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Contractions of Lie algebras and separation of variables

A A Izmest'ev†, G S Pogosyan†, A N Sissakian† and P Winternitz‡

† Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia

‡ Centre de recherches mathématiques, Université de Montréal, CP 6128, succ. Centre Ville, Montréal, Québec, H3C 3J7, Canada

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Abstract. The Wigner–Inonu contraction from the rotation group O(3) to the Euclidean group E(2) is used to relate the separation of variables in the Laplace–Beltrami operators on two corresponding homogeneous spaces. Different realizations of the contraction take the two separable coordinate systems on the sphere S_2 to the four on the plane E_2.

Résumé. La contraction de Wigner–Inonu du groupe de rotation O(3) au groupe euclidien E(2) est utilisée pour établir une relation entre la séparation des variables dans les opérateurs de Laplace–Beltrami sur les espaces homogènes correspondants. Différentes réalisations de la contraction transforment les deux systèmes de coordonnées sur la sphère S_2 en quatre sur le plan E_2.

1. Introduction

It is well known that an intimate relationship exists between the theory of special functions and Lie group theory [1–3]. Virtually all properties of large classes of special functions can be obtained from the representation theory of Lie groups, making use of the fact that the special functions occur as basis functions of irreducible representations, as matrix elements of transformation matrices, as Clebsch–Gordon coefficients, or in some other guise. Recently, the class of functions treatable by group theoretical and algebraic methods has been extended to the so-called q-special functions that have been related to quantum groups [4–8].

One very fruitful application of Lie theory in this context is the algebraic approach to the separation of variables in partial differential equations [9–15]. In this approach separable coordinate systems (for Laplace–Beltrami, Hamilton–Jacobi and other invariant partial differential equations) are characterized by complete sets of commuting second-order operators. These lie in the enveloping algebra of the Lie algebra of the isometry group (or in some cases conformal group) of the corresponding homogeneous space. We mention that the operator approach to separation of variables has also been extended to quantum groups and thus to the separation of variables in differential-difference equations [16].

A question that has so far received little attention in the literature is that of the connections between the separation of variables in different spaces, e.g. in homogeneous spaces of different Lie groups. In particular, it is of interest to study the behaviour of separable coordinates, sets of commuting operators and the corresponding separating eigenfunctions under deformations and contractions of the underlying Lie algebras.

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Two types of Lie algebra contractions exist in the literature. The first are standard Wigner–Inonu contractions [17–19]. They can be interpreted as singular limits of transformations of bases of Lie algebras. More recently, 'graded contractions' have been introduced [20–22]. They are more general than the Wigner–Inonu ones and can be obtained by introducing parameters that modify the structure constants of a Lie algebra L in a manner respecting a certain grading and then taking limits when these parameters go to zero.

Our aim is to perform a study of the connection between the contractions of Lie algebras and the separation of variables. In this first paper we restrict ourselves to the simplest case. We shall consider Wigner–Inonu contractions of the rotation algebra o(3) to the Euclidean algebra e(2). The two separable coordinate systems on the sphere $S_2 \sim 0(3)/0(2)$ will be related to the four separable systems on the plane $E_2 \sim E(2)/0(2)$. The contractions will be followed through on several levels: the Lie algebra, the commuting sets of operators, the coordinate systems and the eigenfunctions of the Laplace–Beltrami operators.

Our motivation comes from several directions. Among them we mention the following. In special function theory contractions provide the possibility of obtaining new asymptotic formulas, new expansions, etc. In the theory of finite-dimensional integrable systems contractions provide relations between such systems in curved and flat spaces. Contractions play a significant role in the theory of quantum groups [23–25] and it is to be expected that methods developed for Lie groups will be generalizable to the case of quantum groups.

In section 2 we first review the two separable systems on the sphere S_2 and the four on the plane E_2. We then introduce geodesical coordinates on S_2 that are well adapted for the contraction limit. Using these coordinates we take the limit $R \to \infty$, where R is the radius of the sphere. Spherical coordinates on S_2 go into polar or Cartesian ones on E_2. Elliptic coordinates on S_2 go into elliptic, parabolic or Cartesian ones on E_2. Section 3 is devoted to the contraction of basis functions. Thus, spherical harmonics go over into Bessel functions or exponentials. Elliptic harmonics, expressed as products of Lamé polynomials, go into exponentials, Mathieu functions, or parabolic cylinder functions.

2. Complete sets of commuting operators, separable coordinates, and their contractions

2.1. Separable coordinates on the sphere S_2

Let us first consider the sphere S_2. Its isometry group is $0(3)$. We choose a standard basis $\{L_1, L_2, L_3\}$ for the Lie algebra o(3):

$$L_i = -\epsilon_{ikj} u_k \frac{\partial}{\partial u_j} \qquad [L_i, L_k] = \epsilon_{ikj} L_j \qquad i, k = 1, 2, 3 \tag{2.1}$$

where u_i are Cartesian coordinates in the ambient space E_3.

