Psi_p^{(\pm)}(x) &= \frac{1}{\Gamma(1+m_1\pm m_2)}\frac{\sqrt{M\sinh(\pi|m_1\pm m_2|)/2R}}{\hbar|\sin\pi(m_1+L_B)|}\\ &\times\left(\frac{1+\tanh\frac{x}{R}}{2}\right)^{\frac{m_1+m_2}{2}}\left(\frac{1-\tanh\frac{x}{R}}{2}\right)^{\frac{m_1-m_2}{2}}\\ &\times{}_2F_1\left(m_1+L_B+1,m_1-L_B;1+m_1\pm m_2;1\pm\tanh\frac{x}{R}\right) \ .\end{aligned} \tag{A.39}$$

B New Super-Integrable Potential in $\mathbb{R}^3$

In this subsection we discuss an additional minimally super-integrable potential in $\mathbb{R}^3$ which we have not mentioned in [80]. It consists of a Coulomb potential in $S^{(2)}$ as a subsystem, which is maximally super-integrable in $S^{(2)}$, hence minimally super-integrable in $\mathbb{R}^3$. It is separable in spherical and conical coordinates which are listed in the following small table (a complete tabulation of coordinate systems in $\mathbb{R}^3$ can be found in [75, 80]):

Coordinate System	Coordinates
Conical, $r \ge 0$ $\tilde\alpha \in [-K, K]$ $\tilde\beta \in [-2K', 2K']$	$x = r\,\mathrm{sn}(\tilde\alpha,k)\,\mathrm{dn}(\tilde\beta,k')$ $y = r\,\mathrm{cn}(\tilde\alpha,k)\,\mathrm{dn}(\tilde\beta,k')$ $z = r\,\mathrm{dn}(\tilde\alpha,k)\,\mathrm{sn}(\tilde\beta,k')$
Conical II, $r \ge 0$ $\tilde\alpha \in [-K, K]$ $\tilde\beta \in [-2K', 2K']$	$x = r[k'\,\mathrm{sn}(\tilde\alpha,k)\,\mathrm{dn}(\tilde\beta,k') + k\,\mathrm{dn}(\tilde\alpha,k)\,\mathrm{sn}(\tilde\beta,k')]$ $y = r\,\mathrm{cn}(\tilde\alpha,k)\,\mathrm{cn}(\tilde\beta,k')$ $z = r[k'\,\mathrm{dn}(\tilde\alpha,k)\,\mathrm{sn}(\tilde\beta,k') - k\,\mathrm{sn}(\tilde\alpha,k)\,\mathrm{dn}(\tilde\beta,k')]$
Spherical $r \ge 0, \vartheta \in (0,\pi)$ $\varphi \in [0, 2\pi)$	$x = r\sin\vartheta\cos\varphi$ $y = r\sin\vartheta\sin\varphi$ $z = r\cos\vartheta$

B.1 Subsystem of Higgs Oscillator on $S^{(2)}$.

We consider the potential

$$V(\mathbf{x}) = \frac{M}{2}\frac{\omega^2}{x^2+y^2+z^2}\frac{x^2+y^2}{z^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2} + \frac{k_3^2-\frac{1}{4}}{z^2}\right) \ . \tag{B.1}$$

This potential has the proper sub-group structure to be a minimally super-integrable potential in $\mathbb{R}^3$. However, it corresponds to the first minimally super-integrable potential in [80] with $F(r) \equiv 0$, and therefore gives nothing new.

B.2 Subsystem of Coulomb potential on $S^{(2)}$.

We consider the potential

$$V_9(\mathbf{x}) = -\frac{\alpha}{x^2+y^2+z^2}\frac{z}{\sqrt{x^2+y^2}} + \frac{\hbar^2}{4M\sqrt{x^2+y^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{x^2+y^2}+x} + \frac{k_2^2-\frac{1}{4}}{\sqrt{x^2+y^2}-x}\right) \tag{B.2}$$

Spherical ($r>0, \vartheta\in(0,\pi), \varphi\in(0,\pi)$):

$$= \frac{1}{r^2}\left[-\alpha\cot\vartheta + \frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\varphi}{2}} + \frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\varphi}{2}}\right)\right] \tag{B.3}$$

Sphero-Conical II ($r>0, \tilde\alpha\in(0,K), \tilde\beta\in(0,K')$):

$$\begin{aligned} &= \frac{1}{r^2}\Bigg[-\alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta - k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta} + \frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)} \\ &\quad\times\left(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha} + \frac{(k_1^2+k_2^2-\frac{1}{2})k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\right)\Bigg] \ . \end{aligned} \tag{B.4}$$

The constants of motion for V_9 are

$$\left.\begin{aligned} I_1 &= \frac{1}{2M}\mathbf{p}^2 + V_9(\mathrm{u}) \ , \\ I_2 &= \frac{1}{2M}L_3^2 + \frac{1}{2M}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\varphi}{2}} + \frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\varphi}{2}}\right) \ , \\ I_3 &= \frac{1}{2M}\mathbf{L}^2 - \alpha\cot\vartheta + \frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\varphi}{2}} + \frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\varphi}{2}}\right) \ , \end{aligned}\right\} \tag{B.5}$$

and the fourth observable is given by

$$\begin{aligned} I_4 = \frac{1}{2M}(\tfrac{1}{2}\sin 2f\{L_1,L_3\} - \cos 2f L_3^2) - \alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta - k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta} \\ +\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)}\Bigg(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha} \\ +\frac{(k_1^2+k_2^2-\frac{1}{2})k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Bigg) \ . \end{aligned} \tag{B.6}$$

Only the case of spherical coordinates is exactly solvable. We obtain

$$\begin{aligned} &K^{(V_9)}(\mathbf{x}'',\mathbf{x}';T) \\ &= \int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^2\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\Big(\dot r^2 + r^2(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big) + \frac{\alpha}{r^2}\cot\vartheta\right.\right. \\ &\qquad\left.\left. -\frac{\hbar^2}{8Mr^2}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\varphi}{2}} + \frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\varphi}{2}} - 1\right) - 1\right)\right]dt\right\} \end{aligned}$$

$$= \frac{1}{r'r''}\sum_{ml}\Psi_{ml}^{(\alpha)\,*}(\mathbf{s}')\Psi_{ml}^{(\alpha)}(\mathbf{s}'')\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{M}{2}\dot r^2 - \frac{\hbar^2}{2M}\frac{\lambda_3^2-\frac{1}{4}}{r^2}\right)dt\right] \tag{B.7}$$

$$= \frac{1}{\sqrt{r'r''}}\sum_{ml}\Psi_{ml}^{(\alpha)\,*}(\mathbf{s}')\Psi_{ml}^{(\alpha)}(\mathbf{s}'')\int_0^\infty pdp\,\mathrm{e}^{-\mathrm{i}\hbar p^2T/2M}J_{|\lambda_3|}(pr')J_{|\lambda_3|}(pr'') \ , \qquad (\lambda_3^2\geq 0) \ , \tag{B.8}$$

$$= \frac{1}{\sqrt{r'r''}}\sum_{ml}\Psi_{ml}^{(\alpha)\,*}(\mathbf{s}')\Psi_{ml}^{(\alpha)}(\mathbf{s}'')\int_0^\infty \frac{pdp}{\pi^2}\mathrm{e}^{-\mathrm{i}\hbar p^2T/2M}K_{\mathrm{i}|\lambda_3|}(-\mathrm{i}pr'')K_{\mathrm{i}|\lambda_3|}(\mathrm{i}pr') \ , \ (\lambda_3^2<0) \ . \tag{B.9}$$

Here are $\lambda_1 = m + (1 \pm k_1 \pm k_2)/2, \lambda_3^2 = 2ME_l/\hbar^2$ with $E_l = \frac{\hbar^2}{2M}(\tilde{l}^2 - \frac{1}{4}) - M\alpha/2\hbar^2\tilde{l}^2$, and $\tilde{l} = l + \lambda_1 + \frac{1}{2}$. The $\Psi_{ml}^{(\alpha)}(\mathbf{s})$ are the wave-functions of the Coulomb potential on $S^{(2)}$ [36] (in any of the two separating coordinate systems, where however, only in spherical coordinates they are explicitly known). Due to the property of the Coulomb spectrum on $S^{(2)}$ that it can be negative as well as positive we must distinguish between the cases $\lambda_3^2 \geq 0$ and $\lambda_3^2 < 0$. The former case is simple. We just obtain a usual repelling centrifugal potential. The latter case, however, causes the necessity of an analytic continuation due to the strong attractive singularity at the origin. Such terms arise in the treatment of the quantum mechanical motion in pseudo-Euclidean spaces [113, 154], and the corresponding path integral discussions have been done in [75], and c.f. [75] for more details and references therein.

C New Super-Integrable Potential on $S^{(3)}$

In this appendix we discuss two minimally super-integrable potentials in $S^{(3)}$ which we have not mentioned in [81]. They are the Higgs-oscillator and the Coulomb potential in $S^{(2)}$ as subsystems, which are maximally super-integrable in $S^{(2)}2$, hence minimally super-integrable in $S^{(3)}$. Both potentials are separable in spherical and sphero-conical coordinates which are listed in the following small table (a complete tabulation of coordinate systems in $\mathbb{R}^3$ can be found in [75, 81]):

Coordinate System	Coordinates
Sphero-Elliptic	$s_1 = \sin\chi\,\mathrm{sn}(\tilde{\alpha}, k)\,\mathrm{dn}(\tilde{\beta}, k')$
$\chi \in (0, \pi)$	$s_2 = \sin\chi\,\mathrm{cn}(\tilde{\alpha}, k)\,\mathrm{cn}(\tilde{\beta}, k')$
$\tilde{\alpha} \in [-K, K]$	$s_3 = \sin\chi\,\mathrm{dn}(\tilde{\alpha}, k)\,\mathrm{sn}(\tilde{\beta}, k')$
$\tilde{\beta} \in [-2K', 2K']$	$s_4 = \cos\chi$
Sphero-Elliptic rotated	$s_1 = \sin\chi[k'\,\mathrm{sn}(\tilde{\alpha}, k)\,\mathrm{dn}(\tilde{\beta}, k') + k\,\mathrm{dn}(\tilde{\alpha}, k)\,\mathrm{sn}(\tilde{\beta}, k')]$
$\chi \in (0, \pi)$	$s_2 = \sin\chi\,\mathrm{cn}(\tilde{\alpha}, k)\,\mathrm{cn}(\tilde{\beta}, k')$
$\tilde{\alpha} \in [-K, K]$	$s_3 = \sin\chi[k'\,\mathrm{dn}(\tilde{\alpha}, k)\,\mathrm{sn}(\tilde{\beta}, k') - k\,\mathrm{sn}(\tilde{\alpha}, k)\,\mathrm{dn}(\tilde{\beta}, k')]$
$\tilde{\beta} \in [-2K', 2K']$	$s_4 = \cos\chi$
Spherical	$s_1 = \sin\chi\sin\vartheta\cos\varphi$
$\chi \in (0, \pi)$	$s_2 = \sin\chi\sin\vartheta\sin\varphi$
$\vartheta \in (0, \pi)$	$s_3 = \sin\chi\cos\vartheta$
$\varphi \in [0, 2\pi)$	$s_4 = \cos\chi$

C.1 Subsystem of Higgs Oscillator on $S^{(2)}$.

We consider the potential

$$V(\mathbf{s}) = \frac{M}{2}\frac{\omega^2}{s_2^2 + s_3^2 + s_4^2}\frac{s_3^2 + s_4^2}{s_2^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{s_1^2} + \frac{k_2^2 - \frac{1}{4}}{s_2^2} + \frac{k_3^2 - \frac{1}{4}}{s_3^2} + \frac{k_4^2 - \frac{1}{4}}{s_4^2}\right) \ . \qquad \text{(C.1)}$$

It has the proper sup-group structure to be a minimally super-integrable potential on $S^{(3)}$. However, the explicit form in, e.g., spherical coordinates show that V corresponds by an appropriate redefinition of parameters to the generalized Higgs-oscillator on $S^{(3)}$ [81], and is therefore maximally super-integrable.

