JOINT INSTITUTE FOR NUCLEAR RESEARCH

International Workshop

SYMMETRY METHODS IN PHYSICS

IN MEMORY OF PROFESSOR YA.A.SMORODINSKY

Dubna, Russia July 6 – 10 1993

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Volume 1

Dubna 1994

ELLIPSOIDAL BASES FOR ISOTROPIC OSCILLATOR

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1. Introduction

The role of the isotropic harmonic oscillator is rather well known in physics. Many real physical problems in their "zero" approximation lead to a problem of harmonic oscillator or a system of interacting oscillators. Moreover, the problem of isotropic oscillator potential is one of a small number of exactly solvable problems both in classical and quantum mechanics. Therefore, it is natural that this problem found it own place almost in all text–books of classical and quantum physics, and nowadays it is one of the best studied problems.

Though the problem of isotropic oscillator is important and characteristic, its solutions have been studied only in the simplest systems of coordinates (such as spherical, cylindrical and Cartesian). At the beginning of the eighties, the problem of isotropic oscillator has been thoroughly considered in ref.[1] in the prolate and oblate spheroidal and elliptical cylindrical systems of coordinates. In contrast with the simple coordinate systems, these three systems have a dimensional parameter R (which is pure kinematical for a potential of a harmonic oscillator) which turn into simpler ones in the limit $R = \infty$ and $R = 0$.

As is known [2], in the three-dimensional Euclidean space there exist 11 orthogonal coordinate systems admitting a complete separation of variables in the Hamilton–Jacobi equation or in the Helmholtz equation. It is also known that an isotropic oscillator is the only potential admitting a complete separation of variables in the Schrödinger equation in eight of eleven systems of coordinates. These are ellipsoidal, prolate spheroidal, oblate spheroidal, spherical, cylindrical, Cartesian, elliptic cylindrical and sphero–conical systems of coordinates.

In the present paper, the problem of an isotropic oscillator is solved in the ellipsoidal coordinate system.

It is obvious that the most general and complex of all 11 system of coordinates is the ellipsoidal coordinate system. It is a less degenerate system as three families of second order confocal surfaces are the coordinate surfaces. In contrast with all simple systems, the ellipsoidal coordinates are characterized by two dimensional parameters R_1^2 and R_2^2. When the parameters R_1^2 and R_2^2 tend to zero or infinity from the ellipsoidal system, all the remaining 10 coordinate systems are obtained in the limit [3]. In this respect, the solution of the problem with the potential of the isotropic oscillator in the ellipsoidal system of coordinates unifies all seven bases, and therefore, is of special interest.

The paper is organized as follows. The second section contains formulae relevant to the ellipsoidal system of coordinates and to different degenerate cases that appear as a result of the limiting transitions of the parameters R_1^2 and R_2^2. In sect. 3, the separation of variables in the Schrödinger equation is given, and it is shown that the problem of determining the ellipsoidal basis of the isotropic oscillator is reduced to the solution of the generalized Lame' equation with four singularities. Sect.4 contains operators corresponding to two ellipsoidal separation constants which with the system Hamiltonian form a full set of commuting operators for the ellipsoidal basis. By the limiting transitions of the parameters R_1^2 and R_2^2 we have derived all seven nonequivalent sets of commuting operators corresponding to the remaining types of bases for the isotropic oscillator. In sections 5 and 6 we have derived the solution of the ellipsoidal equation and constructed the ellipsoidal basis of the isotropic oscillator. Sect. 7 contains the explicit form of the ellipsoidal basis for some small values of quantum numbers.

2. The ellipsoidal coordinates

Let us describe the ellipsoidal coordinate system in which the Schrödinger equation for the isotropic oscillator potential admits a full separation of variables.

