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OSCILLATOR AS A HIDDEN NON-ABELIAN MONOPOLE

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Systems with nontrivial topology are the most interesting objects of quantum mechanics [1,2], quantum field theory [3,4], and condensed matter physics [5,6].

In pioneering papers [7,8], these systems were merely postulated; later on, they were deduced in quantum field theory from the first principles [9,10].

The problem of generation of systems with nontrivial topology in quantum mechanics was formulated in refs. [11, 12]; while in refs. [13,14], a four-dimensional isotropic oscillator was related to an Abelian system "charge–dyon".

This note is aimed at constructing a non-Abelian $SU(2)$ model that describes the system "charge–dyon".

Our consideration is essentially based on the Hurwitz transformation [15–18]:

$$\begin{aligned} x_0 &= u_0^2+u_1^2+u_2^2+u_3^2-u_4^2-u_5^2-u_6^2-u_7^2 \\ x_1 &= 2(u_0u_4+u_1u_5-u_2u_6-u_3u_7) \\ x_2 &= 2(u_0u_5-u_6u_4+u_2u_7-u_3u_6) \\ x_3 &= 2(u_0u_6+u_1u_7+u_2u_4+u_3u_5) \\ x_4 &= 2(u_0u_7-u_1u_6-u_2u_5+u_3u_4) \end{aligned} \tag{1}$$

Transformation (1) obeys the identity

$$r \equiv (x_0^2+\ldots+x_4^2)^{1/2} = u_0^2+\ldots+u_7^2 \equiv u^2$$

We complement (1) with the angular coordinates [19]

$$\begin{aligned} \alpha_1 &= \frac{i}{2}\ln\frac{(u_0+iu_1)(u_2-iu_3)}{(u_0-iu_1)(u_2+iu_3)} \\ \beta_1 &= 2\arctan\left[\frac{(u_0+iu_1)(u_0-iu_1)}{(u_2+iu_3)(u_2-iu_3)}\right]^{1/2} \\ \gamma_1 &= \frac{i}{2}\ln\frac{(u_0+iu_1)(u_2+iu_3)}{(u_0-iu_1)(u_2-iu_3)} \end{aligned} \tag{2}$$

Transformations (1)–(2) transform the space $R^8(\vec{u})$ into the direct product of the space $R^5(\vec{x})$ and three-dimensional sphere $S^3(\alpha_1,\beta_1,\gamma_1)$. From (2) it follows that $\alpha_1 \in [0,2\pi)$, $\beta_1 \in [0,\pi]$, $\gamma_1 \in [0,4\pi)$.

Consider, in $R^8(\vec{u})$, the quantum oscillator

$$\frac{\partial^2\psi}{\partial u_\mu^2} + \frac{2M}{\hbar^2}\left(E - \frac{M\omega^2 u^2}{2}\right)\psi = 0, \quad u_\mu \in R^8 \tag{3}$$

Formulae (1)–(3) provide the basic result of our paper

$$\frac{1}{2M}\left(-i\hbar\frac{\partial}{\partial x_j} - \hbar A_j^a \hat{T}_a\right)^2 \psi + \frac{\hbar^2}{2Mr^2}\hat{T}^2\psi - \frac{\alpha}{r}\psi = \epsilon\psi \tag{4}$$

Here $j = 1, 2, ..., 5; a = 1, 2, 3$; parameters ϵ and α are connected with the oscillator parameters

$$\epsilon = -M\omega^2/8, \qquad \alpha = E/4$$

Equations (3) and (4) are dual to each other. In equation (3) ω is fixed and E is quantized, whereas in eq. (4) E is fixed and ω is quantized. These situations are alternative to each other: that in eq. (3) having the meaning of energy acquires the meaning of coupling constant in eq. (4), and vice versa. For a fixed E equation (3) selects an infinite set of monoenergetic states with frequencies $\omega_N = E/\hbar(N+4), N \in$ Z. These states form a complete system of functions describing the object connected with eq.(4).

Further, $\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2$, and the operators $\hat{T}_a$ satisfy the commutation relations

$$[\hat{T}_a, \hat{T}_b] = i\epsilon_{abc}\hat{T}_c$$

and are explicitly written as follows:

$$\begin{aligned}
\hat{T}_1 &= i\left(\cos\alpha_1 \cot\beta_1 \frac{\partial}{\partial\alpha_1} + \sin\alpha_1 \frac{\partial}{\partial\beta_1} - \frac{\cos\alpha_1}{\sin\beta_1}\frac{\partial}{\partial\gamma_1}\right)\\
\hat{T}_2 &= i\left(\sin\alpha_1 \cot\beta_1 \frac{\partial}{\partial\alpha_1} - \cos\alpha_1 \frac{\partial}{\partial\beta_1} - \frac{\sin\alpha_1}{\sin\beta_1}\frac{\partial}{\partial\gamma_1}\right) \qquad (5)\\
\hat{T}_3 &= -i\frac{\partial}{\partial\alpha_1}
\end{aligned}$$

The triplet of five–dimensional vectors A_j^a is given by the expressions

$$\begin{aligned}
A_j^1 &= \frac{1}{r(r+x_0)}(0, -x_4, -x_3, x_2, x_1)\\
A_j^2 &= \frac{1}{r(r+x_0)}(0, x_3, -x_4, -x_1, x_2)\\
A_j^3 &= \frac{1}{r(r+x_0)}(0, x_2, -x_1, x_4, -x_3)
\end{aligned}$$

The vectors A_j^a are orthogonal to each other,

$$A_j^a A_j^b = \frac{1}{r^2}\frac{(r-x_0)}{(r+x_0)}\delta_{ab}$$

and also to the vector $\vec{x} = (x_0, ..., x_4)$.