C.2 Subsystem of Coulomb Potential on $S^{(2)}$.

We consider the potential

$$\begin{aligned} V_7(\mathbf{s}) &= -\frac{\alpha}{s_1^2+s_2^2+s_3^2}\frac{s_3}{\sqrt{s_1^2+s_2^2}} \\ &\quad+\frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_2^2-\frac14}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{k_3^2-\frac14}{\sqrt{s_1^2+s_2^2}-s_1}\right)+\frac{\hbar^2}{2M}\frac{k_4^2-\frac14}{s_4^2} \end{aligned} \tag{C.2}$$

Spherical ($\chi\in(0,\pi),\vartheta\in(0,\pi),\varphi\in(0,\pi)$):

$$=\frac{1}{R^2\sin^2\chi}\left[-\alpha\cot\vartheta+\frac{\hbar^2}{8M\sin\vartheta}\left(\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_3^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)\right]+\frac{\hbar^2}{2MR^2}\frac{k_4^2-\frac14}{\cos^2\chi} \tag{C.3}$$

Sphero-Elliptic rotated ($\chi\in(0,\pi),\tilde\alpha\in(0,K),\tilde\beta\in(0,K')$):

$$\begin{aligned} &=\frac{1}{R^2\sin^2\chi}\Bigg[-\alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta-k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta} \\ &\qquad+\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta)}\Bigg(\frac{(k_1^2+k_2^2-\frac12)k'^2+(k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha} \\ &\qquad\qquad+\frac{(k_1^2+k_2^2-\frac12)k^2+(k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Bigg)\Bigg]+\frac{\hbar^2}{2MR^2}\frac{k_4^2-\frac14}{\cos^2\chi}\ . \end{aligned} \tag{C.4}$$

The constants of motion for V_7 are

$$\left.\begin{aligned} I_1&=\frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{K}^2)+V_7(\mathbf{s})\ ,\\ I_2&=\frac{1}{2M}L_3^2+\frac{1}{2M}\left(\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)\ ,\\ I_3&=\frac{1}{2M}\mathbf{L}^2-\alpha\cot\vartheta+\frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)\ , \end{aligned}\right\} \tag{C.5}$$

and the fourth observable is given by

$$\begin{aligned} I_4&=\frac{1}{2M}(\tfrac12\sin2f\{L_1,L_3\}-\cos2fL_3^2)-\alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta-k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha+k'^2\,\mathrm{cn}^2\tilde\beta} \\ &\quad+\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\alpha+k'^2\,\mathrm{cn}^2\beta)}\Bigg(\frac{(k_1^2+k_2^2-\frac12)k'^2+(k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha} \\ &\qquad+\frac{(k_1^2+k_2^2-\frac12)k^2+(k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Bigg) \end{aligned} \tag{C.6}$$

Only the spherical system can be solved. We have

$$\begin{aligned} &K^{(V_7)}(\mathbf{s}'',\mathbf{s}';T) \\ &=\frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\sin^2\chi\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\Big(\dot\chi^2+\sin^2\chi(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big)+\frac{\alpha}{R^2\sin^2\chi}\cot\vartheta\right.\right. \end{aligned}$$

$$-\frac{\hbar^2}{8MR^2\sin^2\chi\sin^2\vartheta}\left(\frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_3^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)-\frac{\hbar^2}{2MR^2}\frac{k_4^2-\frac14}{\cos^2\chi}\Bigg]dt\Bigg\} \tag{C.7}$$

$$= (R\sin\chi'\sin\chi'')^{-1}\sum_{ml}\Psi_{ml}^{(\alpha)*}(\mathbf{s}')\Psi_{ml}^{(\alpha)}(\mathbf{s}'')$$

$$\times\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot\chi^2-\frac{\hbar^2}{2MR^2}\left(\frac{\lambda_3^2-\frac14}{\sin^2\chi}+\frac{k_4^2-\frac14}{\cos^2\chi}\right)\right]dt\right\} \tag{C.8}$$

$$= (\sin\chi'\sin\chi'')^{-1}\sum_{ml}\Psi_{ml}^{(\alpha)*}(\mathbf{s}')\Psi_{ml}^{(\alpha)}(\mathbf{s}'')\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}E_nT/\hbar}\Phi_n^{(\lambda_3,\pm k_4)*}(\chi';R)\Phi_n^{(\lambda_3,\pm k_4)}(\chi'';R)\ , \tag{C.9}$$

$$E_n=\frac{\hbar^2}{2MR^2}\Big[(2n+\lambda_3+\pm k_4+1)^2-1)\Big]\ . \tag{C.10}$$

Here are $\lambda_1 = m+(1\pm k_1\pm k_2)/2, \lambda_3^2 = 2ME_l/\hbar^2$ with $E_l = \frac{\hbar^2}{2M}(\tilde l^2-\frac14)-M\alpha/2\hbar^2\tilde l^2$, $\tilde l = l+\lambda_1+\frac12$, the wave-functions $\Psi_{ml}^{(\alpha)}(\mathbf{s})$ for the Coulomb problem on $S^{(2)}$ [81] as in Section B.2, and the Pöschl-Teller wave-functions $\Phi_n^{(\lambda_3,\pm k_4)}(\chi;R)$ (A.26) endowed with the factor $R^{-3/2}$. Again we must distinguish between two cases, i.e., $\lambda_3^2\geq 0$ and $\lambda_3^2<0$, respectively. For the former case we have written down the solution. For the latter case, however, the proper analytic continuation is not known. It corresponds to one of the 74 coordinate system solutions on the $O(2,2)$ hyperboloid [115]. We will not dwell on this topic any further here, and the issue will be discussed elsewhere [85].

D List of Super-Integrable Potentials in Spaces of Constant Curvature

We briefly list the super-integrable potentials on the spaces of constant curvature in $\mathbb{R}^2$, $\mathbb{R}^3$ (maximally and minimally super-integrable), on $S^{(2)}$ and $S^{(3)}$ (maximally and minimally super-integrable), and on $\Lambda^{(2)}$ according to [80]–[82], respectively. In $\mathbb{R}^2$ we have four super-integrable potentials, $V_1(\mathbf{x}),\dots,V_4(\mathbf{x})$ ($\mathbf{x}\in\mathbb{R}^2$), in $\mathbb{R}^3$ five maximally super-integrable $V_1(\mathbf{x}),\dots,V_5(\mathbf{x})$, and nine minimally super-integrable $V_1(\mathbf{x}),\dots,V_9(\mathbf{x})$ ($\mathbf{x}\in\mathbb{R}^3$) potentials. On $S^{(2)}$ there are two superintegrable potentials $V_1(\mathbf{s})$ and $V_2(\mathbf{s})$ ($\mathbf{s}\in S^{(2)}$), on $S^{(3)}$ there are three maximally super-integrable $V_1(\mathbf{s}),\dots,V_3(\mathbf{s})$ and four minimally super-integrable potentials $V_4(\mathbf{s}),\dots,V_7(\mathbf{s})$ ($\mathbf{s}\in S^{(3)}$). On $\Lambda^{(2)}$ there are five super-integrable potentials $V_1(\mathrm{u}),\dots,V_5(\mathrm{u})$ ($\mathrm{u}\in\Lambda^{(2)}$). The integrable potentials $V_{8,9,10}(\mathbf{s})$ on $S^{(3)}$ are not taken into account.

For each potential we list the separating coordinate system, for the respective definitions c.f. [80]–[82], where the coordinate systems where an explicit analytic solution can be found are underlined. We also state all the corresponding observables.

They are now the following potentials:

Table D.1: Superintegrable Potentials in $\mathbb{R}^2$

Potential $V(\mathbf{x})$	Coordinate Systems	Observables
$V_1(\mathbf{x}) = \frac{M}{2}\omega^2(x^2+y^2) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)$	<u>Cartesian</u>	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_1(\mathbf{x})$
	<u>Polar</u>	$I_2 = \frac{1}{2M}P_1^2 + \frac{M}{2}\omega^2 x_1^2 + \frac{\hbar^2}{2M}\frac{k_1^2-\frac{1}{4}}{x_1^2}$
	Elliptic	$I_3 = \frac{1}{2M}P_2^2 + \frac{M}{2}\omega^2 x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{x_2^2}$
$V_2(\mathbf{x}) = \frac{M}{2}\omega^2(4x^2+y^2) + k_1 x + \frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{y^2}$	<u>Cartesian</u>	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_2(\mathbf{x})$
	Parabolic	$I_2 = \frac{1}{2M}P_1^2 + 2M\omega^2 x_1^2 + k_1 x_1$
		$I_3 = \frac{1}{2M}P_2^2 + \frac{M}{2}\omega^2 x_2^2 + \frac{\hbar^2}{2M}\frac{k_2^2-\frac{1}{4}}{x_2^2}$
$V_3(\mathbf{x}) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\hbar^2}{4M}\frac{1}{\sqrt{x^2+y^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{x^2+y^2}+x} + \frac{k_2^2-\frac{1}{4}}{\sqrt{x^2+y^2}-x}\right)$	<u>Polar</u>	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_3(\mathbf{x})$
	Elliptic II	$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2(\varphi/2)} + \frac{k_2^2-\frac{1}{4}}{\sin^2(\varphi/2)}\right)$
	<u>Parabolic</u>	$I_3 = \frac{1}{4M}\{L_3,P_2\} + \frac{1}{\xi\eta}\Big[-\alpha(\xi-\eta) + (\frac{1}{2}-2k_1^2)\frac{\eta}{\xi} + (2k_2^2-\frac{1}{2})\frac{\xi}{\eta}\Big]$
$V_4(\mathbf{x}) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta_1\sqrt{\sqrt{x^2+y^2}+x} + \beta_2\sqrt{\sqrt{x^2+y^2}-x}}{2\sqrt{x^2+y^2}}$	<u>Mutually-Parabolic</u>	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_4(\mathbf{x})$
		$I_2 = \frac{1}{4M}\{L_3,P_1\} - \frac{\alpha(\lambda-\mu) + \beta_1\mu\sqrt{\lambda} - \beta_2\lambda\sqrt{\mu}}{\lambda+\mu}$
		$I_3 = \frac{1}{4M}\{L_3,P_2\} - \frac{\alpha(\xi-\eta) + (\beta_1+\beta_2)\eta\sqrt{\xi/2} - (\beta_1+\beta_2)\xi\sqrt{\eta/2}}{\xi+\eta}$

Table D.2: Maximally Superintegrable Potentials in $\mathbb{R}^3$

Potential $V(\mathbf{x})$	Coordinate Systems	Observables
$V_1(\mathbf{x}) = \frac{M}{2}\omega^2\mathbf{x}^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{x^2}+\frac{k_2^2-\frac14}{y^2}+\frac{k_3^2-\frac14}{z^2}\right)$	<u>Cartesian</u> <u>Spherical</u> <u>Circular Polar</u> Circular Elliptic Conical Oblate Spheroidal Prolate Spheroidal Ellipsoidal	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_1(\mathbf{x})$ $I_2 = \frac{1}{2M}P_x^2 + \frac{M}{2}\omega^2x^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{x^2}$ $I_3 = \frac{1}{2M}P_y^2 + \frac{M}{2}\omega^2y^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{y^2}$ $I_4 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_5 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left[\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)+\frac{k_3^2-\frac14}{\cos^2\vartheta}\right]$
$V_2(\mathbf{x}) = \frac{M}{2}\omega^2(x^2+y^2+4z^2) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{x^2}+\frac{k_2^2-\frac14}{y^2}\right)$	<u>Cartesian</u> Parabolic <u>Circular Polar</u> Circular Elliptic	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_2(\mathbf{x})$ $I_2 = \frac{1}{2M}P_x^2 + 2M\omega^2x^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{x^2}$ $I_3 = \frac{1}{2M}P_y^2 + \frac{M}{2}\omega^2y^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{y^2}$ $I_4 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_5 = \frac{1}{4M}(\{L_1,P_y\}-\{P_x,L_2\}) - (\xi-\eta)\left[\frac{M}{2}\xi\eta - \frac{\hbar^2}{2M\xi\eta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\right]$
$V_3(\mathbf{x}) = -\frac{\alpha}{\sqrt{x^2+y^2+z^2}} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{x^2}+\frac{k_2^2-\frac14}{y^2}\right)$	Conical <u>Spherical</u> <u>Parabolic</u> Prolate Spheroidal II	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_3(\mathbf{x})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}L_2^2 + \frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{\tan^2\vartheta\cos^2\varphi}$ $I_4 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_5 = \frac{1}{4M}(\{L_1,P_y\}-\{P_x,L_2\}) + (\xi-\eta)\left[\frac{\hbar^2}{2M\xi\eta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)-\frac{\alpha}{\xi+\eta}\right]$