The algebraic form of the ellipsoidal coordinates has the form

$$\begin{aligned} x^2 &= \frac{(\rho_1-a_3)(\rho_2-a_3)(\rho_3-a_3)}{(a_2-a_3)(a_1-a_3)} \\ y^2 &= \frac{(\rho_1-a_2)(\rho_2-a_2)(\rho_3-a_2)}{(a_3-a_2)(a_1-a_2)} \\ z^2 &= \frac{(\rho_1-a_1)(\rho_2-a_1)(\rho_3-a_1)}{(a_3-a_1)(a_2-a_1)} \end{aligned} \qquad (1)$$

where the parameters a_1, a_2, a_3 entering into definition (1) restrict the region of variation of the ellipsoidal variables ρ_1, ρ_2, ρ_3:

$$0 \le a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3 \le \rho_3 < \infty$$

The inverse dependence of the ellipsoidal variables on the Cartesian ones is determined by a third degree equation with respect to ρ_i $(i = 1,2,3)$, and can be derived from the following system of three equations

$$\begin{aligned} (\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3) &= x^2+y^2+z^2 \\ (\rho_1\rho_2+\rho_1\rho_3+\rho_2\rho_3)-(a_1a_2+a_1a_3+a_2a_3) &= (a_2+a_3)z^2+(a_1+a_3)y^2+(a_1+a_2)x^2 \\ \rho_1\rho_2\rho_3-a_1a_2a_3 &= a_1a_2x^2+a_1a_3y^2+a_2a_3z^2 \end{aligned} \qquad (2)$$

The second order surfaces on which $\rho_i = const.$ represent complete families of confocal ellipsoids and one - and two-sheeted hyperboloids whose equations are written, respectively, in the form

$$\begin{aligned} \frac{x^2}{\rho_3-a_3}+\frac{y^2}{\rho_3-a_2}+\frac{z^2}{\rho_3-a_1} &= 1 \\ \frac{y^2}{\rho_2-a_2}+\frac{z^2}{\rho_2-a_1}-\frac{x^2}{a_3-\rho_2} &= 1 \\ \frac{z^2}{\rho_1-a_1}-\frac{y^2}{a_2-\rho_1}-\frac{x^2}{a_3-\rho_1} &= 1 \end{aligned} \qquad (3)$$

It immediately follows from formulae (3) that the ellipsoidal coordinate system has a distinguished axis z. Four focuses of the ellipsoidal system are on the axis z with the coordinates $z = \pm R_1 = \pm\sqrt{a_2-a_1}$ and $z = \pm R = \pm\sqrt{a_3-a_1}$, and two focuses are on the axis y with the coordinates $y = \pm R_2 = \pm\sqrt{a_3-a_2}$. The quantities R_1, R_2, R are dimensional parameters determining the ellipsoidal coordinate system and connected by a simple relation

$$R_1^2 + R_2^2 = R^2$$

which results in that only two parameters of the ellipsoidal system are independent, for example R_1 and R_2. Clearly, a choice like that is not unique and is connected with concrete parametrization of ellipsoidal coordinates.

Relations (1) and (2) connecting the Cartesian and ellipsoidal coordinates are not in the one-to-one correspondence as ρ_i $(i = 1,2,3)$ depend only on (x^2, y^2, z^2) and, consequently, take the same values at eight points $(\pm x, \pm y, \pm z)$. To obtain a one-to-one correspondence between the Cartesian and ellipsoidal coordinates, one can introduce uniformized variables γ, μ, ν determining the position of the point in space by the following relations:

$$\rho_1 = a_1 + (a_2 - a_1)\cos^2\mu, \quad \rho_2 = a_2 + (a_3 - a_2)\sin^2\nu, \quad \rho_3 = a_3 + (a_3 - a_1)\sinh^2\gamma. \tag{4}$$

Using (4) one can write down the ellipsoidal coordinate system in the trigonometric–hyperbolic form

$$\begin{aligned} x &= R\sinh\gamma\sqrt{1 - k^2\cos^2\mu}\cos\nu \\ y &= R\sqrt{k'^2 + \sinh^2\gamma}\sin\mu\sin\nu \\ z &= R\cosh\gamma\cos\mu\sqrt{1 - k'^2\cos^2\nu} \end{aligned} \tag{5}$$

$$0 \le \nu < 2\pi, \quad 0 \le \mu \le \pi, \quad 0 \le \gamma < \infty$$

where

$$k^2 = \frac{a_2 - a_1}{a_3 - a_1} = \frac{R_1^2}{R^2}, \qquad k'^2 = \frac{a_3 - a_2}{a_3 - a_1} = \frac{R_2^2}{R^2}, \qquad k^2 + k'^2 = 1.$$

The trigonometric–hyperbolic form of the ellipsoidal coordinate system is not used in the literature. Note that alongside with the algebraic form (1), the parametrization through the elliptical Jacobi and Weierstrass functions is used in the mathematical literature [4].

