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METHOD OF VARIATIONAL PERTURBATION THEORY

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Approximation of a quantity under consideration by a finite number of terms of a certain series is a standard computational procedure in many problems of physics. In quantum field theory this is conventionally an expansion into a perturbative series. This approach combined with the renormalization procedure is now a basic method for computations. As is well-known, perturbative series for many interesting models including realistic models are not convergent. Nevertheless, at small values of the coupling constant these series may be considered as asymptotic series and could provide a useful information. However, even in the theories with a small coupling constant, for instance, in quantum electrodynamics there exist problems which cannot be solved by perturbative methods. Also, a lot of problems of quantum chromodynamics require nonperturbative approaches. At present, a central problem of quantum field theory is to go beyond the scope of perturbation theory.

A great amount of studies is devoted to the development of nonperturbative methods. Among them is the summation of a perturbative series (see reviews [1,2] and monograph [3]). The main difficulty is that the procedure of summation of asymptotic series is not unique, which is generally a functional arbitrariness, and the correct formulation of a problem of summation is ensured by further information on the sum of a series [4]. At present information of that kind is known only for the simplest field-theoretical models [5].

In refs. [6-10] approaches are proposed which are not directly based on the perturbative series. Thus, the method of Gaussian effective potential has recently become rather popular [11-14]. Many of nonperturbative approaches make use of a variational procedure for finding the leading contribution. However, in this case there is no always an algorithm of calculating corrections to the value found by a variational procedure, and this makes difficult to answer the question how adequate is the so-called main contribution to the object under investigation and what is the range of applicability of the obtained estimations. However, even if the algorithm of calculating corrections, i.e. terms of a certain approximating series, exists, it is not still sufficient. Here of fundamental importance are the properties of convergence of a series. Indeed, unlike the case when even a divergent perturbative series in the weak coupling constant approximates a given object as an asymptotic series, the approximating series in the absence of a small parameter should obey more strict requirements. Reliable information in this case may be obtained only on the basis of convergent series. It is more reliable to deal not with an arbitrary convergent series but just with the Leibniz series (an alternating series with terms decreasing in absolute value). Then it will become possible to compute upper and lower estimates for a given quantity on the basis of first terms of the series. In case of additional free parameters influencing the terms of the series, these estimates may be made as close as possible to each other.

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In this paper, we consider a method of variational perturbation theory (VPT) [15-17]. Despite the word "perturbation" being present in the name of the approach, the VPT method does not use any small parameter. The additions in the VPT method are calculable because this method employs only calculable Gaussian functional quadratures. Besides, a VPT series can be written so that its terms are defined by the usual Feynman diagrams. In this case, the VPT series will surely differ in structure from the conventional perturbation theory, and diagrams will contain a modified propagator.

Here we will apply the VPT method to Green functions of the φ^4-model in the Euclidean d-dimensional space. To this end we write the 2ν-point function in the form

$$G_{2\nu} = \int D\varphi\{\varphi^{2\nu}\}\exp\Big(-S[\varphi]\Big), \tag{1}$$

where

$$\{\varphi^{2\nu}\} = \varphi(x_1)...\varphi(x_n)$$

and the functional of action looks as follows:

$$\begin{aligned} S[\varphi] &= S_0[\varphi] + \frac{m^2}{2}S_2[\varphi] + \lambda S_4[\varphi], \\ S_0[\varphi] &= \frac{1}{2}\int dx\,(\partial\varphi)^2, \qquad S_p[\varphi] = \int dx\,\varphi^p. \end{aligned} \tag{2}$$

We shall construct a VPT series by using the following Gaussian functional quadratures

$$\begin{aligned} &\int D\varphi \exp\Big\{-\Big[\frac{1}{2} < \varphi\hat{K}\varphi > + < \varphi J >\Big]\Big\} = \\ &= \left(\det\frac{\hat{K}}{-\partial^2+m^2}\right)^{-1/2}\exp\Big[\frac{1}{2} < J\hat{K}^{-1}J >\Big]. \end{aligned} \tag{3}$$