Table D.2 (cont.): Maximally Superintegrable Potentials in $\mathbb{R}^3$

Potential $V(\mathbf{x})$	Coordinate Systems	Observables
$V_4(\mathbf{x}) = \frac{\hbar^2}{2M}\left(\frac{k_1^2 x}{y^2\sqrt{x^2+y^2}} + \frac{k_2^2-\frac14}{y^2} + \frac{k_3^2-\frac14}{z^2}\right)$	Spherical	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_4(\mathbf{x})$
$= \frac{\hbar^2}{2M}\left[\frac{1}{\sqrt{x^2+y^2}}\left(\frac{\beta_1^2-\frac14}{\sqrt{x^2+y^2}+x} + \frac{\beta_2^2-\frac14}{\sqrt{x^2+y^2}-x}\right) + \frac{k_3^2-\frac14}{z^2}\right]$	Circular Elliptic II	$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_2^2+k_1^2-\frac14}{4\sin^2\frac{\varphi}{2}} + \frac{k_2^2-k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}\right)$
	Circular Parabolic	$I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left[\frac{1}{\sin^2\vartheta}\left(\frac{k_2^2+k_1^2-\frac14}{4\sin^2\frac{\varphi}{2}} + \frac{k_2^2-k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}\right) + \frac{k_3^2-\frac14}{\cos^2\vartheta}\right]$
$\beta_1^2 = \frac12(k_2^2+k_1^2+\frac14),\quad \beta_2^2 = \frac12(k_2^2-k_1^2+\frac14)$	Circular Polar	$I_4 = \frac{1}{2M}P_z^2 + \frac{\hbar^2}{2M}\frac{k_3^2-1/4}{z^2}$
		$I_5 = \frac{1}{4M}\{L_3, P_y\} + \frac{\hbar^2}{2M}\frac{(k_1^2+k_2^2-1/4)\xi^2 + (k_1^2-k_2^2+1/4)\eta^2}{\xi\eta(\xi+\eta)}$
$V_5(\mathbf{x}) = \frac{\hbar^2}{2M}\left(\frac{k_1^2 x}{y^2\sqrt{x^2+y^2}} + \frac{k_2^2-\frac14}{y^2}\right) - k_3 z$	Circular Polar	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_5(\mathbf{x})$
$= \frac{\hbar^2}{2M\sqrt{x^2+y^2}}\left(\frac{\beta_1^2-\frac14}{\sqrt{x^2+y^2}+x} + \frac{\beta_2^2-\frac14}{\sqrt{x^2+y^2}-x}\right) - k_3 z$	Circular Elliptic II	$I_2 = \frac{1}{2M}P_z^2 - k_3 z$
	Circular Parabolic	$I_3 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_2^2+k_1^2-\frac14}{4\sin^2\frac{\varphi}{2}} + \frac{k_2^2-k_1^2-\frac14}{4\cos^2\frac{\varphi}{2}}\right)$
$\beta_1^2 = \frac12(k_2^2+k_1^2+\frac14),\quad \beta_2^2 = \frac12(k_2^2-k_1^2+\frac14)$	Parabolic	$I_4 = \frac{1}{4M}(\{L_1, P_y\} - \{P_x, L_2\}) + \frac{k_3}{2}\xi\eta + \frac{\hbar^2}{2M}\frac{(\xi-\eta)(k_1^2\cos^2\varphi + k_2^2 - 1/4)}{\xi\eta\sin^2\varphi}$
		$I_5 = \frac{1}{4M}\{L_3, P_y\} + \frac{\hbar^2}{2M}\frac{(k_1^2+k_2^2-1/4)\xi^2 + (k_1^2-k_2^2+1/4)\eta^2}{\xi\eta(\xi+\eta)}$

Table D.3: Minimally Superintegrable Potentials in $\mathbb{R}^3$

Potential $V(\mathbf{x})$	Coordinate Systems	Observables
$V_1(\mathbf{x}) = F(r) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2} + \frac{k_3^2-\frac{1}{4}}{z^2}\right)$	<u>Spherical</u> Conical	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_1(\mathbf{x})$ $I_2 = \frac{1}{2M}L_1^2 + \frac{\hbar^2}{2M}\left(\frac{k_2^2-\frac{1}{4}}{\tan^2\vartheta\sin^2\varphi} + \frac{(k_3^2-\frac{1}{4})\sin^2\varphi}{\cot^2\vartheta}\right)$ $I_3 = \frac{1}{2M}L_2^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\tan^2\vartheta\cos^2\varphi} + \frac{(k_2^2-\frac{1}{4})\cos^2\varphi}{\cot^2\vartheta}\right)$ $I_4 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$
$V_2(\mathbf{x}) = \frac{M}{2}\omega^2(x^2+y^2) + F(z) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)$	<u>Cartesian</u> <u>Circular Polar</u> Circular Elliptic	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_2(\mathbf{x}), \quad I_2 = \frac{1}{2M}P_x^2 + \frac{M}{2}\omega^2x^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{x^2}$ $I_3 = \frac{1}{2M}P_y^2 + \frac{M}{2}\omega^2y^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{y^2}$ $I_4 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$
$V_3(\mathbf{x}) = \frac{M}{2}\omega^2(4x^2+y^2) + F(z) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)$	<u>Cartesian</u> Circular Parabolic	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_3(\mathbf{x}), \quad I_2 = \frac{1}{2M}P_x^2 + 2M\omega^2x^2$ $I_3 = \frac{1}{2M}P_y^2 + \frac{M}{2}\omega^2y^2 + \frac{\hbar^2}{2M}\frac{k_1^2-1/4}{y^2}$ $I_4 = \frac{1}{4M}\{L_3, P_y\} - \frac{M}{2}\omega^2\xi\eta(\xi-\eta) + \frac{\hbar^2}{2M}(\xi-\eta)\frac{k_2^2-1/4}{\xi\eta}$
$V_4(\mathbf{x}) = F(z) - \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\hbar^2}{2M}\left(\frac{k_1^2 x}{y^2\sqrt{x^2+y^2}} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)$	<u>Circular Polar</u> Circular Elliptic II <u>Circular Parabolic</u>	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_4(\mathbf{x}), \quad I_2 = \frac{1}{2M}P_z^2 + F(z)$ $I_3 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\frac{k_1^2\cos\varphi + k_2^2 - 1/4}{\sin^2\varphi}$ $I_4 = \frac{1}{4M}\{L_3, P_y\} + \frac{1}{\xi+\eta}\left[\alpha(\xi-\eta) + \frac{\hbar^2}{2M}\left(\eta\frac{k_1^2-k_2^2+1/4}{\xi} + \xi\frac{k_1^2+k_2^2-1/4}{\eta}\right)\right]$
$V_5(\mathbf{x}) = \frac{M}{2}\omega^2(x^2+y^2+z^2) + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{z^2} + \frac{F(y/x)}{x^2+y^2}\right)$	<u>Spherical</u> <u>Circular Polar</u> Oblate Spheroidal Prolate Spheroidal	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_5(\mathbf{x})$ $I_2 = \frac{1}{2M}P_z^2 + \frac{M}{2}\omega^2z^2 + \frac{\hbar^2}{2M}\frac{k_3^2-\frac{1}{4}}{z^2}$ $I_3 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}F(\tan\varphi)$ $I_2 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta} + \frac{F(\tan\varphi)}{\sin^2\vartheta}\right)$

Table D.3 (cont.): Minimally Superintegrable Potentials in $\mathbb{R}^3$

Potential $V(\mathbf{x})$	Coordinate Systems	Observables
$V_6(\mathbf{x}) = \frac{M}{2}\omega^2(x^2+y^2+4z^2) + \frac{\hbar^2}{2M}\frac{F(y/x)}{x^2+y^2}$	Circular Polar Parabolic	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_6(\mathbf{x}), \qquad I_2 = \frac{1}{2M}P_z^2 + 2M\omega^2 z^2$ $I_3 = \frac{1}{2M}L_3^2 + F(\tan\varphi)$ $I_4 = \frac{1}{4M}(\{L_1,P_y\} - \{P_x,L_2\}) - \frac{M}{2}\omega^2\xi\eta(\xi-\eta) + \frac{\hbar^2}{2M}(\xi-\eta)\frac{F(\tan\varphi)}{\xi\eta}$
$V_7(\mathbf{x}) = -\frac{\alpha}{r} + \frac{\hbar^2}{2M(x^2+y^2)}\left(\frac{k_1^2 z}{r} + F\left(\frac{y}{x}\right)\right)$	Spherical Parabolic Prolate Spheroidal II	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_7(\mathbf{x}), \qquad I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\frac{k_1^2\cos\vartheta + F(\tan\varphi)}{\sin^2\vartheta}$ $I_4 = \frac{1}{4M}(\{L_1,P_y\} - \{P_x,L_2\}) - \alpha\frac{\xi-\eta}{\xi+\eta} + \frac{\hbar^2}{2M}\left(\frac{k_1^2(\xi^2-\eta^2)}{\xi\eta(\xi+\eta)} + F(\tan\varphi)\frac{\xi-\eta}{\xi\eta}\right)$
$V_8(\mathbf{x}) = -\frac{\alpha}{\rho} + \sqrt{\frac{2}{\rho}}\left(\beta_1\cos\frac{\phi}{2} + \beta_2\sin\frac{\phi}{2}\right) + F(z)$	Mutually- Circular Parabolic	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_8(\mathbf{x}), \qquad I_2 = \frac{1}{2M}P_z^2 + F(z)$ $I_2 = \frac{1}{4M}\{L_3,P_1\} - \frac{\alpha(\lambda-\mu) + \beta_1\mu\sqrt{\lambda} - \beta_2\lambda\sqrt{\mu}}{\lambda+\mu}$ $I_4 = \frac{1}{4M}\{L_3,P_2\} - \frac{\alpha(\xi-\eta) + (\beta_1+\beta_2)\eta\sqrt{\xi/2} - (\beta_1+\beta_2)\xi\sqrt{\eta/2}}{\xi+\eta}$
$V_9(\mathbf{x}) = -\frac{\alpha}{x^2+y^2+z^2}\frac{z}{\sqrt{x^2+y^2}}$ $+\frac{\hbar^2}{4M\sqrt{x^2+y^2}}\left(\frac{k_1^2-\frac14}{\sqrt{x^2+y^2}+x} + \frac{k_2^2-\frac14}{\sqrt{x^2+y^2}-x}\right)$	Spherical Conical*	$I_1 = \frac{1}{2M}\mathbf{P}^2 + V_9(\mathbf{x}), \qquad I_2 = \frac{1}{2M}L_3^2 + \frac{1}{2M}\left(\frac{k_2^2-\frac14}{\cos^2(\varphi/2)} + \frac{k_1^2-\frac14}{\sin^2(\varphi/2)}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 - \alpha\cot\vartheta + \frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_2^2-\frac14}{\cos^2(\varphi/2)} + \frac{k_1^2-\frac14}{\sin^2(\varphi/2)}\right)$ $I_4 = \frac{1}{2M}\left(\frac12\sin 2f\{L_1,L_3\} - \cos 2f L_3^2\right)$ $+\alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta - k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta}$ $-\frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)}$ $\times\Big((k^2 + k'^2\,\mathrm{cn}^2\tilde\beta)\frac{(k_1^2+k_2^2-\frac12)k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha}$ $+k^2\,\mathrm{sn}^2\tilde\alpha\frac{(k_1^2+k_2^2-\frac12)k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\Big)$

* after appropriate rotation, $\sin^2 f = k^2$.