On the sphere S_2 we have

$$u_1^2 + u_2^2 + u_3^2 = R^2.$$

The Laplace–Beltrami operator and metric on S_2 in curvilinear coordinates are

$$\begin{aligned} \Delta_{\mathrm{LB}} &= \frac{1}{R^2}(L_1^2 + L_2^2 + L_3^2) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \\ \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \qquad g = \det(g_{\mu\nu}) \qquad g_{\alpha\mu} g^{\mu\nu} = \delta_\alpha^\nu. \end{aligned} \tag{2.2}$$

Following the general method [9–15] (that has in particular been applied to the sphere

S_2) [26] we look for separated eigenfunctions of the Laplace–Beltrami operator satisfying

$$\Delta_{\mathrm{LB}}\Psi = \frac{l(l+1)}{R^2}\Psi \qquad X\Psi = k\Psi \qquad \Psi_{lk}(\alpha,\beta) = \Xi_{lk}(\alpha)\Phi_{lk}(\beta) \tag{2.3}$$

where X is a second-order operator in the enveloping algebra of o(3):

$$X = a_{ik}L_iL_k \qquad a_{ik} = a_{ki}. \tag{2.4}$$

Two operators X and X' will be considered equivalent $X \sim X'$, if they are related by a rotation and a linear combination with the Laplacian

$$X \sim X' = \left(g^{\mathrm{T}}ag\right)_{ik} L_iL_k + \mu\Delta \qquad g^{\mathrm{T}}g = I. \tag{2.5}$$

The matrix a_{ik} can be diagonalized to give

$$X(a_1, a_2, a_3) \equiv X = a_1L_1^2 + a_2L_2^2 + a_3L_3^2. \tag{2.6}$$

For $a_1 = a_2 = a_3$ we have $X \sim 0$. If two eigenvalues of a_{ik} are equal, e.g. $a_1 = a_2 \neq a_3$, we can transform X into $X(0,0,1) = L_3^2$ and the corresponding separable coordinates on S_2 are the usual spherical ones

$$u_1 = R\sin\vartheta\cos\varphi \qquad u_2 = R\sin\vartheta\sin\varphi \qquad u_3 = R\cos\vartheta. \tag{2.7}$$

They correspond to the group reduction $0(3) \supset 0(2)$ and $X = L_3^2$ is invariant under $0(2)$ and under reflections in all coordinate planes.

When all three eigenvalues a_i are different, then the separable coordinates in (2.3) are elliptic ones [26–29]. These can be written in algebraic form, as

$$\begin{aligned} u_1^2 &= R^2\frac{(\rho_1-a_1)(\rho_2-a_1)}{(a_2-a_1)(a_3-a_1)} \\ u_2^2 &= R^2\frac{(\rho_1-a_2)(\rho_2-a_2)}{(a_3-a_2)(a_1-a_2)} \\ u_3^2 &= R^2\frac{(\rho_1-a_3)(\rho_2-a_3)}{(a_1-a_3)(a_2-a_3)} \end{aligned} \tag{2.8}$$

with $a_1 \leqslant \rho_1 \leqslant a_2 \leqslant \rho_2 \leqslant a_3$.

In trigonometric form we put

$$\rho_1 = a_1 + (a_2-a_1)\cos^2\phi \qquad \rho_2 = a_3 - (a_3-a_2)\cos^2\theta \tag{2.9}$$

and obtain

$$\left.\begin{aligned} u_1 &= R\sqrt{1-k'^2\cos^2\theta}\cos\phi \\ u_2 &= R\sin\theta\sin\phi \\ u_3 &= R\sqrt{1-k^2\cos^2\phi}\cos\theta \end{aligned}\right\} \qquad 0 \leqslant \phi < 2\pi \quad 0 \leqslant \theta \leqslant \pi \tag{2.10}$$

where

$$k^2 = \frac{a_2-a_1}{a_3-a_1} = \sin^2 f \qquad (k')^2 = \frac{a_3-a_2}{a_3-a_1} = \cos^2 f. \tag{2.11}$$

Finally, the Jacobi elliptic version of elliptic coordinates is obtained by putting

$$\rho_1 = a_1 + (a_2-a_1)\,\mathrm{sn}^2(\alpha,k) \qquad \rho_2 = a_2 + (a_3-a_2)\,\mathrm{cn}^2(\beta,k'). \tag{2.12}$$

We obtain

$$\left.\begin{aligned} u_1 &= R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') \\ u_2 &= R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k') \\ u_3 &= R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k'). \end{aligned}\right\} \qquad -K \leqslant \alpha \leqslant K \quad -2K' \leqslant \beta \leqslant 2K' \tag{2.13}$$

where $\mathrm{sn}(\alpha, k)$, $\mathrm{cn}(\alpha, k)$ and $\mathrm{dn}(\beta, k)$ are the Jacobi elliptic functions with modulus k and K and K' are the complete elliptic integrals [30, 31]. The moduli are given in (2.11), we have $k^2 + k'^2 = 1$ and $2fR$ is the interfocal distance for the ellipses on the upper hemisphere.

Elliptic coordinates corresponding to the reduction $0(3) \supset D_2$, where D_2 is the dihedral group (rotations through π about all three axes and reflections in a coordinate plane). Indeed, the operator (2.6) is invariant only under D_2, rather then $0(2)$, for $a_1 \neq a_2 \neq a_3 \neq a_1$.

2.2. *Separable coordinates on the Euclidean plane E_2*

Let us consider the Lie algebra e(2) in the basis

$$L_3 = u_2\partial_{u_1} - u_1\partial_{u_2} \qquad P_1 = \partial_{u_1} \qquad P_2 = \partial_{u_2}. \tag{2.14}$$

Separated eigenfunctions of the Laplace operator $\Delta = P_1^2 + P_2^2$ satisfy

$$\Delta\Phi_{k,\lambda} = k^2\Phi_{k,\lambda} \qquad X\Phi_{k,\lambda} = \lambda\Phi_{k,\lambda} \qquad \Phi_{k,\lambda}(\alpha, \beta) = \Sigma_{k,\lambda}(\alpha)\Psi_{k,\lambda}(\beta) \tag{2.15}$$

where X is the second-order operator

$$X = aL_3^2 + b(L_3P_1 + P_1L_3) + c(L_3P_2 + P_2L_3) + dP_1^2 + eP_2^2 + 2fP_1P_2\,. \tag{2.16}$$