The VPT series for the Green functions (1) is constructed in the following way:

$$G_{2\nu} = \sum_{n=0}^{\infty} G_{2\nu,n}, \tag{4}$$

$$G_{2\nu,n} = \frac{(-1)^n}{n!}\int D\varphi\Big\{\varphi^{2\nu}\Big\}\Big(\lambda S_4[\varphi] - \tilde{S}[\varphi]\Big)^n \exp\Big(-S_0[\varphi] - \frac{m^2}{2}S_2[\varphi] - \tilde{S}[\varphi]\Big). \tag{5}$$

The variational functional $\tilde{S}[\varphi]$ will be taken to be dependent on certain parameters, but the total sum (4) surely will not depend on these parameters. Their choice can be such as to provide the expansion (4) being optimal.

The functional $\tilde{S}[\varphi]$ should be defined so that the terms of the VPT series (4) be calculable, i.e. the form of $\tilde{S}[\varphi]$ should be such that the functional integral in (5) can be reduced to the Gaussian quadratures (3). This requirement does not mean that the functional $\tilde{S}[\varphi]$ must be quadratic in fields. We can pass to the Gaussian functional integral by using the Fourier transformation.

We choose here, for example, the sum of harmonic and anharmonic functionals being $\tilde{S}[\varphi]$,i.e.:

$$\tilde{S}[\varphi] = \frac{M^2}{2}S_2[\varphi] + \theta^2 S_2^2[\varphi], \tag{6}$$

where M and θ are the certain parameters through which the VPT series is optimised. We obtain

$$\begin{aligned} G_{2\nu,n} &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{1}{l!(n-k-l)!}\int_{-\infty}^{\infty}\frac{du}{2\sqrt{\pi}}\exp\Big(-\frac{u^2}{4}\Big) \\ &\times\ \theta^{2l}\Big(M^2-m^2\Big)^{n-k-l}\Big(-\frac{\partial}{\partial M^2}\Big)^{n+l-k}\tilde{g}_{2\nu}^{(k)}(\chi^2), \end{aligned} \tag{7}$$

where

$$\tilde{g}^{(k)}_{2\nu,n}(\chi^2) = \frac{1}{k!}\int D\varphi\{\varphi^{2\nu}\}\Big(-\lambda S_4[\varphi]\Big)^k \exp\Big\{-\Big[S_0[\varphi]+\frac{\chi^2}{2}S_2[\varphi]\Big]\Big\}. \quad (8)$$

The latter expression can be written as follows

$$\tilde{g}^{(k)}_{2\nu,n}(\chi^2) = \det\left[\frac{-\partial^2+\chi^2}{-\partial^2+m^2}\right]^{-1/2} g^{(k)}_{2\nu,n}(\chi^2), \quad (9)$$

where $g^{(k)}_{2\nu,n}(\chi^2)$ are calculated on the basis of diagrams of the $k-th$ order of conventional perturbation theory with the propagator $\Delta(p,\chi^2) = (p^2+\chi^2)^{-1}$. A new mass parameter χ^2 is depended on u and variational parameters M^2 and θ. Thus, the $N-th$ order of the VPT expansion (4) can be constructed with the same diagrams as the conventional perturbation $N-th$ order is made up.

Let us consider a case of the quantum-mechanical anharmonic oscillator (AO) as an example of exploiting the VPT method. The AO from a point of view of the path integral formalism is a one-dimensional φ^4-model. The connection between the ground state energy E_0 and the dimensionless four-point Green function $G_4(0,0,0,0)$ takes the form .

$$\frac{\partial E_0}{\partial \lambda} = \lambda^{-2/3} G_4 . \quad (10)$$

For calculating Green function G_4 we will use the two-parameters anharmonical VPT functional

$$\tilde{S}[\varphi] = [\theta S_0[\varphi] + \chi S_2[\varphi]]^2 . \quad (11)$$