Table D.4: Superintegrable Potentials on $S^{(2)}$

Potential $V(\mathbf{s})$	Coordinate Systems	Observables
$V_1(\mathbf{s}) = \frac{M}{2}\omega^2 \frac{s_1^2+s_2^2}{s_3^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2} + \frac{k_2^2-\frac{1}{4}}{s_2^2}\right)$ $\lambda^2 = \frac{M^2\omega^2}{\hbar^2}R^4 + \frac{1}{4}$	Spherical Elliptic	$I_1 = \frac{1}{2MR^2}\mathbf{L}^2 + V_1(\mathbf{s})$ $I_2 = \frac{1}{2M}L_3^2 + \left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\left(L_1^2 + k'^2 L_2^2\right) + \frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)}\left[(k_1^2-\tfrac{1}{4})\left(\frac{1}{\mathrm{sn}^2\tilde\alpha} - \frac{k^2}{\mathrm{dn}^2\tilde\beta}\right) + (k_2^2-\tfrac{1}{4})\left(\frac{k'^2}{\mathrm{cn}^2\tilde\alpha} + \frac{k^2}{\mathrm{cn}^2\tilde\beta}\right) - (\lambda^2-\tfrac{1}{4})\left(\frac{k'^2}{\mathrm{dn}^2\tilde\alpha} - \frac{1}{\mathrm{sn}^2\tilde\beta}\right)\right]$
$V_2(\mathbf{s}) = -\frac{\alpha}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}+s_1} + \frac{k_2^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}-s_1}\right)$	Spherical Elliptic*	$I_1 = \frac{1}{2MR^2}\mathbf{L}^2 + V_2(\mathbf{s})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{1}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2(\varphi/2)} + \frac{k_2^2-\frac{1}{4}}{\cos^2(\varphi/2)}\right)$ $I_3 = \frac{1}{2M}\left(\tfrac{1}{2}\sin 2f\{L_1, L_3\} - \cos 2f L_3^2\right) - \alpha R\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta - k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta} + \frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta)} \times\left(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha} + \frac{(k_1^2+k_2^2-\frac{1}{2})k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\right)$

* after appropriate rotation, $\sin^2 f = k^2$.

Table D.5: Maximally Superintegrable Potentials on $S^{(3)}$

Potential $V(\mathbf{s})$	Coordinate Systems	Observables
$V_1(\mathbf{s}) = \frac{M}{2}\omega^2 R^2 \frac{s_1^2+s_2^2+s_3^2}{s_4^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{s_1^2}+\frac{k_2^2-\frac14}{s_2^2}+\frac{k_3^2-\frac14}{s_3^2}\right)$	Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2)+V_1(\mathbf{s})$
	Cylindrical	$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
	Sphero-Elliptic Oblate Elliptic	$I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left[\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\right]$
	Prolate Elliptic Ellipsoidal	$I_4 = \frac{1}{2M}(L_1^2+k'^2L_2^2)+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\operatorname{sn}^2\tilde\alpha\operatorname{dn}^2\tilde\beta}+\frac{k_2^2-\frac14}{\operatorname{cn}^2\tilde\alpha\operatorname{cn}^2\tilde\beta}+\frac{k_3^2-\frac14}{\operatorname{dn}^2\tilde\alpha\operatorname{sn}^2\tilde\beta}\right)$
		$I_5 = \frac{1}{2M}N_3^2 + \frac{M^2\omega^2}{\hbar^2}R^4\tan^2\varphi_2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi_1}+\frac{k_2^2-\frac14}{\sin^2\varphi_1}+\frac{k_3^2-\frac14}{\cos^2\varphi_2}\right)$
$V_2(\mathbf{s}) = -\frac{\alpha}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{s_1^2}+\frac{k_2^2-\frac14}{s_2^2}\right)$ $\mathbf{A} = \frac{1}{2R}(\mathbf{L}\times\tilde{\mathbf{K}}-\tilde{\mathbf{K}}\times\mathbf{L})+\frac{\alpha\mathbf{s}}{\lvert\mathbf{s}\rvert}, \quad \mathbf{s}=(s_1,s_2,s_3)$	Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2)+V_2(\mathbf{s})$
	Sphero-Elliptic	$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
	Prolate Elliptic*	$I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left[\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)\right]$
		$I_4 = \frac{1}{2M}(L_1^2+k'^2L_2^2)-\alpha R\frac{k'\operatorname{sn}\alpha\operatorname{dn}\alpha-k\operatorname{sn}\beta\operatorname{dn}\beta}{k^2\operatorname{cn}^2\alpha+k'^2\operatorname{cn}^2\beta}+\frac{\hbar^2}{2M\operatorname{cn}^2\alpha\operatorname{cn}^2\beta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
		$I_5 = \frac{1}{2M}(\cos 2f + R\sin 2f A_3)+\frac{\hbar^2}{2M\sin^2\chi\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2\varphi}+\frac{k_2^2-\frac14}{\sin^2\varphi}\right)$
$V_3(\mathbf{s}) = \frac{\hbar^2}{2M}\frac{k_3^2-\frac14}{s_3^2} + \frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{k_2^2-\frac14}{\sqrt{s_1^2+s_2^2}-s_1}\right)$	Spherical	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2)+V_2(\mathbf{s})$
	Cylindrical	$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)$
	Sphero-Elliptic*,+	$I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left[\frac{k_3^2-\frac14}{\cos^2\vartheta}+\frac{1}{4\sin^2\vartheta}\left(\frac{k_1^2-\frac14}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac14}{\sin^2(\varphi/2)}\right)\right]$
		$I_4 = \frac{1}{2M}N_3^2 + \frac{\hbar^2}{2M}\left[\frac{k_3^2-\frac14}{\cos^2\varphi_2}+\frac14\left(\frac{k_1^2-\frac14}{\cos^2(\varphi_1/2)}+\frac{k_2^2-\frac14}{\cos^2(\varphi_1/2)}\right)\right]$
		$I_5^+ = \frac{1}{2M}(\frac12\sin 2f\{L_1,L_3\}-\cos 2f L_3^2)+\frac{\hbar^2}{2M(k^2\operatorname{cn}^2\tilde\alpha+k'^2\operatorname{cn}^2\tilde\beta)} \times\left(\frac{(k_1^2+k_2^2-\frac12)k'^2+(k_2^2-k_1^2)k'\operatorname{sn}\tilde\alpha\operatorname{dn}\tilde\alpha}{\operatorname{cn}^2\tilde\alpha}+\frac{(k_1^2+k_2^2-\frac12)k^2+(k_2^2-k_1^2)k\operatorname{sn}\tilde\beta\operatorname{dn}\tilde\beta}{\operatorname{cn}^2\tilde\beta}\right)$

* after appropriate rotation, $\sin^2 f = k^2$; + only for $k_3^2 - 1/4 = 0$, otherwise only minimally superintegrable.

Table D.6: Minimally Superintegrable Potentials on $S^{(3)}$

Potential $V(\mathbf{s})$	Coordinate Systems	Observables
$V_4(\mathbf{s}) = \frac{M}{2}\omega^2 R^2 \frac{s_1^2+s_2^2+s_3^2}{s_4^2} + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{s_3^2} + \frac{F(s_2/s_1)}{s_1^2+s_2^2}\right)$	Spherical Cylindrical Oblate Elliptic Prolate Elliptic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2) + V_4(\mathbf{s}), \quad I_2 = \frac{1}{2M}L_3 + \frac{\hbar^2}{2M}F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{k^2-1/4}{\cos^2\vartheta} + \frac{F(\tan\varphi)}{\sin^2\vartheta}\right)$ $I_4 = \frac{1}{2M}N_3^2 + \frac{M\omega^2R^4}{2\cos^2\vartheta\sin^2\varphi_2} + \frac{\hbar^2}{2M}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\vartheta\cos^2\varphi_2} + \frac{F(\tan\varphi_1)}{\sin^2\vartheta}\right)$
$V_5(\mathbf{s}) = -\frac{\alpha}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}} + \frac{\hbar^2}{2M(s_1^2+s_2^2)}\left[\frac{\beta s_3}{\sqrt{s_1^2+s_2^2+s_3^2}} + F\left(\frac{s_2}{s_1}\right)\right]$	Spherical Prolate Elliptic II*	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2) + V_5(\mathbf{s}), \quad I_2 = \frac{1}{2M}L_3 + \frac{\hbar^2}{2M}F(\tan\varphi)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\frac{\beta\cos\vartheta + F(\tan\varphi)}{\sin^2\vartheta}$ $I_4 = \frac{1}{2M}[\cos 2f\mathbf{L}^2 - \frac{1}{2}\sin 2f(\{L_2,K_1\} - \{L_1,K_2\})] + V_5(\alpha,\beta,\varphi)$
$V_6(\mathbf{s}) = F(\chi) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2} + \frac{k_2^2-\frac{1}{4}}{s_2^2} + \frac{k_3^2-\frac{1}{4}}{s_3^2} + \frac{k_4^2-\frac{1}{4}}{s_4^2}\right)$	Spherical Sphero-Elliptic	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2) + V_6(\mathbf{s}), \quad I_2 = \frac{1}{2M}L_3 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 + \frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta} + \frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)\right)$ $I_4 = \frac{1}{2M}(L_1^2 + k'^2 L_2^2) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\tilde\alpha\,\mathrm{dn}^2\tilde\beta} + \frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\tilde\alpha\,\mathrm{cn}^2\tilde\beta} + \frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\tilde\alpha\,\mathrm{sn}^2\tilde\beta}\right)$
$V_7(\mathbf{s}) = -\frac{\alpha}{s_1^2+s_2^2+s_3^2}\frac{s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\hbar^2}{2M}\frac{k_4^2-\frac{1}{4}}{s_4^2} + \frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_2^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}+s_1} + \frac{k_3^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}-s_1}\right)$	Spherical Sphero-Elliptic*	$I_1 = \frac{1}{2MR^2}(\mathbf{L}^2+\mathbf{N}^2) + V_7(\mathbf{s}), \quad I_2 = \frac{1}{2M}L_3^2 + \frac{1}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2-\frac{1}{4}}{\sin^2\frac{\varphi}{2}}\right)$ $I_3 = \frac{1}{2M}\mathbf{L}^2 - \alpha\cot\vartheta + \frac{\hbar^2}{8M\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2(\varphi/2)} + \frac{k_2^2-\frac{1}{4}}{\sin^2(\varphi/2)}\right)$ $I_4 = \frac{1}{2M}(\frac{1}{2}\sin 2f\{L_1,L_3\} - \cos 2f L_3^2) + \alpha\frac{k'\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta - k\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{k^2\,\mathrm{cn}^2\tilde\alpha + k'^2\,\mathrm{cn}^2\tilde\beta} - \frac{\hbar^2}{2M(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)}\left(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\tilde\alpha\,\mathrm{dn}\tilde\alpha}{\mathrm{cn}^2\tilde\alpha} + \frac{(k_1^2+k_2^2-\frac{1}{2})k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\tilde\beta\,\mathrm{dn}\tilde\beta}{\mathrm{cn}^2\tilde\beta}\right)$

* after appropriate rotation, $\sin^2 f = k^2$.