By means of Euclidean transformations, and linear combination with Δ, we can take X into precisely one of the following operators

$$X_{\mathrm{S}} = L_3^2 \qquad (a \neq 0,\, D = 0) \tag{2.17}$$

$$X_{\mathrm{C}} = P_1^2 - P_2^2 \qquad (a = b = c = 0) \tag{2.18}$$

$$X_{\mathrm{P}} = L_3P_1 + P_1L_3 \qquad (a = 0,\ b^2 + c^2 \neq 0) \tag{2.19}$$

$$X_{\mathrm{E}} = L_3^2 + \tfrac{1}{2}D^2(P_1^2 - P_2^2) \qquad (a \neq 0,\, D \neq 0) \tag{2.20}$$

where

$$D^2 = \frac{1}{2a^2}\{4b^2c^2 + [a(b-e) - b^2 + c^2\}^{1/2}\,. \tag{2.21}$$

Each of the operators (2.17)–(2.20) corresponds to a different separable coordinate system in (2.15). Thus X_{C} corresponds to Cartesian coordinates (x, y), X_{S} to polar ones

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \tag{2.22}$$

X_{P} to parabolic coordinates

$$x = \tfrac{1}{2}(u^2 - v^2) \qquad y = uv \tag{2.23}$$

and X_{E} to elliptic ones

$$x = D\cosh\xi\cos\eta \qquad y = D\sinh\xi\sin\eta \tag{2.24}$$

where $2D$ is the focal distance.

2.3. *The contractions*

We shall use R^{-1} as the contraction parameter. To realize the contraction explicitly, let us introduce geodesical coordinates on the sphere [32], putting

$$x_\mu = R\frac{u_\mu}{u_3} = \frac{u_\mu}{\sqrt{1-(u_1^2+u_2^2)/R^2}} \qquad \mu = 1, 2\,. \tag{2.25}$$

The 0(3) generators can then be expressed as

$$-\frac{L_1}{R} \equiv \pi_2 = p_2 + \frac{1}{R^2}x_2(x_1p_1 + x_2p_2) \tag{2.26}$$

$$\frac{L_2}{R} \equiv \pi_1 = p_1 + \frac{1}{R^2}x_1(x_1p_1 + x_2p_2) \tag{2.27}$$

$$L_3 = x_2p_1 - x_1p_2 = x_2\pi_1 - x_1\pi_2. \tag{2.28}$$

The commutation relations are

$$[L_3, \pi_1] = \pi_2 \qquad [L_3, \pi_2] = -\pi_1 \qquad [\pi_1, \pi_2] = \frac{L_3}{R^2} \tag{2.29}$$

so that for $R \to \infty$ the o(3) algebra contracts to the e(2) algebra. Moreover the momenta π_μ contract to $p_\mu = \partial/\partial x_\mu$ $(\mu = 1, 2)$.

The o(3) Laplace–Beltrami operator (2.2) contracts to the e(2) operator:

$$\Delta_{\mathrm{LB}} = \pi_1^2 + \pi_2^2 + \frac{L_3^2}{R^2} \to \Delta = (p_1^2 + p_2^2). \tag{2.30}$$

Let us now consider the contractions of the operator (2.6) and of the corresponding coordinates.

2.3.1. *Spherical coordinates on S_2 to polar coordinates on E_2.*

We choose $a_1 = a_2$ in (2.6) and put

$$\tan\theta = \frac{r}{R}. \tag{2.31}$$

In the limit $R \to \infty$, $\theta \to 0$ we have

$$X = L_3^2 \to X_{\mathrm{S}} = L_3^2 \tag{2.32}$$

and

$$\begin{aligned} x_1 &= R\frac{u_1}{u_3} \to x = r\cos\phi \\ x_2 &= R\frac{u_2}{u_3} \to y = r\sin\phi\,. \end{aligned} \tag{2.33}$$

2.3.2. *Spherical coordinates on S_2 to Cartesian on E_2.*

We choose $a_2 = a_3 \sim 0$ in (2.6) so that the coordinates (2.7) permute into

$$u_1 = R\cos\theta' \qquad u_2 = R\sin\theta'\cos\phi' \qquad u_3 = R\sin\theta'\sin\phi'. \tag{2.34}$$

Putting

$$\cos\theta' = \frac{x}{R} \qquad \cos\phi' = \frac{y}{R} \tag{2.35}$$

and taking the limit $R \to \infty$, $\theta' \to \frac{\pi}{2}$, $\phi' \to \frac{\pi}{2}$, we obtain

$$X\left(\tfrac{1}{R^2}, 0, 0\right) = \frac{L_1^2}{R^2} = \pi_1^2 \to X = P_1^2 \sim X_{\mathrm{C}} \tag{2.36}$$

and

$$x_1 = R\,\frac{\cot\theta'}{\sin\phi'} \to x \qquad x_2 = R\,\cot\phi' \to y. \tag{2.37}$$

2.3.3. Elliptic coordinates on S_2 to elliptic coordinates on E_2. We take X in its general form, equivalent to

$$X = L_3^2 - \left(\frac{a_2 - a_1}{a_3 - a_2}\right) L_1^2. \tag{2.38}$$

We put

$$\frac{R^2}{a_3 - a_1} = \frac{D^2}{a_2 - a_1} \qquad L_1 = -R\pi_2 \tag{2.39}$$

and in the limit $R^2 \sim a_3 \to \infty$ obtain

$$X \to L_3^2 - D^2 P_2^2 \sim X_{\mathrm{E}}. \tag{2.40}$$

For the coordinates we put

$$\rho_1 = a_1 + (a_2 - a_1)\cos^2\eta \qquad \rho_2 = a_1 + (a_2 - a_1)\cosh^2\xi \tag{2.41}$$

and for $R^2 \sim a_3 \to \infty$, using (2.39), we obtain (2.24), i.e. elliptic coordinates on the plane E_2.