It is clearly seen from formula (5) than in particular cases $k^2 = 0$ and $k^2 = 1$ the ellipsoidal coordinate system turns into the oblate spheroidal coordinates with the symmetry axis along the axis x and into the prolate spheroidal coordinates with the symmetry axis along the axis z, respectively. If the parameter R will further be tending to zero or infinity, we can obtain either a spherical or a cylindrical coordinate systems, respectively. The limiting transition to the remaining coordinate systems can easily be traced from the system of equations (2) in terms of the variables $\overline{\rho}_i = \rho_i - a_2$. Now if we let R_1 and R_2 tend to zero and the ratio R_1/R is put finite equal to k^2, then one can easily see that the ellipsoidal coordinates degenerates into the sphero–conical one and upon substitution $k^2 = 0$ or $k^2 = 1$ turns into the spherical coordinates.

Further, as $R_1 \to \infty$ we arrive at the system of an elliptical cylinder, whose particular cases are the Cartesian and cylindrical coordinate systems. Thus, one can obtain all seven possible degenerate forms of the ellipsoidal coordinate system (see the table) without shifting the origin of coordinates.

If the center of the Cartesian system of coordinates is placed at the focus of the ellipsoidal system of coordinates R_1, i.e. one make translation $z' \to z - R_1$ in (1), then in the limit $R_1, R_2 \to \infty$ we obtain the paraboloidal coordinates a particular case of which is the rotational parabolic and the parabolic cylindrical coordinates.

The volume element and the Laplacian in the ellipsoidal coordinate system have the form

$$\begin{aligned} \Delta = 4\Bigg\{ &\frac{\sqrt{P(\rho_1)}}{(\rho_1 - \rho_2)(\rho_1 - \rho_3)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1} + \frac{\sqrt{P(\rho_2)}}{(\rho_2 - \rho_3)(\rho_2 - \rho_1)}\frac{\partial}{\partial\rho_2}\sqrt{P(\rho_2)}\frac{\partial}{\partial\rho_2} \\ &+ \frac{\sqrt{P(\rho_3)}}{(\rho_3 - \rho_1)(\rho_3 - \rho_2)}\frac{\partial}{\partial\rho_3}\sqrt{P(\rho_3)}\frac{\partial}{\partial\rho_3}\Bigg\} \end{aligned}$$