Every term of the triplet coincides with the vector potential of a 5–dimensional Dirac monopole with a unit magnetic charge and the line of singularity extending along the negative axis x_0, including point $x_0 = 0$.

Equation (4) describes a 5–dimensional system "charge–dyon". The operators $\hat{T}_a$ represent components of the total momentum operator of that system and are generators of the $SU(2)$ group of invariance of equation (4). The term $\hbar^2\hat{T}^2/2Mr^2$ represents the correction introduced by Goldhaber [20].

In the space $R^5(\vec{x})$ we introduce the hyperspherical coordinates

$$\begin{aligned} x_0 &= r\cos\theta \\ x_1 + ix_2 &= r\sin\theta\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}} \\ x_3 + ix_4 &= r\sin\theta\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} \end{aligned}$$

It is to be noted that $\theta \in [0, \pi]$, $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 4\pi)$.

Equation (4) in these coordinates assumes the form

$$\left(\Delta_{r\theta} - \frac{\hat{J}^2}{r^2\sin^2\theta/2} - \frac{\hat{L}^2}{r^2\cos^2\theta/2}\right)\psi + \frac{2M}{\hbar^2}\left(\epsilon + \frac{\alpha}{r}\right)\psi = 0 \qquad (6)$$

where

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2, \quad \hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2, \quad \hat{L}_a = \hat{J}_a + \hat{T}_a$$

and the operators $\hat{J}_a$ follow from formulae (5) under the change $\alpha_1 \to \alpha$, $\beta_1 \to \beta, \gamma_1 \to \gamma$. Then,

$$\Delta_{r\theta} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right)$$

By G we shall denote common eigenfunctions of the commuting operators $\hat{L}^2$ and $\hat{J}^2$ corresponding to the eigenvalues $L(L+1)$ и $J(J+1)$.

If we look for the solution to eq. (4) in the form

$$\Psi = R(r)Z(\theta)G(\alpha, \beta, \gamma; \alpha_1, \beta_1, \gamma_1)$$

then upon separating the variables r and θ and introducing the constant of separation $\lambda(\lambda+3)$ (for the time being λ is arbitrary), we arrive at two ordinary differential equations whose solutions are given by the following expressions

$$Z(\theta) = (1-\cos\theta)^J(1+\cos\theta)^L$$

$$_2F_1\left(-\lambda+J+L, \lambda+J+L+3, 2J+2; \frac{1-\cos\theta}{2}\right)$$

$$R_{n_r\lambda}(r) = const(2\kappa r)^\lambda e^{-\kappa r} F\left(\lambda+2-\frac{1}{r_0\kappa}, 2\lambda+4; 2\kappa r\right)$$

Here we introduced the notation:

$$r_0 = \hbar^2/M\alpha^2, \qquad \kappa^2 = -2M\epsilon/\hbar^2$$

From standard conditions it follows that λ should be either integer (including $\lambda = 0$) or half–integer and the energy spectrum of the 5–dimensional bound system "charge–dyon" is of the form

$$\epsilon = -\frac{M\alpha^2}{2\hbar^2(n_r+\lambda+2)^2}, \quad n_r \in \mathrm{Z}, \quad \lambda \in \mathrm{Z} \tag{7}$$

The energy levels (7) are degenerate with the multiplicity

$$g_N = \frac{(N+7)!}{7!N!}, \qquad N = 2(n_r+\lambda)$$

Instead of an oscillator, one could consider a system with the potential energy

$$M\omega^2u^2/2 + V(u^2)$$

where $V(u^2)$ is an arbitrary function. Then, an extra term $V(r)/4r$ would arise in equation (4), though equations (3) and (4) would remain dual to each other.

There are no a priori reasons to consider that equation (3) is in a way related to the problem of magnetic charge. To obtain a non–Abelian monopole from an oscillator, we should apply to the nontrivial system of transformations (1)–(2); and therefore, we called the $8D$ oscillator the hidden non–Abelian monopole.

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Мардоян Л.Г., Сисакян А.Н., Тер-Антонян В.М. E2-96-24
Осциллятор как скрытый неабелевый монополь

Из восьмимерного изотропного осциллятора преобразованием Гурвица построена неабелева $SU(2)$-модель, описывающая пятимерную связанную систему заряд-дион. Сформулирован принцип дион-осцилляторной дуальности. Вычислен энергетический спектр и волновые функции системы заряд-дион.

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Oscillator as a Hidden Non-Abelian Monopole

A non-Abelian $SU(2)$ model is constructed for a five-dimensional bound system «charge-dyon» on the basis of the Hurwitz-transformed eight-dimensional isotropic quantum oscillator. The principle of dyon-oscillator duality is formulated; the energy spectrum and wave functions of the system «charge-dyon» are calculated.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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