2.3.4. Elliptic coordinates on S_2 to Cartesian coordinates on E_2. We start from the coordinates (2.8) but change the ordering of the parameters a_i i.e. put

$$a_1 \leqslant \rho_1 \leqslant a_3 \leqslant \rho_2 \leqslant a_2 \tag{2.42}$$

and choose $a_3 - a_1 = a_2 - a_3 \equiv a$. We than have

$$\frac{1}{aR^2} X = (\pi_1^2 - \pi_2^2) \to P_1^2 - P_2^2. \tag{2.43}$$

For the coordinates we put

$$\frac{a_3 - \rho_1}{a} = \xi_1 \qquad \frac{\rho_2 - a_3}{a} = \xi_2. \tag{2.44}$$

Using equations (2.25) and (2.8) we have

$$x_1^2 = R^2 \frac{(1 - \xi_1)(1 + \xi_2)}{2\xi_1\xi_2} \qquad x_2^2 = R^2 \frac{(1 + \xi_1)(1 - \xi_2)}{2\xi_1\xi_2}. \tag{2.45}$$

From equation (2.45) we obtain

$$\xi_{1,2} = \frac{R^2}{R^2 + x_1^2 + x_2^2}\left\{\left[1 + \frac{x_1^2 + x_2^2}{R^2} + \frac{(x_1^2 - x_2^2)^2}{4R^4}\right]^{1/2} \mp \frac{x_1^2 - x_2^2}{2R^2}\right\}. \tag{2.46}$$

In the limit $R \to \infty$ we have

$$\xi_1 \to 1 - \frac{x^2}{R^2} \qquad \xi_2 \to 1 - \frac{y^2}{R^2} \tag{2.47}$$

and hence x_1 and x_2 of (2.45) go into Cartesian coordinates:

$$x_1 \to x \qquad x_2 \to y. \tag{2.48}$$

2.3.5. Elliptic coordinates on S_2 to parabolic coordinates on E_2. We take the operator (2.6) with $a_1 \leqslant \rho_1 \leqslant a_2 \leqslant \rho_2 \leqslant a_3$ and choose $a_3 - a_2 = a_2 - a_1 \equiv a$. We must first 'undo' the diagonalization (2.5) by a rotation through $\pi/4$. The operator (2.6) transforms into

$$\frac{1}{aR} X_{\mathrm{S}} = -\frac{1}{aR}(L_1 L_3 + L_3 L_1) = L_3 \pi_2 + \pi_2 L_3 \tag{2.49}$$

with the correct limit for $R \to \infty$. The coordinates (2.13) on S_2 are rotated into

$$\begin{aligned} u_1 &= \frac{R}{\sqrt{2}}(\operatorname{sn}\alpha \operatorname{dn}\beta + \operatorname{dn}\alpha \operatorname{sn}\beta) \\ u_2 &= R \operatorname{cn}\alpha \operatorname{cn}\beta \\ u_3 &= \frac{R}{\sqrt{2}}(\operatorname{dn}\alpha \operatorname{sn}\beta - \operatorname{sn}\alpha \operatorname{dn}\beta) \end{aligned} \tag{2.50}$$

with modulus $k = k' = 1/\sqrt{2}$ for all Jacobi elliptic functions.

From equation (2.50) we obtain

$$\begin{aligned} \operatorname{sn}\alpha &= \frac{1}{\sqrt{2}}\left[\left(1+\frac{u_1}{R}\right)^{1/2}\left(1-\frac{u_3}{R}\right)^{1/2} - \left(1-\frac{u_1}{R}\right)^{1/2}\left(1+\frac{u_3}{R}\right)^{1/2}\right] \\ \sqrt{2}\operatorname{dn}\beta &= \frac{1}{\sqrt{2}}\left[\left(1+\frac{u_1}{R}\right)^{1/2}\left(1-\frac{u_3}{R}\right)^{1/2} + \left(1-\frac{u_1}{R}\right)^{1/2}\left(1+\frac{u_3}{R}\right)^{1/2}\right]. \end{aligned} \tag{2.51}$$

Equation (2.51) suggest the limiting procedure. Indeed we put

$$\operatorname{sn}\alpha = -1 + \frac{u^2}{2R} \qquad \sqrt{2}\operatorname{dn}\beta = 1 + \frac{v^2}{2R}. \tag{2.52}$$

In the limit $R \to \infty$ we than obtain

$$x_1 \to x = \frac{u^2 - v^2}{2} \qquad x_2 \to y = uv \tag{2.53}$$

i.e. the parabolic coordinates (2.23).

3. Contraction of basis functions

Having established the contraction properties of separable coordinates and the corresponding complete sets of commuting operators. We shall now consider the behaviour of eigenfunctions.

3.1. Spherical basis on S_2 to polar basis on E_2

We start from the standard spherical functions $Y_{lm}(\theta, \phi)$ as basis functions of irreducible representations of the group O(3) (see, e.g., [33])

$$\begin{aligned} \Psi_{lm}(\theta,\phi) &= \frac{1}{\sqrt{R}} Y_{lm}(\theta,\phi) = \frac{(-1)^{(m+|m|)/2}}{\sqrt{R}}\left[\frac{2l+1}{2}\frac{(l+|m|)!}{(l-|m|)!}\right]^{1/2} \frac{(\sin\theta)^{|m|}}{2^{|m|}|m|!} \\ &\quad \times {}_2F_1\left(-l+|m|, l+|m|+1; |m|+1; \sin^2(\tfrac{1}{2}\theta)\right) \frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}. \end{aligned} \tag{3.1}$$

In the contraction limit $R \to \infty$ we put

$$\tan\theta \sim \theta \sim \frac{r}{R} \qquad l \sim kR. \tag{3.2}$$

Using the asymptotic formulae

$$\lim_{R\to\infty} {}_2F_1\left(-kR, kR; |m|+1; \tfrac{r^2}{4R^2}\right) = {}_0F_1\left(|m|+1; -\tfrac{k^2r^2}{4}\right)$$
$$\lim_{z\to\infty} \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta} \tag{3.3}$$

and the formula

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} {}_0F_1\left(\nu+1; -\tfrac{z^2}{4}\right) \tag{3.4}$$

we obtain

$$\lim_{\substack{R\to\infty\\ \theta\to 0}} \frac{1}{\sqrt{R}} Y_{lm}(\theta,\phi) = (-1)^{(m+|m|)/2}\sqrt{k} J_{|m|}(kr)\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}. \tag{3.5}$$

The result (3.5) is not new [33]. The point is that this asymptotic formula is obtained very naturally in the context of group contractions applied to the separation of variables.