Table: The degenerations of the ellipsoidal coordinate system

prolate spheroidal system	$x = \frac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi$ $y = \frac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi$ $z = \frac{R}{2}\xi\eta$	$R_2^2 \to 0$ $R_1^2 = R^2/4$	$\bar{\rho}_1 \to \frac{R^2}{4}(\eta^2-1)$ $\bar{\rho}_2 \to R_2^2\sin^2\phi$ $\bar{\rho}_3 \to \frac{R^2}{4}(\xi^2-1)$
oblate spheroidal system	$x = \frac{\bar{R}}{2}\bar{\xi}\bar{\eta}$ $y = \frac{\bar{R}}{2}\sqrt{(\bar{\xi}^2+1)(1-\bar{\eta}^2)}\sin\phi$ $z = \frac{\bar{R}}{2}\sqrt{(\bar{\xi}^2+1)(1-\bar{\eta}^2)}\cos\phi$	$R_1^2 \to 0$ $R_2^2 = \bar{R}^2/4$	$\bar{\rho}_1 \to -R_1^2\sin^2\phi$ $\bar{\rho}_2 \to \frac{\bar{R}^2}{4}(1-\bar{\eta}^2)$ $\bar{\rho}_3 \to \frac{\bar{R}^2}{4}(1+\bar{\xi}^2)$
Spherical system	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$R_2^2 \to 0$ $R_1^2 \to 0$ $R_2^2/R_1^2 \to 0$	$\bar{\rho}_1 \to -R_1^2\sin^2\theta$ $\bar{\rho}_2 \to R_2^2\sin^2\phi$ $\bar{\rho}_3 \to r^2$
Sphero - conical system	$x = r\cos\psi\sqrt{(1-k^2\cos^2\bar{\psi})}$ $y = r\sin\psi\sin\bar{\psi}$ $z = r\cos\bar{\psi}\sqrt{(1-k'^2\cos^2\psi)}$	$R_1^2 \to 0$ $R_2^2 \to 0$ $R_1^2/R^2 \to k^2$	$\bar{\rho}_1 \to -R_1^2\sin^2\bar{\psi}$ $\bar{\rho}_2 \to R_2^2\sin^2\psi$ $\bar{\rho}_3 \to r^2$
Circular elliptic system	$x = \frac{R}{2}\sinh\mu\sin\nu$ $y = \frac{R}{2}\cosh\mu\cos\nu$ $z = z'$	$R_1^2 \to \infty$ $R_2^2 = R^2/4$	$\bar{\rho}_1 \to (z'^2 - R_1^2)$ $\bar{\rho}_2 \to \frac{R^2}{4}\cos^2\nu$ $\bar{\rho}_3 \to \frac{R^2}{4}\cosh^2\mu$
Circular polar system	$x = \rho\sin\phi$ $y = \rho\cos\phi$ $z = z'$	$R_1^2 \to \infty$ $R_2^2 \to 0$	$\bar{\rho}_1 \to (z'^2 - R_1^2)$ $\bar{\rho}_2 \to R_2^2\cos^2\phi$ $\bar{\rho}_3 \to \rho^2$
Cartesian system	$x = x'$ $y = y'$ $z = z'$	$R_1^2 \to \infty$ $R_2^2 \to \infty$ $R_2^2/R_1^2 \to 0$	$\bar{\rho}_1 \to (z'^2 - R_1^2)$ $\bar{\rho}_2 \to (x'^2 + R_2^2)$ $\bar{\rho}_3 \to y'^2$

$$dV = \frac{1}{8}\frac{(\rho_2-\rho_1)(\rho_3-\rho_2)(\rho_3-\rho_1)}{\sqrt{-P(\rho_1)P(\rho_2)P(\rho_3)}}\,d\rho_1\,d\rho_2\,d\rho_3$$

where

$$P(\rho) = (\rho - a_1)(\rho - a_2)(\rho - a_3)$$

To separate variables in the Schrödinger equation in the ellipsoidal coordinate system, a potential is to be of the following form:

$$V(\rho_1,\rho_2,\rho_3) = \frac{V_1(\rho_1)}{(\rho_1-\rho_2)(\rho_1-\rho_3)} + \frac{V_2(\rho_2)}{(\rho_2-\rho_3)(\rho_2-\rho_1)} + \frac{V_3(\rho_3)}{(\rho_3-\rho_1)(\rho_3-\rho_2)}$$

In particular, for the isotropic oscillator potential in the ellipsoidal variables we get the following expression:

$$\begin{aligned} V(x,y,z) &= \frac{m\omega^2(x^2+y^2+z^2)}{2} \\ &= \frac{m\omega^2}{2}\left\{\frac{P_1(\rho_1)}{(\rho_1-\rho_2)(\rho_1-\rho_3)} + \frac{P_2(\rho_2)}{(\rho_2-\rho_3)(\rho_2-\rho_1)} + \frac{P_3(\rho_3)}{(\rho_3-\rho_1)(\rho_3-\rho_2)}\right\} \end{aligned}$$

It is to be emphasized that if the Schrödinger equation with the potential $V(x,y,z)$ admits separation of variables in the ellipsoidal coordinates, it is also separated in all seven limiting coordinate systems (this fact does not concern boundary conditions imposed on the wave function).