The application of the asymptotic optimization that requires the contribution of the remote terms in the VPT series to be minimal allows one to find the relation between the parameter θ and χ : $16\theta\chi^3 = 9$. The remaining variational parameter is fixed on the basis of a finite number of VPT expansion terms. For the ground state energy in the first order of VPT we get the strong coupling expansion

$$E_0^{(1)} = \lambda^{1/3}[0.663 + 0.1407\omega^2 - 0.0085\omega^4 + \cdots] , \quad (12)$$

where the dimensionless parameter $\omega^2 = m^2\lambda^{-2/3}$. We have to compare the obtained result with the exact value [18]

$$E_0^{exact} = \lambda^{1/3}[0.668 + 0.1437\omega^2 - 0.0088\omega^4 + \cdots] . \quad (13)$$

We can also calculate the mass parameter μ^2 connected with the two-point Green function: $\mu^{-2} = G_2(p=0)$. In the strong coupling limit we obtain $\mu^2 = 3.078\lambda^{2/3}$, whereas the exact value is $\mu^2_{exact} = 3.009\lambda^{2/3}$. We can estimate the energy of the first excited level E_1. Defining the energy shift $\mu_1 = E_1 - E_0$ and using the spectral representation for the propagator we arrive at the following estimate for μ_1 :$\mu_1 \le \mu_1^{(+)}$, where

$$\mu_1^{(+)} = 2G_2(x=0)/G_2(p=0) . \quad (14)$$

By analogy with the sum rules, we may expect a sufficietly rapid saturation of the spectral representation, which brings μ_1 and $\mu_1^{(+)}$ closer to each other. In the first order of the one-parameter VPT in the strong coupling limit we get $\mu_1^{(+)} = 1.763\lambda^{1/3}$, whereas exact value is $\mu_1^{exact} = 1.726\lambda^{1/3}$ [18]. The effective potential and corresponded numerical characteristics for AO was computed in [16].

Now consider a massless $\varphi^4_{(4)}$ theory in the four-dimensional Euclidean space. For the generating functional of the Green functions $W[J]$ with the variational addition taken in the form $g\,t\,S_0^2[\varphi]$ we obtain the following VPT series

$$
\begin{aligned}
W[J] &= \int D\varphi \exp\Big\{-S[\varphi] + < J\varphi >\Big\} = \sum_{n=0}^{\infty} W_n[J,t], \\
W_n[J,t] &= \frac{(-g)^n}{n!} \int D\varphi \left[\frac{S_4}{4C_s} - t\,S_0^2\right]^n \exp\Big\{-S_0 - g\,t\,S_0^2 + < J\varphi >\Big\}.
\end{aligned} \tag{15}
$$

Here we made use of the constant $C_s = 4!/(16\pi)^2$ from the Sobolev inequality (see, for instance, refs. [3,19]):

$$
S_4[\varphi]/S_0^2[\varphi] \leq 4\,C_s \tag{16}
$$

and set $\lambda = g/(4C_s)$.

In the given case we are interested in the problem of convergence of the series. Asymptotic estimate of remote expansion terms can be made by the functional saddle-point method [1-3,20]. To this end, we represent $W_n[g,t]$ in the form

$$
W_n[J,t] = (-g)^n \frac{n^n}{n!} \int D\varphi \exp\Big\{-nS_{eff} - n^{1/2}S_0 + n^{1/4} < J\varphi >\Big\}, \tag{17}
$$

where

$$
\begin{aligned}
S_{eff}[\varphi] &= g\,t\,S_0^2[\varphi] - \ln D[\varphi], \\
D[\varphi] &= \frac{S_4[\varphi]}{4\,C_s} - t\,S_0^2[\varphi].
\end{aligned} \tag{18}
$$