3.2. Spherical basis on S_2 to Cartesian basis on E_2

We start from the coordinates (2.34), but drop the primes, and write the corresponding spherical functions as

$$Y_{lm}(\theta,\phi) = \frac{\sqrt{2l+1}}{2\pi}\mathrm{e}^{\mathrm{i}m\phi}(\sin\theta)^{|m|}$$
$$\times\begin{cases} (-1)^{(l+m)/2}\left[\dfrac{\Gamma(\frac{l+m+1}{2})\Gamma(\frac{l-m+1}{2})}{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{l-m+2}{2})}\right]^{\frac12} F\left(-\frac{l-m}{2}, \frac{l+m+1}{2}; \frac12; \cos^2\theta\right) \\ (-1)^{(l+m-1)/2}\left[\dfrac{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{l-m+2}{2})}{\Gamma(\frac{l+m+1}{2})\Gamma(\frac{l-m+1}{2})}\right]^{\frac12} 2\cos\theta\, F\left(-\frac{l-m-1}{2}, \frac{l+m+2}{2}; \frac32; \cos^2\theta\right) \end{cases} \tag{3.6}$$

for $l+m$ even and odd, respectively, we now put

$$l \sim kR \qquad m \sim k_2R \tag{3.7}$$

and

$$\sin\theta \to 1 \qquad \cos\theta \to \frac{x}{R} \qquad \phi \to \frac{y}{R} - \frac{\pi}{2}. \tag{3.8}$$

The $F \equiv {}_2F_1$ hypergeometric functions simplify to ${}_0F_1$ ones, the Γ functions also simplify and we find

$$Y_{lm}(\theta,\phi) \stackrel{R\to\infty}{\to} (-1)^{m/2}\sqrt{2kR}\,\frac{\mathrm{e}^{\mathrm{i}k_2y}}{2\pi}\begin{cases} (-1)^{(l+m)/2}\left(\dfrac{2}{k_1R}\right)^{\frac12} {}_0F_1\left(\frac12; \frac{-k_1^2x^2}{4}\right) \\ (-1)^{(l+m-1)/2}\dfrac{2x}{R}\left(\dfrac{k_1R}{2}\right)^{\frac12} {}_0F_1\left(\frac32; \frac{-k_1{}^2x^2}{4}\right) \end{cases} \tag{3.9}$$

with $k_1{}^2 + k_2{}^2 = k^2$.

The ${}_0F_1$ functions are in this case expressible in terms of the Bessel functions $J_{\pm 1/2}$ that are, essentialy, trigonometric functions. The final result is that under the contraction we have

$$\lim_{R\to\infty}(-1)^{l/2}Y_{lm}(\theta,\phi) = \frac{\mathrm{e}^{\mathrm{i}k_2y}}{\sqrt{2\pi}}\begin{cases}\cos k_1x \\ -\mathrm{i}\sin k_1x\end{cases} \tag{3.10}$$

for $l + m$ even and odd, respectively, where the limit is taken as in (3.7). The parity properties of Y_{lm} under the exchange $\theta \to \pi - \theta$ have led to the appearance of $\cos k_1 x$ and $\sin k_1 x$ in (3.9), instead of the usual Cartesian coordinate solution $\exp \mathrm{i}(k_1 x + k_2 y)$.

3.3. Solutions of the Lamé equation

Let us consider (2.3) on the sphere S_2 and separate variables in the elliptic coordinates (2.8). We obtain two ordinary differential equation of the form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\rho^2} + \frac{1}{2}\left\{\frac{1}{\rho - a_1} + \frac{1}{\rho - a_2} + \frac{1}{\rho - a_3}\right\}\frac{\mathrm{d}\psi}{\mathrm{d}\rho} + \frac{1}{4}\left\{\frac{\lambda - l(l+1)\rho}{(\rho - a_1)(\rho - a_2)(\rho - a_3)}\right\}\psi = 0 \tag{3.11}$$

or equivalently

$$4\sqrt{P(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\sqrt{P(\rho)}\frac{\mathrm{d}\psi}{\mathrm{d}\rho} - \{l(l+1)\rho - \lambda\}\psi = 0 \tag{3.12}$$

where

$$P(\rho) = (\rho - a_1)(\rho - a_2)(\rho - a_3).$$

Equation (3.11) is the Lamé equation in algebraic form. It is a Fuchsian type equation with four regular singularities (at a_1, a_2, a_3 and ∞) [26–30, 34].

Its general solution can be represented by a series expansion about any one of the singular points a_k as

$$\psi(\rho) = (\rho - a_1)^{\alpha_1/2}(\rho - a_2)^{\alpha_2/2}(\rho - a_3)^{\alpha_3/2}\sum_{t=0}^{\infty} b_t^{(k)}(\rho - a_k)^t \tag{3.13}$$

where we have

$$\alpha_j(\alpha_j - 1) = 0 \qquad j = 1, 2, 3$$

and can choose k equal to 1, 2, or 3.