Table D.7: Superintegrable Potentials on the Two-Dimensional Hyperboloid

Potential $V(\mathrm{u})$	Coordinate System	Observables
$V_1(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{u_1^2} + \frac{k_2^2-\frac14}{u_2^2}\right)$	Spherical Equidistant Elliptic Hyperbolic	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2) + V_1(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{\cos^2\varphi} + \frac{k_2^2-\frac14}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}K_2^2 - \frac{M}{2}\frac{\omega^2 R^4}{\cosh^2\tau_2} + \frac{\hbar^2}{2M}\frac{k_1^2-\frac14}{\sinh^2\tau_2}$
$V_2(\mathrm{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2+u_2^2}} - 1\right) + \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{u_1^2+u_2^2}+u_1} + \frac{k_2^2-\frac14}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Spherical Elliptic-Parabolic Elliptic* Semi-Hyperbolic	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2) + V_2(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac14}{\sin^2\frac{\varphi}{2}} + \frac{k_2^2-\frac14}{\cos^2\frac{\varphi}{2}}\right)$ $I_3 = \frac{1}{2M}\{K_1,L_3\} - \alpha R\frac{\sqrt{1+\mu_1^2}+\sqrt{1+\mu_2^2}}{\mu_1+\mu_2} + \frac{\hbar^2}{4MR^2(\mu_1+\mu_2)}$ $\times\left[(k_1^2+k_2^2-\frac12)\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) + (k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1^2}}{\mu_1} - \frac{\sqrt{1+\mu_2^2}}{\mu_2}\right)\right]$
$V_3(\mathrm{u}) = \frac{\alpha}{(u_0-u_1)^2} + \frac{M}{2}\omega^2\frac{R^2+4u_2^2}{(u_0-u_1)^4} - \lambda\frac{u_2}{(u_0-u_1)^3}$	Horicyclic Semi-Circular-Parabolic	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2) + V_3(\mathrm{u})$ $I_2 = \frac{1}{2M}(K_1-L_3)^2 + \alpha + 2M\omega^2x^2 - \lambda x$ $I_3 = \frac{1}{2M}(\{K_1,K_2\} - \{K_2,L_3\})$ $+\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left[\alpha(\xi^2+\eta^2) + \frac{\lambda}{2}(\xi^4-\eta^4) + \frac{M}{2}(\xi^6+\eta^6)\right]$
$V_4(\mathrm{u}) = \frac{M}{2}\frac{\omega^2}{(u_0-u_1)^2} + \frac{\hbar^2}{2M}\frac{\kappa^2-\frac14}{u_2^2}$ $\lvert\kappa\rvert = 1/2$ $\omega = 0$	Equidistant Horicyclic Elliptic-Parabolic Hyperbolic-Parabolic Semi-Circular-Parabolic all systems except IX.	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2) + V_4(\mathrm{u})$ $I_2 = \frac{1}{2M}(K_1-L_3)^2 + \frac{\hbar^2}{2M}\frac{\kappa^2-\frac14}{x^2}$ $I_3 = \frac{1}{2M}K_2^2 + \frac{M}{2}\omega^2 e^{2\tau_2}$
$V_5(\mathrm{u}) = \alpha R\frac{u_2}{\sqrt{u_0^2-u_1^2}}$	Equidistant Semi-Circular-Parabolic	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2) + V_5(\mathrm{u})$ $I_2 = \frac{1}{2M}(\{K_1,K_2\} - \{K_2,L_3\}) + \alpha R\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right)$ $I_3 = K_2^2$

* after appropriate rotation, $\sin^2 f = k^2$.

References

[1] Y.Aharonov and D.Bohm: Significance of Electromagnetic Potentials in the Quantum Theory. *Phys.Rev.* **115** (1959) 485–491.

[2] H.H.Aly and R.M.Spector: Some Solvable Potentials for the Schrödinger Equation. *Nuovo Cimento* **38** (1965) 149–152.

[3] O.Babelon and M.Talon: Separation of Variables for the Classical and Neumann Model. *Nucl.Phys.* **B 379** (1992) 321–339.

[4] M.Bander and C.Itzykson: Group Theory and the Hydrogen Atom (I,II). *Rev.Mod.Phys.* **38** (1968) 330–345 and 346–358.

[5] V.Bargmann: Zur Theorie des Wasserstoffatoms. Bemerkungen zur gleichnamigen Arbeit von V.Fock. *Zeitschr.Phys.* **99** (1936) 576–582, reprinted in [6], 411–417.

[6] A.O.Barut, A.Bohm and Y.Ne'eman and (Eds.): Dynamical Groups and Spectrum Generating Algebras Vol.I (*Academic Press*, New York, 1987).

[7] A.O.Barut, A.Inomata and G.Junker: Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature. *J.Phys.A: Math.Gen.* **20** (1987) 6271–6280.

[8] A.O.Barut, A.Inomata and G.Junker: Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature II. *J.Phys.A: Math.Gen.* **23** (1990) 1179–1190.

[9] A.O.Barut, C.K.E.Schneider and R.Wilson: Quantum Theory of Infinite Component Fields. *J.Math.Phys.* **20** (1979) 2244–2256.

[10] C.C.Bernido Path Integral Evaluation of a Charged Particle Interacting With a Kaluza-Klein Monopole. *Nucl.Phys.* **B 321** (1979) 108–120.

[11] G.Bessis and N.Bessis: Electronic Wavefunctions in a Space of Constant Curvature. *J.Phys.A: Math.Gen.* **12** (1979) 1991–1997

[12] M.Böhm and G.Junker: The $SU(1,1)$ Propagator as a Path Integral Over Noncompact Groups. *Phys.Lett.* **A 117** (1986) 375–380.

[13] M.Böhm and G.Junker: Path Integration Over Compact and Noncompact Rotation Groups. *J.Math.Phys.* **28** (1987) 1978–1994.

[14] D.Bonatos, C.Daskaloyannis and K.Kokkotas: Quantum-Algebraic Description of Quantum Superintegrable Systems in Two Dimensions. *Phys.Rev.* **A 48** (1993) R3407–R3410.

[15] A.K.Bose: A Class of Solvable Potentials. *Nuovo Cimento* **32** (1964) 679–688.

[16] C.P.Boyer, E.G.Kalnins and W.Miller, Jr.: Symmetry and Separation of Variables for the Helmholtz and Laplace Equations. *Nagoya Math.J.* **60** (1976) 35–80.

[17] P.A.Brown and E.A.Solov'ev: The Stark Effect for the Hydrogen Atom in a Magnetic Field. *Sov.Phys.JETP* **59** (1984) 38–46.

[18] M.V.Carpio-Bernido: Path Integral Quantization of Certain Noncentral Systems with Dynamical Symmetries. *J.Math.Phys.* **32** (1991) 1799–1807.

[19] M.V.Carpio-Bernido: Green Function for an Axially Symmetric Potential Field: A Path Integral Evaluation in Polar Coordinates. *J.Phys.A: Math.Gen.* **24** (1991) 3013–3019.

[20] M.V.Carpio-Bernido and C.C.Bernido: An Exact Solution of a Ring-Shaped Oscillator Plus a $c\sec^2\theta/r^2$ Potential. *Phys.Lett.* **A 134** (1989) 395–399. Algebraic Treatment of a Double Ring-Shaped Oscillator. *Phys.Lett.* **A 137** (1989) 1–3.

[21] M.V.Carpio-Bernido, C.C.Bernido and A.Inomata: Exact Path Integral Treatment of Two Classes of Axially Symmetric Potentials. In the *Proceedings of the Third International Conference on "Path Integrals From meV to MeV"*, Bangkok 1989; eds.: V.Sa-yakanit, W.Sritrakool, J.-O. Berananda, M.C.Gutzwiller, A.Inomata, S.Lundqvist, J.R.Klauder and L.S.Schulman (*World Scientific*, Singapore, 1989), 442–459.

[22] M.V.Carpio-Bernido and A.Inomata: Path Integral Treatment of the Hartmann Potential. In "Path Integrals From meV to MeV", Bielefeld 1985; eds.: M.C.Gutzwiller, A.Inomata, J.R.Klauder and L.Streit (*World Scientific*, Singapore, 1986), 261–270.

[23] D.P.L.Castrigiano and F.Stärk: New Aspects of the Path Integrational Treatment of the Coulomb Potential. *J.Math.Phys.* **30** (1989) 2785–2788.

[24] L.Chetouani, L.Guechi and T.F.Hammann: Exact Path Integral for the Ring Potential. *Phys.Lett.* **A 125** (1987) 277–281. Algebraic Treatment of a General Noncentral Potential. *J.Math.Phys.* **33** (1992) 3410–3418.

[25] L.Chetouani, L.Guechi and T.F.Hammann: Exact Path Integral Solution of the Coulomb Plus Aharonov-Bohm Potential. *J.Math.Phys.* **30** (1989) 655–658.

[26] L.Chetouani and T.F.Hammann: Coulomb Green's Function, in a n-Dimensional Euclidean Space. *J.Math.Phys.* **27** (1986) 2944–2948.

[27] L.Chetouani and T.F.Hammann: Traitement Exact des Système Coulombiens, dans le Formalisme des Intégrales de Feynman, en Coordonnées Parabolique. *Nuovo Cimento* **B 98** (1987) 1–24.

[28] A.Cisneros and H.V.McIntosh: Symmetry of the Two-Dimensional Hydrogen Atom. *J.Math.Phys.* **10** (1969) 277–286.

[29] B.Cordani, G.Fehér and P.A.Horvàthy: O(4,2) Dynamical Symmetry of the Kaluza-Klein Monopole. *Phys.Lett.* **B 201** (1988) 481–486.

[30] C.A.Coulson and A.Joseph: A Constant of the Motion for the Two-Centre Kepler Problem. *Int.J. Quantum Chem.* **1** (1967) 337–347.

[31] C.A.Coulson and P.D.Robinson: Wave Functions for the Hydrogen Atom in Spheroidal Coordinates I: The Derivation and Properties of the Functions. *Proc.Phys.Soc.* **71** (1958) 815–827.

[32] C.Dane and Y.A.Verdiyev: Integrable Systems of Group SO(1,2) and Green's Functions. *J.Math. Phys.* **37** (1996) 39–60.

[33] L.S.Davtyan, L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan: Hidden Symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *J.Phys.A: Math.Gen.* **18** (1985) 455–466.

[34] L.S.Davtyan, L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan: Generalised KS Transformation: From Five-Dimensional Hydrogen Atom to Eight-Dimensional Isotrope Oscillator. *J.Phys.A: Math.Gen.* **20** (1987) 6121–6125.

[35] L.S.Davtyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan: Two-Dimensional Hydrogen Atom. Reciprocal Expansions of the Polar and Parabolic Bases of the Continuous Spectrum. *Theor.Math.Phys.* **66** (1986) 146–153. On the Hidden Symmetry of a One-Dimensional Hydrogen Atom: *J.Phys.A: Math.Gen.* **20** (1987) 2765–2772.

[36] L.S.Davtyan, G.S.Pogosyan, A.N.Sisakyan and V.M.Ter-Antonyan: Transformations Between Parabolic Bases of the Two-Dimensional Hydrogen Atom in the Continuous Spectrum. *Theor. Math.Phys.* **74** (1988) 157–161.
L.G.Mardoyan, G.S.Pogosyan, A.N.Sisakyan and V.M.Ter-Antonyan: Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *Theor.Math.Phys.* **63** (1985) 597–603. Spheroidal Corrections to the Spherical and Parabolic Bases of the Hydrogen Atom. *Theor. Math.Phys.* **64** (1985) 762–764. Elliptic Basis for a Circular Oscillator. *Theor.Math.Phys.* **65** (1985) 1113–1122.

[37] Yu.N.Demkov: Symmetry Group of the Isotropic Oscillator. *Sov.Phys.JETP* **26** (1954) 757. Ibidem **36** (1959) 63–66.