The main contribution to the integral (17) in the leading order in the large saddle-point parameter n comes from the functions φ_0 obeying the equation $\delta S_{eff}/\delta\varphi = 0$,i.e.,

$$
\partial^2\varphi_0 + \frac{a}{3!}\varphi_0^3 = 0, \tag{19}
$$

and leaving the action functional to be finite. Their explicit form is as follows

$$
\varphi_0(x) = \sqrt{\frac{48}{a}}\,\frac{\mu}{(x-x_0)^2 + \mu^2}, \tag{20}
$$

$$
a = \frac{32\pi^2}{tS_0[\varphi_0]\{1 + gD[\varphi_0]\}}. \tag{21}
$$

Arbitrary parameters x_0 and μ in (20) reflect the translational and scale invariance of the theory under consideration. From (20) and (21) it follows that $a^2 = g\,t\,(32\pi^2)^2$; as a result we obtain

$$
W_n[J,t] \sim (-1)^n \frac{n^n}{n!} \left(\frac{1-t}{t}\right)^n \exp\Big\{-n - \sqrt{\frac{n}{g\,t}} + n^{1/4} < J\varphi_0 >\Big\}. \tag{22}
$$

From this expression it is clear that irrespective of the values of the coupling constant g, the VPT series (15) absolutely converges when $t > 1/2$ and when $t > 1$, as follows from the Sobolev inequality (16), that series is of positive sign. In the interval $1/2 < t < 1$ at large n the series (15) is the Leibniz series. Here again the value $t = 1$ corresponds both to the change of the regime of the VPT series and to its asymptotic optimization. Note is to be made that the expression (22) determines only the leading contribution to the functional dependence of W_n on the large parameter n. In particular, in (22) we do not reproduce a certain multiplier that appears in the next to leading order in n. However, the properties of convergence of the series can be quite well analyzed in the leading order in n.

Let us consider the VPT method in view of point of connection between the VPT and the method of Gaussian effective potential (GEP) [11-14]. In the following, we shall have in mind the dimensional regularization setting $n = d - 2\varepsilon$, where d is an integer number. We separate the classical contribution in the generating functional of Green functions $W[J]$ by writing

$$W[J] \;=\; \int D\varphi \exp\Big\{i\Big[S[\varphi] + <J\varphi>\Big]\Big\} = \exp\Big\{i\Big[S[\varphi_c] + <J\varphi_c>\Big]\Big\}D[J], \qquad (23)$$

where

$$D[J] = \int D\varphi \exp\Big\{-iP[\varphi]\Big\}, \qquad (24)$$

$$P[\varphi] = \int dx\,[\frac{1}{2}\varphi\,(\partial^2 + m^2 + 12\lambda\varphi_c^2)\,\varphi + 4\lambda\varphi_c\varphi^3 + \lambda\varphi^4] \qquad (25)$$

and the function φ_c satisfies a classical equation of motion $\delta S/\delta\varphi_c = -J$.

In the standard classical approximation one would retain only the addends quadric in fields in expression (25) for the quantity $P[\varphi]$. In this case the functional integral for $D[J]$ becomes Gaussian and for $W[J]$ the ordinary one-loop representation arises.

Let us now calculate the quantity $D[J]$ by using the anharmonic variation of the action functional. We choose the VPT functional in the form $\tilde{S}[\varphi] = R^2[\varphi]$, where

$$R[\varphi] = \frac{\chi}{2\Omega^{1/2}} \int dx\,\varphi^2(x).$$

The space volume Ω appears here because V_{eff} is derived from the effective action by using the constant-field configurations. Thus, the parameter χ, optimizing the VPT series, does not depend on Ω.

Any power of $R^2[\varphi]$ in the VPT expansion can be obtained by the corresponding number of differentiations of the expression $\exp(-i\varepsilon R^2[\varphi])$ with respect to ε with putting $\varepsilon = 1$ at the end. As to the addend $R^2[\varphi]$ in the exponential, giving rise to a nongaussian form of the functional integral, the problem is easily to solve by implementing the Fourier transformation, due to which only the first power of $R[\varphi]$ emerges in the exponential.