Substituting in the Lamé equation (3.11) we obtain a three-term recursion relation for b_t^k:

$$\beta_t^k b_{t+1}^k + [\gamma_t^k + \lambda - l(l+1)a_k]b_t^k + (2t + \alpha - l - 2)(2t + \alpha + l - 1)b_{t-1}^k = 0 \tag{3.14}$$

with

$$\begin{aligned} &\alpha = \alpha_1 + \alpha_2 + \alpha_3 \qquad \alpha_{ik} = \alpha_i - \alpha_k \qquad b_{-1} = 0 \\ &\beta_t^{(k)} = 4(a_i - a_k)(a_j - a_k)(t+1)(t + \alpha_k + \tfrac{1}{2}) \qquad (i, j, k \text{ cyclic}) \\ &\gamma_t^{(k)} = -(a_i - a_k)(2t + \alpha_k + \alpha_j)^2 - (a_j - a_k)(2t + \alpha_k + \alpha_i)^2\,. \end{aligned} \tag{3.15}$$

The expansion (3.13) represents a Lamé function. Since we are interested in representations of SO(3), the sum in $\psi(\rho)$ must be a polynomial of order N, i.e. we must have

$$b_N \neq 0 \qquad b_{N+1} = b_{N+2} = \cdots = 0$$

for some N. The condition for this is that we have

$$l = 2N + \alpha \tag{3.16}$$

and we obtain a secular equation for the eigenvalues λ, i.e. the separation constant in elliptic coordinates, by requiring that the determinant of the homogeneous linear system (3.14) for $\{b_0, b_1, \cdots, b_N\}$ should vanish. Since N and l must be integers, equation (3.16) implies that α and l must have the same parity.

Numerous further properties of the Lamé polynomials, in the context of representations of the group O(3), in the O(3)⊃D_2 basis, were established, e.g., in [26-29].

Here let us just represent the basis functions as

$$\Psi^{pq}_{l\lambda}(\rho_1,\rho_2) = A^{pq}_{l\lambda}\psi^{pq}_{l\lambda}(\rho_1)\psi^{pq}_{l\lambda}(\rho_2) \tag{3.17}$$

where $A^{pq}_{l\lambda}$ is some normalization constant. The labels p, q take values ± 1 and identify representations of D_2. For each value of l the values of p, q and λ label $(2l+1)$ different states. Since a given representations (p, q) of D_2 can figure more than once in the reduction of a representation of O(3) corresponding to a given l, we are faced with a 'missing label problem', resolved by the quantum number λ, i.e. the operator X of (2.6).

The expansions that we shall use for the Lamé polynomials in (3.17) are as in (3.13), but the summation over t is from $t = 0$ to $t = N$.

3.4. Elliptic basis on S_2 to Cartesian basis on E_2

We choose elliptic coordinates on S_2 as in (2.8), but with $a_1 < a_3 < a_2$, as in (2.42). We write the basis functions as in (3.17) with

$$\begin{aligned}
\psi_{l\lambda}(\rho_1) &= (\rho_1 - a_1)^{\alpha_1/2}(\rho_1 - a_2)^{\alpha_2/2}(\rho_1 - a_3)^{\alpha_3/2}\sum_{t=0}^{N} b^{(1)}_t(\rho_1 - a_1)^t \\
\psi_{l\lambda}(\rho_2) &= (\rho_2 - a_1)^{\alpha_1/2}(\rho_2 - a_2)^{\alpha_2/2}(\rho_2 - a_3)^{\alpha_3/2}\sum_{t=0}^{N} b^{(2)}_t(\rho_2 - a_2)^t
\end{aligned} \tag{3.18}$$

as in (3.13). The coefficients b^j_t $(j = 1, 2)$ satisfy the recursion relation (3.14) and we have $N = (l - \alpha)/2$. We use the coordinates ξ_1 and ξ_2 introduced in (2.44) (for $a \equiv a_3 - a_1 = a_2 - a_3$). Equation (3.18) reduces to

$$\begin{aligned}
\psi_{l\lambda}(\xi_1) &= (-1)^{(\alpha_2+\alpha_3)/2}a^{\alpha/2}(1-\xi_1)^{\alpha_1/2}(1+\xi_1)^{\alpha_2/2}\xi_1^{\alpha_3/2}\sum_{t=0}^{N} C^1_t(1-\xi_1)^t \\
\psi_{l\lambda}(\xi_2) &= (-1)^{\alpha_2/2}a^{\alpha/2}(1-\xi_2)^{\alpha_2/2}(1+\xi_2)^{\alpha_1/2}\xi_2^{\alpha_3/2}\sum_{t=0}^{N} C^2_t(1-\xi_2)^t
\end{aligned} \tag{3.19}$$

with $C^{(1)}_t = a^t b_t,\ C^{(2)}_t = (-a)^t b_t$.

The recursion relations (3.14) now imply

$$\begin{aligned}
&8(t+1)(t+\alpha_1+\tfrac{1}{2})C^{(1)}_{t+1} + \{\mu^{(1)} - 2(2t+\alpha_1+\alpha_3)^2 - (2t+\alpha_1+\alpha_2)^2\}C^{(1)}_t \\
&\qquad +(2t+\alpha-l-2)(2t+\alpha+l-1)C^{(1)}_{t-1} = 0 \\
&-8(t+1)(t+\alpha_2+\tfrac{1}{2})C^{(2)}_{t+1} + \{\mu^{(2)} + 2(2t+\alpha_2+\alpha_3)^2 + (2t+\alpha_1+\alpha_2)^2\}C^{(2)}_t \\
&\qquad -(2t+\alpha-l-2)(2t+\alpha+l-1)C^{(2)}_{t-1} = 0
\end{aligned} \tag{3.20}$$

where

$$\mu^{(j)} = \frac{1}{a}[\lambda - a_j l(l+1)] \qquad j = 1, 2\,. \tag{3.21}$$

The contraction limit is taken using equation (2.47) to relate $\xi_{1,2}$ to the Cartesian coordinates on E_2. Taking $l \sim kR$ we find

$$\mu^{(1)} \to 2R^2k_1^2 \qquad \mu^{(2)} \to -2R^2k_2^2 \qquad k = \sqrt{k_1^2 + k_2^2}. \tag{3.22}$$