[38] B.S.DeWitt: Dynamical Theory in Curved Spaces. I. A Review of the Classical and Quantum Action Principles. *Rev.Mod.Phys.* **29** (1957) 377–397.

[39] J.C.D'Olivo and M.Torres: The Canonical Formalism and Path Integrals in Curved Spaces. *J.Phys.A: Math.Gen.* **21** (1988) 3355–3363.

[40] I.H.Duru: Path Integrals Over SU(2) Manifold and Related Potentials. *Phys.Rev.* **D 30** (1984) 2121–2127.

[41] I.H.Duru: On the Path Integral for the Potential $V = ar^{-2}+br^2$. *Phys.Lett.* **A 112** (1985) 421–423.

[42] I.H.Duru and H.Kleinert: Solution of the Path Integral for the H-Atom. *Phys.Lett.* **B 84** (1979) 185–188. Quantum Mechanics of H-Atoms From Path Integrals. *Fortschr.Phys.* **30** (1982) 401–435.

[43] L.P.Eisenhart: Enumeration of Potentials for Which One-Particle Schroedinger Equations Are Separable. *Phys.Rev.* **74** (1948) 87–89. The Selberg Zeta-Function for Compact Discrete Subgroups of PSL$(2,\mathbb{C})$. *Banach Center Publications* **17** (1985) 83–120. Zeta-Functions of Binary Hermitian Forms and Special Values of Eisenstein Series on Three-Dimensional Hyperbolic Spaces. *Math.Ann.* **277** (1987) 655–708.

[44] N.W.Evans: Superintegrability in Classical Mechanics. *Phys.Rev.* **A 41** (1990) 5666–5676. Group Theory of the Smorodinsky-Winternitz System. *J.Math.Phys.* **32** (1991) 3369–3375. Super-Integrability of the Winternitz System. *Phys.Lett.* **A 147** (1990) 483–486.

[45] R.P.Feynman: Space-Time Approach to Non-Relativistic Quantum Mechanics. *Rev.Mod.Phys.* **20** (1948) 367–387.

[46] R.P.Feynman and A.Hibbs: Quantum Mechanics and Path Integrals (*McGraw Hill*, New York, 1965).

[47] W.Fischer, H.Leschke and P.Müller: Changing Dimension and Time: Two Well-Founded and Practical Techniques for Path Integration in Quantum Physics. *J.Phys.A: Math.Gen.* **25** (1992) 3835–3853.

[48] W.Fischer, H.Leschke and P.Müller: Path Integration in Quantum Physics by Changing the Drift of the Underlying Diffusion Process: Application of Legendre Processes. *Ann.Phys.(N.Y.)* **227** (1993) 206–221.

[49] V.Fock: Zur Theorie des Wasserstoffatoms. *Zeitschr.Phys.* **98** (1935) 145–154, reprinted in [6], 400–410.

[50] A.Frank and K.B.Wolf: Lie Algebras for Systems With Mixed Spectra.I. The Scattering Pöschl-Teller Potential. *J.Math.Phys.* **25** (1985) 973–983.

[51] J.Friš, V.Mandrosov, Ya.A.Smorodinsky, M.Uhlir and P.Winternitz: On Higher Symmetries in Quantum Mechanics. *Phys.Lett.* **16** (1965) 354–356.
J.Friš, Ya.A.Smorodinskiĭ, M.Uhlíř and P.Winternitz: Symmetry Groups in Classical and Quantum Mechanics. *Sov.J.Nucl.Phys.* **4** (1967) 444–450.

[52] O.F.Gal'bert, Ya.I.Granovskii and A.S.Zhedanov: Dynamical Symmetry of Anisotropic Singular Oscillator. *Phys.Lett.* **A 153** (1991) 177–180.

[53] C.C.Gerry: Dynamical Group for a Ring Potential. *Phys.Lett.* **A 118** (1986) 445–447.

[54] J.-L.Gervais and A.Jevicki: Point Canonical Transformations in the Path Integral. *Nucl.Phys.* **B 110** (1976) 93–112.

[55] G.W.Gibbons and N.S.Manton: Classical and Quantum Dynamics of the BPS Monopoles. *Nucl.Phys.* **B 274** (1986) 183–224.

[56] M.J.Goovaerts: Path-Integral Evaluation of a Nonstationary Calogero Model. *J.Math.Phys.* **16** (1975) 720–723.

[57] I.S.Gradshteyn and I.M.Ryzhik: Table of Integrals, Series, and Products (*Academic Press*, New York, 1980).

[58] Ya.A.Granovsky, A.S.Zhedanov and I.M.Lutzenko: Quadratic Algebra as a 'Hidden' Symmetry of the Hartmann Potential. *J.Phys.A: Math.Gen.* **24** (1991) 3887–3894.

[59] Ya.A.Granovsky, A.S.Zhedanov and I.M.Lutzenko: Quadratic Algebras and Dynamics in Curved Spaces. I. Oscillator. *Theor.Math.Phys.* **91** (1992) 474–480.

[60] Ya.A.Granovsky, A.S.Zhedanov and I.M.Lutzenko: Quadratic Algebras and Dynamics in Curved Spaces. II. The Kepler Problem. *Theor.Math.Phys.* **91** (1992) 604–612.

[61] C.Grosche: The Product Form for Path Integrals on Curved Manifolds. *Phys.Lett.* **A 128** (1988) 113-122.

[62] C.Grosche: Path Integral Solution of a Class of Potentials Related to the Pöschl-Teller Potential. *J.Phys.A: Math.Gen.* **22** (1989) 5073–5087.

[63] C.Grosche: The Path Integral on the Poincaré Disc, the Poincaré Upper Half-Plane and on the Hyperbolic Strip. *Fortschr.Phys.* **38** (1990) 531–569.

[64] C.Grosche: Path Integration on the Hyperbolic Plane with a Magnetic Field. *Ann.Phys.(N.Y.)* **201** (1990) 258–284.

[65] C.Grosche: The Path Integral for the Kepler Problem on the Pseudosphere. *Ann.Phys.(N.Y.)* **204** (1990) 208–222.

[66] C.Grosche: Separation of Variables in Path Integrals and Path Integral Solution of Two Potentials on the Poincaré Upper Half-Plane. *J.Phys.A: Math.Gen.* **23** (1990) 4885–4901.

[67] C.Grosche: Kaluza-Klein Monopole System in Parabolic Coordinates by Functional Integration. *J.Phys.A: Math.Gen.* **24** (1991) 1771–1783.

[68] C.Grosche: Coulomb Potentials by Path-Integration. *Fortschr.Phys.* **40** (1992) 695–737.

[69] C.Grosche: Path Integral Solution of a Non-Isotropic Coulomb-Like Potential. *Phys.Lett.* **A 165** (1992) 185–190.

[70] C.Grosche: Path Integration on Hyperbolic Spaces. *J.Phys.A: Math.Gen.* **25** (1992) 4211–4244.

[71] C.Grosche: Path Integral Solution of Two Potentials Related to the $SO(2,1)$ Dynamical Algebra. *J.Phys.A: Math.Gen.* **26** (1993) L279–L287.

[72] C.Grosche: Path Integral Solution of Scarf-Like Potentials. *Nuovo Cimento* **B 108** (1993) 1365–1376.

[73] C.Grosche: On the Path Integral in Imaginary Lobachevsky Space. *J.Phys.A: Math. Gen.* **27** (1994) 3475–3489.

[74] C.Grosche: Path Integration and Separation of Variables in Spaces of Constant Curvature in Two and Three Dimensions. *Fortschr.Phys.* **42** (1994) 509–584.

[75] C.Grosche: Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulæ (*World Scientific*, Singapore, 1996).

[76] C.Grosche: Conditionally Solvable Path Integral Problems. *J.Phys.A: Math.Gen.* **28** (1995) 5889–5902. Conditionally Solvable Path Integral Problems II: Natanzon Potentials. *J.Phys.A: Math.Gen.* **29** (1996) 365–383.

[77] C.Grosche: The General Besselian and Legendrian Path Integral. *J.Phys.A: Math.Gen.* **29** (1996) L183–L189.

[78] C.Grosche, Kh.Karayan, G.S.Pogosyan and A.N.Sissakian: Quantum Motion on the Three-Dimensional Sphere: The Ellipso-Cylindrical Bases. *DESY Report*, DESY 95–218, November 1995, 28pp.

[79] C.Grosche, Kh.Karayan, G.S.Pogosyan and A.N.Sissakian: Quantum Motion on the Three-Dimensional Sphere: Interbasis Expansion of the Ellipsoidal Basis. *DESY Report*, in preparation.
R.Airapetyan, Kh.Karayan, G.S.Pogosyan and A.N.Sissakian: Quantum Motion on the Three-Dimensional Sphere: The Ellipsoidal Basis. In the *Proceedings of the International Workshop on Finite Dimensional Integrable Systems*, Dubna 1995; eds.: G.S.Pogosyan and A.N.Sissakian. (*JINR Publications*, Dubna, 1996), 1–11.

[80] C.Grosche, G.S.Pogosyan and A.N.Sissakian: Path Integral Discussion for Smorodinsky-Winternitz Potentials: I. Two- and Three-Dimensional Euclidean Space. *Fortschr.Phys.* **43** (1995) 453–521.

[81] C.Grosche, G.S.Pogosyan and A.N.Sissakian: Path Integral Discussion for Smorodinsky-Winternitz Potentials: II. The Two- and Three-Dimensional Sphere. *Fortschr.Phys.* **43** (1995) 523–563.

[82] C.Grosche, G.S.Pogosyan and A.N.Sissakian: Path-Integral Approach to Superintegrable Potentials on the Two-Dimensional Hyperboloid. *Phys.Part.Nucl.* **27** (1996) 244-278.

[83] C.Grosche, G.S.Pogosyan and A.N.Sissakian: On the Interbasis Expansion of the Kaluza-Klein Monopole System: *DESY Report*, DESY 96–033, February 1996, 16pp.

[84] C.Grosche, G.S.Pogosyan and A.N.Sissakian: On the O(4) Symmetry of the Kaluza-Klein Monopole System: Interbasis Expansion of the Spheroidal Basis. *DESY Report*, in preparation.

[85] C.Grosche, G.S.Pogosyan and A.N.Sissakian: On the Two-Center Coulomb Problem on the Hyperboloid. *DESY Report*, in preparation.

[86] C.Grosche, G.S.Pogosyan and A.N.Sissakian: Interbasis Expansions, Generic Coordinate Systems, and Path Integrals. *DESY Report*, in preparation.

[87] C.Grosche and F.Steiner: The Path Integral on the Poincaré Upper Half Plane and for Liouville Quantum Mechanics. *Phys.Lett.* **A 123** (1987) 319–328.

[88] C.Grosche and F.Steiner: Path Integrals on Curved Manifolds. *Zeitschr.Phys.* **C 36** (1987) 699–714.

[89] C.Grosche and F.Steiner: The Path Integral on the Pseudosphere. *Ann.Phys.(N.Y.)* **182** (1988) 120–156.

[90] C.Grosche and F.Steiner: Classification of Solvable Feynman Path Integrals. In the *Proceedings of the IV. International Conference on "Path Integrals from meV to MeV"*, Tutzing, Germany 1992; eds.: H.Grabert, A.Inomata, L.S.Schulman and U.Weiss (*World Scientific*, Singapore, 1993), 276–288.

[91] C.Grosche and F.Steiner: How to Solve Path Integrals in Quantum Mechanics. *J.Math.Phys.* **36** (1995) 2354–2385.

[92] C.Grosche and F.Steiner: Table of Feynman Path Integrals. To appear in: *Springer Tracts in Modern Physics* (1996).

[93] A.Guha and S.Mukherjee: Exact Solution of the Schrödinger Equation with Noncentral Parabolic Potentials. *J.Math.Phys.* **28** (1987) 840–843.

[94] M.C.Gutzwiller: Chaos in Classical and Quantum Mechanics (*Springer*, Berlin, 1990).