As a result, the VPT series takes the form

$$\begin{aligned} D[J] \;=\; & \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} \frac{(-i)^{n-k}}{(n-k)!n!}\left[\frac{d}{d\varepsilon}\right]^{n-k} \\ \times\; & \sqrt{\Omega}\int_{-\infty}^{\infty} \frac{dv}{2\sqrt{\pi}} \exp\Big\{i\Omega\frac{v^2}{4} - i\frac{\pi}{4}\Big\}\left[\det\frac{\partial^2 + M^2}{\partial^2}\right]^{-1/2} \\ \times\; & \Big[\lambda\int dx\Big(4\varphi_c\hat{\varphi}^3 + \hat{\varphi}^4\Big)\Big]^k \exp\Big[-\frac{i}{2} <j\Delta j>\Big]_{j=0}, \end{aligned} \qquad (26)$$

where

$$\begin{aligned} \Delta(p) &= (p^2 - M^2 + i0)^{-1}, \\ \hat{\varphi}(x) &= i\frac{\delta}{\delta j(x)}, \end{aligned}$$

$$M^2 = m^2 + 12\lambda\varphi_c^2 + \sqrt{\varepsilon}\chi v.$$

The integral over v in (26) contains the large parameter Ω and, hence, can be evaluated by using the method of a stationary phase. Then, the effective potential in the first nontrivial

VPT-order looks as

$$\begin{aligned} V_{eff} &= V_c + V_0 + V_1, \\ V_0 &= \frac{1}{n} M^2 \Delta_0 - \frac{\chi^2}{4} \Delta_0^2, \\ V_1 &= -\frac{\chi^2}{4} \Delta_0^2 + 3\lambda \Delta_0^2, \end{aligned} \tag{27}$$

where Δ_0 is the Euclidean propagator $\Delta_{Eucl}(x=0, M^2)$ written with the help of the dimensional regularization. Here M^2 is the massive parameter taken at $\varepsilon = 1$ and $v = v_0$, where v_0 is the stationary phase point in the integral (26). The corresponding equation reads

$$M^2 = m^2 + 12\lambda\varphi^2 + \chi^2\Delta_0(M^2). \tag{28}$$

One can apply now the following optimization versions (see [15,16]): (i) The requirement $\min|V_1|$ (here there exists the solution to the equation $V_1 = 0$); (ii) $\partial V_{eff}/\partial\chi^2 = 0$. It is easy to find out that these different versions give rise to the same optimal value of the parameter χ^2: $\chi^2 = 12\lambda$. As a result, the effective potential (27) with the condition (28) yields the GEP in n-dimensional space [21].

It is interesting that the first order of the VPT for $D[J]$ give us GEP by various ways of constructing the variational procedure. However, despite the same result in the first order, other properties of the series are different. In the case of the harmonic variational procedure the VPT series is the asymptotic series. In the case of the anharmonic VPT procedure we can obtain the convergence series. It is important in the point of view of the stability of the results obtained as the leading contribution.

When the variational addition is harmonic, the VPT series is asymptotic and its higher-order terms behave like the terms of standard perturbation theory. Nevertheless, the harmonic variational addition produces a certain stabilization of the results for the further radiative corrections. In the regions where the partial sums of conventional perturbation theory suffer of oscillations specific for asymptotic series, the VPT series gives a stable result. However, the harmonic way of varying the action though rather making pass to large coupling constants does not lead essentially off the weak-coupling region. This can be achieved by passing to the variational addition of the anharmonic type, which can be explained as follows: For higher-order terms of the VPT series the major contribution to the functional integral comes from the field configurations that are proportional to the positive degree of the large saddle-point parameter. Therefore, the effective interaction $\lambda S_4[\varphi] - \tilde{S}[\varphi]$ is dominated by the conventional term $\lambda S_4[\varphi]$ that, as in perturbation theory, leads to an asymptotic series. A different picture arises when the action is varied with the help of an anharmonic functional. Here the degrees of fields in $\lambda S_4[\varphi]$ and $\tilde{S}[\varphi]$ are the same and the variational addition greatly influences the asymptotic behavior of higher-order terms of the VPT series. In this paper we have shown that there exists a finite region of values of the variational parameters where the VPT series converges for all positive coupling constants. We would like to mention one more interesting possibility of the VPT method, namely, construction of the Leibniz series as the VPT series. When a searched quantity is represented by this series, even and odd partial sums of the series define the estimates of upper and lower bounds. In other words, for a case under consideration we can obtain nonperturbative estimates of the upper and lower bounds; and which is more, when we have free parameters defining the VPT series terms, we can govern the accuracy of estimates.