For $R \to \infty$ the recursion relations (3.20) simplify to two term ones that can be solved to obtain

$$C_t^{(j)} = \frac{R^{2t}}{(\alpha_j + \frac{1}{2})_t} \left(\frac{-k_j^2}{4} \right)^t \frac{1}{t!} \tag{3.23}$$

with

$$\left(\alpha_j + \tfrac{1}{2}\right)_t = \left(\alpha_j + \tfrac{1}{2}\right)\left(\alpha_j + \tfrac{3}{2}\right) \cdots \left(\alpha_j - \tfrac{3}{2} + t\right) \qquad t \geqslant 1 \quad \left(\alpha_j + \tfrac{1}{2}\right)_0 = 1.$$

Substituting (3.23) into (3.19) we obtain

$$\psi_{l\lambda}(\xi_1) = (-1)^{(\alpha_2+\alpha_3)/2} \frac{a^{\alpha/2}}{R^{\alpha_1}} x^{\alpha_1} {}_0F_1 \left(\alpha_1 + \tfrac{1}{2}; -\tfrac{k_1^2 x^2}{4} \right)$$

$$\psi_{l\lambda}(\xi_2) = (-1)^{\alpha_2/2} \frac{a^{\alpha/2}}{R^{\alpha_2}} y^{\alpha_2} {}_0F_1 \left(\alpha_2 + \tfrac{1}{2}; -\tfrac{k_2^2 y^2}{4} \right) . \tag{3.24}$$

Using equation (3.4) and the explicit expressions for $J_{\pm 1/2}$, we find the contraction limit: $A_{l\lambda}^{pq}(R)\psi_{l\lambda}(\xi_1, \xi_2) \to A_{l\lambda}^{pq}\psi_{k_1}(x)\psi_{k_2}(y)$

$$= A_{l\lambda}^{pq}(-1)^{\alpha_3/2}a^{\alpha/2} \begin{cases} \cos k_1 x \cos k_2 y & \alpha_1 = 0, \alpha_2 = 0 \\ -\dfrac{1}{k_2 R} \cos k_1 x \sin k_2 y & \alpha_1 = 0, \alpha_2 = 1 \\ -\dfrac{1}{k_1 R} \sin k_1 x \cos k_2 y & \alpha_1 = 1, \alpha_2 = 0 \\ -\dfrac{1}{k_1 k_2 R^2} \sin k_1 x \sin k_2 y & \alpha_1 = 1, \alpha_2 = 1. \end{cases} \tag{3.25}$$

Though these formulae are quite simple, to our knowledge they are new.

3.5. Elliptic basis on S_2 to elliptic basis on E_2

Let us start from the elliptic coordinates (2.8) with $a_1 \leqslant \rho_1 \leqslant a_2 \leqslant \rho_2 \leqslant a_3$. We take the limit $R \to \infty$, $a_3 \to \infty$ with $\sqrt{a_3}/R$, a_1 and a_2 finite. We introduce a constant D as in (2.39). Elliptic coordinates on the plane E_2 are introduced via (2.41), so that the Cartesian coordinates (x, y) are expressed in terms of the elliptic ones (ξ, η) as in (2.24). Let us first take the limit in the separated equations (3.12). Going over to the variables (ξ, η) from (ρ_1, ρ_2), for $R \to \infty$ we obtain:

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}\eta^2} + \left\{ \mu - \frac{k^2 D^2}{2} \left(\frac{a_2 + a_1}{a_2 - a_1} \right) - \frac{k^2 D^2}{2} \cos 2\eta \right\} \psi_1 = 0 \tag{3.26}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}\xi^2} + \left\{ \mu - \frac{k^2 D^2}{2} \left(\frac{a_2 + a_1}{a_2 - a_1} \right) - \frac{k^2 D^2}{2} \cosh 2\xi \right\} \psi_2 = 0 \tag{3.27}$$

with

$$\mu = \frac{\lambda}{a_3} \qquad l \sim kR.$$

In equation (3.26) we recognize the standard form of the Mathieu equation, whereas equation (3.27) is a modified Mathieu equation [35]. Thus, in the contraction limit, Lamé functions will go over into Mathieu ones. Moreover, periodic solutions of the Lamé equation go over into periodic solutions of (3.26).

The contraction limit can also be taken directly in the Lamé polynomials, using the expansion (3.13) (cut off at $t = N$). The result that we obtain is

$$\lim_{R\to\infty} \frac{\Psi_{l\lambda}(\rho_1)}{R^{\alpha_3}} = (a_2 - a_1)^{\alpha/2} \frac{(-1)^{(\alpha_2+\alpha_3)/2}}{D^{\alpha_3}} (\cos\eta)^{\alpha_1} (\sin\eta)^{\alpha_2} \sum_{t=0}^{\infty} C_t (\cos\eta)^{2t} \tag{3.28}$$

$$\lim_{R\to\infty} \frac{\Psi_{l\lambda}(\rho_2)}{R^{\alpha_3}} = (a_2 - a_1)^{\alpha/2} \frac{(-1)^{\alpha_3/2}}{D^{\alpha_3}} (\cosh\xi)^{\alpha_1} (\sinh\xi)^{\alpha_2} \sum_{t=0}^{\infty} C_t (\cosh\xi)^{2t} \tag{3.29}$$

where the expansion coefficients C_t satisfy recursion relations obtained from (3.14), namely

$$4(t+1)(t+1/2+\alpha_1)C_t + \{\mu - (2t+\alpha_1+\alpha_2)^2\}C_t - k^2 D^2 C_t = 0. \tag{3.30}$$

Depending on the values of α_1 and α_2, defined in (3.18), (3.30) takes one of four different forms.