[95] H.Hartmann: Bewegung eines Körpers in einem ringförmigen Potentialfeld. *Theoret.Chim.Acta* **24** (1972) 201–206.

[96] D.A.Hejhal: The Selberg Trace Formula for PSL(2,$\mathbb{R}$) I. *Lecture Notes in Mathematics* **548** (*Springer*, Berlin, 1976).

[97] D.R.Herrick: Symmetry of the Quadratic Zeemann Effect for Hydrogen. *Phys.Rev.* **A 26** (1982) 323–329.

[98] J.Hietarinta: Direct Methods for the Search of the Second Invariant. *Phys.Rep.* **147** (1987) 87–154.

[99] P.W.Higgs: Dynamical Symmetries in a Spherical Geometry. *J.Phys.A: Math.Gen.* **12** (1979) 309–323.

[100] D.B.Hodge: Eigenvalues and Eigenfunctions of the Spheroidal Wave Equation. *J.Math.Phys.* **11** (1970) 2308–3312.

[101] C.R.Holt: Construction of New Integrable Hamiltonians in Two Degrees of Freedom. *J.Math.Phys.* **23** (1982) 1037–1046.

[102] M.Ikeda and N.Katayama: On Generalization of Betrand's Theorem to Spaces of Constant Curvature. *Tensor* **38** (1982) 37–48.

[103] L.Infeld: On a New Treatment of Some Eigenvalue Problems. *Phys.Rev.* **59** (1941) 737–747.

[104] L.Infeld and A.Schild: A Note on the Kepler Problem in a Space of Constant Negative Curvature. *Phys.Rev.* **67** (1945) 121–122.

[105] A.Inomata: Exact Path-Integration for the Two Dimensional Coulomb Problem. *Phys.Lett.* **A 87** (1982) 387–390. Alternative Exact-Path-Integral-Treatment of the Hydrogen Atom. *Phys.Lett.* **A 101** (1984) 253–257.

[106] A.Inomata and G.Junker: Quantization of the Kaluza-Klein Monopole System by Path Integration. *Phys.Lett.* **B 234** (1990) 41–44. Path-Integral Quantization of Kaluza-Klein Monopole Systems. *Phys.Rev.* **D 43** (1991) 1235–1242.

[107] A.Inomata and G.Junker: Path Integral Realization of Dynamical Symmetries. In the *The Second International Wigner Symposium*, Goslar 1991; eds.: H.D.Doebner, W.Scherer and F.Schroek, Jr. (*World Scientific*, Singapore, 1993), 334–337. Path Integrals and Lie Groups. In *Noncompact Lie Groups and Some of Their Applications*, San Antonio, Texas, USA, 1993; eds.: E.A.Tanner and R.Wilson (*Kluwer Academic Publishers*, Dordrecht, 1993), 199–224.

[108] A.Inomata, H.Kuratsuji and C.C.Gerry: Path Integrals and Coherent States of SU(2) and SU(1,1) (*World Scientific*, Singapore, 1992).

[109] A.Inomata and R.Wilson: Path Integral Realization of a Dynamical Group. *Lecture Notes in Physics* **261**, 42–47 (*Springer-Verlag*, Berlin, 1985).

[110] A.A.Izmest'ev, G.S.Pogosyan, A.N.Sissakian and P.Winternitz: Contraction of Lie Algebras and Separation of Variables. *JINR Communications*, Dubna 1996, JINR E2–96–39, 15pp.

[111] W.Kallies, I.Lukáč, G.S.Pogosyan and A.N.Sissakian: Ellipsoidal Basis for Isotropic Oscillator. In the *Proceedings of the "International Workshop on 'Symmetry Methods in Physics' in Memory of Prof. Ya. A. Smorodinsky"*, Dubna, July 1993; eds.: A.N.Sissakian, G.S.Pogosyan and S.I.Vinitsky, *JINR Publications*, E2–94–347, Dubna 1994, 206–217.

[112] E.G.Kalnins: Separation of Variables for Riemannian Spaces of Constant Curvature (*Longman Scientific & Technical*, Essex, 1986).

[113] E.G.Kalnins: On the Separation of Variables for the Laplace Equation $\Delta\psi + K^2\psi = 0$ in Two- and Three-Dimensional Minkowski Space. *SIAM J.Math.Anal.* **6** (1975) 340–374.

[114] E.G.Kalnins and W.Miller, Jr.: Lie Theory and Separation of Variables. 4. The Groups $SO(2,1)$ and SO(3). *J.Math.Phys.* **15** (1974) 1263–1274.

[115] E.G.Kalnins and W.Miller, Jr.: The Wave Equation, $O(2,2)$, and Separation of Variables on Hyperboloids. *Proc.Roy.Soc.Edinburgh* **A 79** (1977) 227–256.

[116] E.G.Kalnins and W.Miller, Jr.: Lie Theory and the Wave Equation in Space-Time. 1. The Lorentz Group. *J.Math.Phys.* **18** (1977) 1–16.

[117] E.G.Kalnins and W.Miller, Jr.: Lie Theory and the Wave Equation in Space-Time. 4. The Klein-Gordon Equation and the Poincaré Group. *J.Math.Phys.* **19** (1978) 1233–1246.

[118] E.G.Kalnins, W.Miller, Jr. and G.S.Pogosyan: Superintegrability in Two-Dimensional Euclidean Space and Associated Polynomial Solutions. *J.Math.Phys.*, to appear.

[119] E.G.Kalnins, W.Miller Jr. and P.Winternitz: The Group O(4), Separation of Variables and the Hydrogen Atom. *SIAM J.Appl.Math.* **30** (1976) 630–664.

[120] N.Katayama: A Note on a Quantum-Mechanical Harmonic Oscillator in a Space of Constant Curvature. *Nuovo Cimento* **B 107** (1992) 763–768.

[121] N.Katayama: On Generalized Runge-Lenz Vector and Conserved Symmetric Tensor for Central-Potential Systems with a Monopole Field on Spaces of Constant Curvature. *Nuovo Cimento* **B 108** (1993) 657–667.

[122] M.Kibler, G.-H.Lamot and P.Winternitz: Classical Trajectories for Two Ring-Shaped Potentials. *Int.J.Quantum Chem.* **43** (1992) 625–645.
M.Kibler and C.Campigotto: Classical and Quantum Study of a Generalized Kepler-Coulomb System. *Int.J.Quantum Chem.* **45** (1993) 209–224.
M.Kibler and P.Winternitz: Periodicity and Quasi-Periodicity for Super-Integrable Hamiltonian Systems. *Phys.Lett.* **147** (1990) 338–342.

[123] M.Kibler, L.G.Mardoyan and G.S.Pogosyan: On a Generalized Kepler-Coulomb System: Interbasis Expansions. *Int.J.Quantum Chem.* **52** (1994) 1301–1316.

[124] M.Kibler, L.G.Mardoyan and G.S.Pogosyan: On a Generalized Oscillator System: Interbasis Expansions. *Int.J.Quantum Chem.* (1996), to appear.

[125] M.Kibler and T.Negadi: Motion of a Particle in a Ring-Shaped Potential: An Approach Via a Nonbijective Canonical Transformation. *Int.J.Quantum Chem.* **26** (1984) 405–410.

[126] M.Kibler and T.Negadi: Motion of a Particle in a Coulomb Field Plus Aharonov-Bohm Potential. *Phys.Lett.* **A 124** (1987) 42–46.

[127] M.Kibler and P.Winternitz: Dynamical Invariance Algebra of the Hartmann Potential. *J.Phys.A: Math.Gen.* **20** (1987) 4097–4108.

[128] M.Kibler and P.Winternitz: Periodicity and Quasi-Periodicity for Super-Integrable Hamiltonian Systems. *Phys.Lett.* **A 147** (1990) 338–342.

[129] H.M.Kiefer and D.M.Fradkin: Comments on Separability Operators, Invariance Ladder Operators, and Quantization of the Kepler Problem in Prolate-Spheroidal Coordinates. *J.Math.Phys.* **9** (1968) 627–632.

[130] H.Kleinert: How to do the Time Sliced Path Integral for the H Atom. *Phys.Lett.* **A 120** (1987) 361–366.

[131] H.Kleinert: Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics (*World Scientific*, Singapore, 1990).

[132] H.Kleinert and I.Mustapic: Summing the Spectral Representations of Pöschl-Teller and Rosen-Morse Fixed-Energy Amplitudes. *J.Math.Phys.* **33** (1992) 643–662.

[133] H.A.Kramers and G.P.Ittmann: Zur Quantelung des asymmetrischen Kreisels (I-III). *Zeitschr. Phys.* **53** (1929) 553–565. Ibidem **58** (1929) 217–231, **60** (1930) 663–681.

[134] R.Kubo: Path Integration on the Upper Half-Plane. *Prog.Theor.Phys.* **78** (1987) 755–759. Geometry, Heat Equation and Path Integrals on the Poincaré Upper Half-Plane. *Prog.Theor.Phys.* **79** (1988) 217–226.

[135] Ю.А.Курочкин и В.С.Отчик: Аналог вектора Рунге-Ленца и спектр знергий в задаче Кеплера на трехмерной цфере. *ДАН БССР* **23** (1979) 987–990.
[Yu.A.Kurochkin and V.S.Otchik: Analogue of the Runge-Lenz Vector and Energy Spectrum in the Kepler Problem on the Three-Dimensional Sphere. *DAN BSSR* **23** (1979) 987–990].

[136] M.Lakshmanan and H.Hasegawa: On the Canonical Equivalence of the Kepler Problem in Coordinate and Momentum Space. *J.Phys.A: Math.Gen.* **17** (1984) L889–L893.

[137] H.I.Leemon: Dynamical Symmetries in a Spherical Geometry. *J.Phys.A: Math.Gen.* **A 12** (1979) 489–501.

[138] I.Lukàčs: A Complete Set of the Quantum-Mechanical Observables on a Two-Dimensional Sphere. *Theor.Math.Phys.* **14** (1973) 271–281

[139] I.Lukàč: Complete Sets of Observables on the Sphere in Four-Dimensional Euclidean Space. *Theor.Math.Phys.* **31** (1977) 457–461.

[140] I.Lukàčs and Ya.A.Smorodinskiĭ: Wave Functions for the Asymmetric Top. *Sov.Phys.JETP* **30** (1970) 728–730.

[141] I.Lukàčs and Ya.A.Smorodinskiĭ: Separation of Variables in a Spheroconical Coordinate System and the Schrödinger Equation for a Case of Noncentral Forces. *Theor.Math.Phys.* **14** (1973) 125–131.

[142] M.Luming and E.Predazzi: A Set of Exactly Solvable Potentials. *Nuovo Cimento* **B 44** (1966) 210–212.

[143] И.В.Луценко, Л.Г.Мардоян, Г.С.Погосян и А.Н.Сисакян: Обобщение межбазисного осцилляторного разложения “цилиндр–сфера” в поле кольцеобразного потенциала. *ОИЯИ препринт* P2–89–814, Дубна 1989, 8pp.
[I.V.Lutsenko, L.G.Mardoyan, G.S.Pogosyan and A.N.Sissakian: The Generalization of Interbasis Oscillator Expansions “Cylinder-Sphere” in the Field of a Ring-Shaped Potential. *JINR Communications*, P2-89-814, Dubna, 1989, unpublished].

[144] I.V.Lutsenko, G.S.Pogosyan, A.N.Sisakyan and V.M.Ter-Antonyan: Hydrogen Atom as Indicator of Hidden Symmetry of a Ring-Shaped Potential. *Theor.Math.Phys.* **83** (1990) 633–639.

[145] A.J.MacFarlane: The Quantum Neumann Model with the Potential of Rosochatius. *Nucl.Phys.* **B 386** (1992) 453–467.

[146] D.W.McLaughlin and L.S.Schulman: Path Integrals in Curved Spaces. *J.Math.Phys.* **12** (1971) 2520–2524.