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References

[1] J.Zinn-Justin, Phys.Rep. **70** (1981) 109.

[2] D.I.Kazakov, D.V.Shirkov, Fortschr.Phys. **28** (1980) 465.

[3] J.Zinn-Justin: Quantum Field Theory and Critical Phenomena. Clarendon Press. Oxford, 1989.

[4] M.Reed, B.Simon: Methods of modern mathematical physics. Vol. 4 Analysis of operators. New York, San Francisco, London: Academic Press, 1978.

[5] J.Climm, A.Jaffe: Quantum Physics. A Functional Integral Point of View. New York, Heidelberg, Berlin: Springer 1981.

[6] G.V.Efimov, Comm. Math. Phys. **65** (1979) 15; Problems of Quantum Theory of Nonlocal Interactions, Moscow,1985.

[7] I.J.Halliday, P.Suranyi, Phys. Lett. **B 85** (1979) 421; Phys. Rev. **D 21** (1980) 1529; W.E.Caswell, Ann. Phys. **123** (1979) 153.

[8] G.Parisi, Phys. Lett. **B 69** (1977) 329; P.Castoldi, C.Schomblond, Nucl. Phys. **B 139** (1978) 269; C.M.Bender et al, Phys. Rev. **D 19** (1979) 1865.

[9] H.M.Fried, Nucl. Phys. **B 169** (1980) 329; J.P.Ader, B.Bonner, M.Hontebeyrie, Nucl. Phys. **B 170** (1980) 165; R.B.Abbot, Z. Phys. **C 26** (1984) 393.

[10] L.Polley, U.Ritschel, Phys.Lett. **B 221** (1989) 44; M.H.Thoma, Z.Phys. **C 44** (1991) 343; J.Soto, Nucl.Phys **B 316** (1989) 141.

[11] T.Barnes, G.I.Ghandour, Phys.Rev. **D 22** (1980) 924.

[12] P.M.Stevenson, Phys.Rev. **D 30** (1985) 1712; Phys.Rev. **D 32** (1985) 1389.

[13] G.A.Hajj, P.M.Stevenson, Phys.Rev. **D 37** (1988) 413; A.Okopinska, Phys.Rev. **D 35** (1987) 1835; Phys.Rev. **D 36** (1987) 2415; I.Stancu, P.M.Stevenson, Phys.Rev. **D 42** (1990) 2710.

[14] U.Ritschel, Phys.Lett. **B 227** (1989) 251; U.Ritschel, Z.Phys. **C 47** (1990) 457; Z.Phys. **C 51** (1991) 469.

[15] A.N.Sissakian, I.L.Solovtsov, Phys.Lett. **A 157** (1991) 261;

[16] A.N.Sissakian, I.L.Solovtsov, Z.Phys. **C 54** (1992) 263; L.D.Korsun, A.N.Sissakian, I.L.Solovtsov, Theor. and Math.Phys. **90** (1992) 37.

[17] A.N.Sissakian, I.L.Solovtsov, O.Yu.Shevchenko, Phys.Lett. **B 297** (1992) 305.

[18] F.T.Hioe, E.W.Montroll, J.Math.Phys. **16** (1975) 1945; F.T.Hioe, D.Mac-Millen, E.W. Montroll, Phys.Rep. **C 43** (1978) 305; L.I.Schiff, Phys.Rev. **92** (1953) 766.

[19] G.Parisi, Phys.Lett. **B 68** (1977) 361; N.N.Khuri, Phys.Lett. **B 150** (1985) 199.

[20] L.N. Lipatov, JETP **72** (1977) 411.

[21] C.G.Bollini, J.J. Giambiagi, Nuovo Cim. **93A** (1986) 113.