3.6. *Elliptic basis on S_2 to parabolic basis on E_2*

Let us consider the contraction limit for the Lamé equations (3.12). To do this we use equations (2.12) with $a_3 - a_2 = a_2 - a_1 = a$ i.e. $k = k' = 1/\sqrt{2}$, together with (2.52), to obtain

$$\rho_1 \sim a_1 + a\left(-1 + \frac{u^2}{2R}\right) \qquad \rho_2 \sim a_1 + a\left(1 + \frac{v^2}{R}\right). \tag{3.31}$$

Equation (3.12) for $\rho = \rho_1$ and $\rho = \rho_2$ in the limit $R \to \infty$, with $l^2 \sim k^2R^2$ and $\lambda - a_2 l(l+1) = \mu R a$, yields the two equations

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}u^2} + (k^2u^2 + \mu)\psi_1 = 0 \tag{3.32}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}v^2} + (k^2v^2 - \mu)\psi_2 = 0 \tag{3.33}$$

respectively.

Thus the Lamé equations in the contraction limit go over into equations (3.32), (3.33) for parabolic cylinder functions [36]. The same is of course true for solutions. The expansion (3.13) is not suitable for the contraction limit. In view of (2.52) we need expansions in terms of the variables $(1 + \mathrm{sn}\,\alpha)$ and $(1 - \sqrt{2}\,\mathrm{dn}\,\beta)$. This is not hard to do, following for instance methods used in [37] to relate the wavefunctions of a two-dimensional hydrogen atom, calculated in different coordinate systems. The formulae are cumbersome, so we shall not present them here.

4. Conclusions

In this paper we have presented a new aspect of the theory of Lie group and Lie algebra contractions: the relation between separable coordinate systems in curved and flat spaces, related by the contraction of their isometry groups. So far we have considered the simplest meaningful example, namely the original Wigner–Inonu contraction from a $O(3)$ to $E(2)$, as applied to the sphere S_2 and Euclidean plane E_2.

We have followed through the contraction $R \to \infty$ (where R is the radius of the sphere S_2) at all levels: the Lie algebras as realized by vector fields, the Laplace–Beltrami operators in the two spaces, the second-order operators in the enveloping algebras, characterizing separable systems, the separable coordinate systems themselves, the separated (ordinary) differential equations and the separated eigenfunctions of the invariant operators.

In particular, we have shown how different limiting procedures lead from two separable systems on S_2, to four systems on the plane E_2.

The contraction parameter in this paper was a geometrical one, namely the radius R of the S_2 sphere. The results of this paper are also applicable in a completely different context, namely that of high energy physics. Indeed, an elementary particle can be described by a wavefunction, transforming according to an irreducible unitary representation of the Poincare group [38, 39]. Its spin is associated with the Pauli–Lubanski vector

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho} M^{\nu\lambda} P^\rho .$$

If the particle is massive, the linear momentum is timelike $p^2 = m^2 > 0$. In the rest frame we have $p_0 = m$, $\boldsymbol{p} = 0$ and the space components of W_μ generate an o(3) algebra. If the particle is massless, then its momentum is lightlike, $p^2 = 0$ and no rest frame exists. We can choose a frame with $p = (\omega, 0, 0, \omega)$ and the three lineary independent components of W_μ generate an e(2) algebra. Let us parametrize the linear momentum of a massive particle in spherical coordinates on the mass shell

$$p_0 = m\cosh a \qquad p_1 = m\sinh a\sin\theta\cos\phi$$
$$p_2 = m\sinh a\sin\theta\sin\phi \qquad p_3 = m\sinh a\cos\theta .$$

At very high energies $p_0 \to \infty$ we have

$$\cosh a \sim \sinh a \sim \tfrac{1}{2}e^a$$

and the momentum p approaches a lightlike one.

All results of this paper are applicable in this case with a reinterpretation of the contraction parameter. This role is now played by the particle energy p_0, or more precisely by the 'energy' coordinate a. The physical relevance of such a contraction is obvious: in the ultrarelativistic limit [40] massive particles start to behave like massless ones, for which spin considerations, for example, reduce to helicity ones.

We mention that the realization (2.26)–(2.28) of o(3) in terms of vector fields in two variables corresponds to the O(3) subgroup of the group $SL(3,\mathbb{R})$ figuring as the group of projective transformations of $\mathbb{R}^2$. Indeed, more generally, $sl(n+1,\mathbb{R})$ can be realized as

$$\begin{aligned} &P_\mu = \frac{\partial}{\partial x_\mu} \qquad L_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} \qquad C_\mu = x_\mu D \\ &D \equiv \sum_{\alpha=1}^{n} x_\alpha \frac{\partial}{\partial x_\alpha} \qquad 1 \leqslant \mu, \quad \nu \leqslant n. \end{aligned} \tag{4.1}$$

This shows that the methods of this paper can easily be adapted to treat other contraction problems. For $n = 2$ we can construct the $o(2,1) \subset sl(3,\mathbb{R})$ subalgebra and consider contractions of separable coordinates on one- or two-sheeted hyperboloids and their contractions to coordinates on Euclidean or pseudo-Euclidean planes. For $n \geqslant 3$ we can again extract o(n), or o(p,q) subalgebras of $sl(n,\mathbb{R})$ and consider similar contraction problems. This, however, lies beyond the scope of the present paper.

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