[147] W.Magnus, F.Oberhettinger and R.P.Soni: Formulas and Theorems for the Special Functions of Mathematical Physics (*Springer-Verlag*, Berlin, 1966).

[148] A.A.Makarov, J.A.Smorodinsky, Kh.Valiev and P.Winternitz: A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries. *Nuovo Cimento* **A 52** (1967) 1061–1084.

[149] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan: Spheroidal Analysis of Hydrogen Atom. *J.Phys.A: Math.Gen.* **16** (1983) 711–718. Spheroidal Corrections to the Spherical and Parabolic Bases of the Hydrogen Atom. *Theor.Math.Phys.* **64** (1985) 762–764.

[150] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan: Elliptic Basis for a Circular Oscillator. *Theor.Math.Phys.* **65** (1985) 1113–1122. Interbasis Expansions in a Circular Oscillator. *Nuovo Cimento* **A 86** (1985) 324–336.

[151] L.G.Mardoyan, G.S.Pogosyan, A.N.Sisakyan and V.M.Ter-Antonyan: Two-Dimensional Hydrogen Atom: I. Elliptic Bases. *Theor.Math.Phys.* **61** (1984) 1021–1034. Hidden Symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *J.Phys.A: Math.Gen.* **18** (1985) 455–466. Interbasis Expansion in a Two-Dimensional Hydrogen Atom. *Theor. Math.Phys.* **63** (1985) 597–603.

[152] I.M.Mayes and J.S.Dowker: Canonical Functional Integrals in General Coordinates. *Proc.Roy. Soc.(London)* **A 327** (1972) 131–135. Hamiltonian Orderings and Functional Integrals. *J.Math. Phys.* **14** (1973) 434–439. The Canonical Quantization of Chiral Dynamics. *Nucl.Phys.* **B 29** (1971) 259–268.

[153] J.Meixner and F.W.Schäfke: Mathieusche Funktionen und Sphäroidfunktionen (*Springer-Verlag*, Berlin, 1954).

[154] W.Miller, Jr.: Symmetry and Separation of Variables (*Addison-Wesley*, Reading, 1977).

[155] M.M.Mizrahi: The Weyl Correspondence and Path Integrals. *J.Math.Phys.* **16** (1975) 2201–2206.

[156] F.Moon and D.Spencer: Theorems on Separability in Riemannian n-Space. *Proc.Amer.Math.Soc.* **3** (1952) 635-642.

[157] P.M.Morse: Diatomic Molecules According to the Wave Mechanics II. Vibrational Levels. *Phys.Rev.* **34** (1929) 57–64.

[158] P.M.Morse and H.Feshbach: Methods of Theoretical Physics (*McGraw-Hill*, New York, 1953).

[159] C.Neumann: De Problemate Quodam Mechanico, Quod ad Primam Integralium Ultraellipticorum Classem Revocatur. *J.Reine Angew.Math.* **56** (1859) 46–63.

[160] Y.Nishino: On Quadratic First Integrals in the 'Central Potential' Problem for the Configuration Space of Constant Curvature. *Math.Japonica* **17** (1972) 59–67.

[161] E.Noether: Invariante Variationsprobleme. *Nachr.König.Gesellsch.Wiss.Göttingen* (1918) 235–257.

[162] М.Н.Олевский: Триортогональные системы в пространствах постоянной кривизны, в которых уравнение $\Delta_2 u + \lambda u = 0$ допускает полное разделение переменных. *Мат.Сб.* **27** (1950) 379–426.
[M.N.Olevskiĭ: Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables. *Math.Sb.* **27** (1950) 379–426].

[163] M.Omote: Point Canonical Transformations and the Path Integral. *Nucl.Phys.* **B 120** (1977) 325–332.

[164] V.S.Otchik: Symmetry and Separation of the Coulomb Two-Center Problem in Three-Dimensional Space with Constant Curvature. *Dokl.AN BSSR* **35** (1991) 420–424 (in Russian); On the Two Coulomb Centers Problem in a Spherical Geometry. In the *Proceedings of the International Workshop on Symmetry Methods in Physics in Memory of Professor Ya.A.Smorodinsky*, Dubna 1993, G.S.Pogosyan, A.N.Sissakian and S.I.Vinitsky (eds.), JINR Publications, Dubna 1994, 384–388.

[165] В.С.Отчик и В.М.Редьков: Квантовомеханическая задача Кеплера в пространствах постоянной кривизны, *Минск Препринт No.298*, 1983, 47pp.
[V.S.Otchik and V.M.Red'kov: Quantum Mechanical Kepler Problem in Space with Constant Curvature. *Minsk Preprint No.298*, 1983, 47pp., unpublished].

А.А.Богуш, В.С.Отчик и В.М.Редьков: Разделение переменных в уравнении Шредингера и нормированные функции состояний для задачи Кеплера в трехмерных пространствах постоянной кривизны. *Вести Акад.Наук.БССР* **3** (1983) 56–62. [A.A.Bogush, V.S.Otchik and V.M.Red'kov: Separation of Variables in the Schrödinger Equation and Normalized State Functions for the Kepler Problem in the Three-Dimensional Spaces of Constant Curvature. *Vesti Akad.Nauk.BSSR* **3** (1983) 56–62].

А.А.Богуш, Ю.А.Курочкин и В.С.Отчик: О квантовомеханической задаче Кеплера в трехмерном пространстве Лобачевского. *ДАН БССР* **24** (1980) 19–22. [A.A.Bogush, Yu.A.Kurochkin and V.S.Otchik: On the Quantum-Mechanical Kepler Problem in Three-Dimensional Lobachevskii Space. *DAN BSSR* **24** (1980) 19–22].

[166] N.K.Pak and I.Sökmen: General New-Time Formalism in the Path Integral. *Phys.Rev.* **A 30** (1984) 1629–1635.

[167] J.Patera and P.Winternitz: A New Basis for the Representations of the Rotation Group. Lamé and Heun Polynomials. *J.Math.Phys.* **14** (1973) 1130–1139.

[168] W.Pauli: Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik. *Zeitschr.Phys.* **36** (1926) 336–363, reprinted in [6], 371–399.

[169] D.Peak and A.Inomata: Summation Over Feynman Histories in Polar Coordinates. *J.Math.Phys.* **10** (1969) 1422–1428.

[170] G.S.Pogosyan and A.N.Sissakian: On the Kepler-Coulomb Problem in Three-Dimensional Space with Constant Curvature. In the *Proceedings of the "Barut's Memorial Conference on Group Theory in Physics*, Edirne, Turkey, December 1995, 9pp.; ed.: I.H.Duru (*World Scientific*, Singapore, 1996).

[171] G.S.Pogosyan, A.N.Sissakian and S.I.Vinitsky: Interbasis "Sphere-Cylinder" Expansions for the Oscillator in the Three-Dimensional Space of Constant Curvature. In *Frontiers of Fundamental Physics*; eds.: M.Barone and F.Selleri (*Plenum Publishing*, New York, 1994), 429–436.

[172] C.Quesne: A New Ring-Shaped Potential and its Dynamical Invariance Algebra. *J.Phys.A: Math. Gen.* **21** (1988) 3093–3103.

[173] E.Schrödinger: A Method of Determining Quantum Mechanical Eigenvalues and Eigenfunctions; *Proc.Roy.Irish Soc.* **46** (1941) 9–16. Further Studies on Solving Eigenvalue Problems by Factorization; *Proc.Roy.Irish Soc.* **46** (1941) 183–206. The Factorization of the Hypergeometric Equation; *Proc.Roy.Irish Soc.* **47** (1941) 53–54.

[174] L.S.Schulman: Techniques and Applications of Path Integration (*John Wiley& Sons*, New York, 1981).

[175] I.Sökmen: Exact Path-Integral Solution for a Charged Particle in a Coulomb Plus Aharonov-Bohm Potential. *Phys.Lett.* **A 132** (1988) 65–68.

[176] Ya.A.Smorodinsky and I.I.Tugov: On Complete Sets of Observables. *Sov.Phys.JETP* **23** (1966) 434–437.

[177] E.A.Solov'ev: The Hydrogen Atom in a Weak Magnetic Field. *Sov.Phys.JETP* **55** (1982) 1017–1022.

[178] R.D.Spence: Angular Momentum in Sphero-Conal Coordinates. *Amer.J.Phys.* **27** (1959) 329–335.

[179] F.Steiner: Space-Time Transformations in Radial Path Integrals. *Phys.Lett.* **A 106** (1984) 356–362.

[180] F.Steiner: Exact Path Integral Treatment of the Hydrogen Atom. *Phys.Lett.* **A 106** (1984) 363–367.

[181] V.V.Stepanov: Sur l'équation de Laplace et certain système triples orthogonaux. *Math.Sb.* **11** (1942) 204–238.

[182] A.F.Steveson: Note on the “Kepler Problem” in a Spherical Space, and the Factorization Method of Solving Eigenvalue Problems; *Phys.Rev.* **59** (1941) 842–843.

[183] S.N.Storchak: Path Reparametrization in a Path Integral on a Finite-Dimensional Manifold. *Theor. Math.Phys.* **75** (1988) 610–618. Rheonomic Homogeneous Point Transformation and Reparametrization in the Path Integral. *Phys.Lett.* **A 135** (1989) 77–85.

[184] E.Teller: Über das Wasserstoffmolekülion. *Zeitschr.Phys.* **61** (1930) 458–480.

[185] E.Titchmarsh: Eigenfunctions Expansions, Vol.I (*Oxford University Press*, London, 1962).

[186] A.G.Ushveridze: Special Case of the Separation of Variables in the Multidimensional Schrödinger Equation. *J.Phys.A: Math.Gen.* **21** (1988) 1601–1605.

[187] A.N.Vaidya and H.Boschi-Filho: Algebraic Calculation of the Green's Function of the Hartmann Potential. *J.Math.Phys.* **31** (1990) 1951–1954.

[188] D.A.Varshalovich, A.N.Moskalev and V.K.Khersonskii: Quantum Theory of Angular Momentum. (*World Scientific*, Singapore, 1988).

[189] N.Ja.Vilenkin: Special Functions Connected with Class 1 Representations of Groups of Motions in Spaces of Constant Curvature. *Trans.Moscow Math.Soc.* (1963) 209–295.

[190] S.I.Vinitsky, L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and T.A.Strizh: A Hydrogen Atom in the Curved Space. Expansion over Free Solutions on the Three-Dimensional Sphere. *Phys.At.Nucl.* **56** (1993) 321–327.

[191] S.I.Vinitskiĭ, V.N.Pervushin, G.S.Pogosyan and A.N.Sisakyan: Equation for Quasiradial Functions in the Momentum Representation on a Three-Dimensional Sphere. *Phys.At.Nucl.* **56** (1993) 1027–1034.

[192] P.Winternitz, I.Lukač and Ya.A.Smorodinskiĭ: Quantum Numbers in the Little Group of the Poincaré Group. *Sov.J.Nucl.Phys.* **7** (1968) 139–145.

[193] P.Winternitz, Ya.A.Smorodinskiĭ, M.Uhlir and I.Fris: Symmetry Groups in Classical and Quantum Mechanics. *Sov.J.Nucl.Phys.* **4** (1967) 444–450.

[194] S.Wojciechovski: Superintegrability of the Calogero-Moser System. *Phys.Lett.* **A 95** (1983) 279–281.

[195] A.Young and C.DeWitt-Morette: Time Substitution in Stochastic Processes as a Tool in Path Integration. *Ann.Phys.* **169** (1986) 140–166.

[196] A.S.Zhedanov: The “Higgs Algebra” as a ‘Quantum’ Deformation of SU(2). *Mod.Phys.Lett.* **A 7** (1992) 507–512.

[197] A.S.Zhedanov: Hidden Symmetry Algebra and Overlap Coefficients for Two Ring-Shaped Potentials. *J.Phys.A: Math.Gen.* **26** (1993) 4633–4641.