3. Separation of variables

Having written down the Schrödinger equation for the isotropic oscillator potential

$$\mathcal{H}\psi = \left\{-\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2 r^2}{2}\right\}\psi = E\psi$$

in the ellipsoidal coordinate system (1) and using the identity

$$\frac{2mE}{\hbar^2} \equiv \frac{2mE}{\hbar^2}\sum_{i=1}^{3}\frac{\rho_i^2}{\prod_{n=1}^{3}(\rho_i-\rho_n)}$$

we arrive at the following equation

$$\sum_{i=1}^{3}\frac{1}{\prod_{n=1}^{3}(\rho_i-\rho_n)}\left\{4\sqrt{P(\rho_i)}\frac{\partial}{\partial\rho_i}\sqrt{P(\rho_i)}\frac{\partial}{\partial\rho_i} + \frac{2mE}{\hbar^2}\rho^2 - \frac{m^2\omega^2}{\hbar^2}P(\rho_i)\right\}\psi = 0,$$

which after the substitution

$$\psi(\rho_1,\rho_2,\rho_3;a_1,a_2,a_3) = \psi(\rho_1;a_1,a_2,a_3)\psi(\rho_2;a_1,a_2,a_3)\psi(\rho_3;a_1,a_2,a_3)$$

and introduction of the ellipsoidal separation constants λ_1 and λ_2 is divided into three identical differential equations

$$4\sqrt{P(\rho_i)}\frac{d}{d\rho_i}\sqrt{P(\rho_i)}\frac{d\psi_i}{d\rho_i} + \left\{2a_0^2\varepsilon\rho_i^2 - \lambda_1\rho_i - \lambda_2 - a_0^4P(\rho_i)\right\}\psi_i = 0, \quad i=1,2,3 \tag{6}$$

or in a more standard form ($\rho \equiv \rho_i$)

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{2}\left\{\frac{1}{\rho-a_1} + \frac{1}{\rho-a_2} + \frac{1}{\rho-a_3}\right\}\frac{d\psi}{d\rho} + \frac{1}{4}\left\{\frac{2a_0^2\varepsilon\rho^2 - \lambda_1\rho - \lambda_2 - a_0^4P(\rho)}{(\rho-a_1)(\rho-a_2)(\rho-a_3)}\right\}\psi = 0 \tag{7}$$

where the notation $\varepsilon = E/\hbar\omega$ and $a_0^2 = m\omega/\hbar$ has been introduced.

Equation (7), derived by separating variables in the ellipsoidal coordinate system, falls into a class of the generalized Lame' equations [5] and has four singularities $\{a_1,a_2,a_3,\infty\}$; moreover, the points (a_1,a_2,a_3) are elementary singularities with indices $(0,1/2)$ and a singularity at infinity is irregular. Apart from the algebraic form of eq.(6), like in the theory of Lame' equations, there may exist either the Jacobian form or the Weierstrass form.

As $\omega \to 0$ ($a_0 \to 0, a_0\varepsilon \neq 0$) eq.(7) turns into the differential equation for the ellipsoidal wave functions [4], known in the mathematical physics, (this equation is derived after the separation of variables in the Helmholtz equation in the ellipsoidal coordinate system).

Equation (7) can be considered as a degenerate form of the Fuchs equation with seven singularities [5]. It is a general enough equation that can lead, after different limiting transitions of the parameters a_1,a_2,a_3 (or confluence of singularities), to many second–order differential equations known in the mathematical physics.

Each of the separated equations (7) contains apart from the energy ε also two constants λ_1 and λ_2 depending in the general case on three dimensional parameters a_1,a_2,a_3 (or R_1,R_2) determining singularities of the given equation. Therefore, unlike the standard one-dimensional spectral problem, the main problem consists in calculating simultaneously (or quantizing) the energy spectrum of the isotropic oscillator and ellipsoidal separation constants.

4. Integrals of motion

Let us explicitly write down the operators (ellipsoidal integrals of motion) Λ_1 and Λ_2 whose eigenvalues are the ellipsoidal separation constants λ_1 and λ_2 . Eliminating the energy from the system of equations (6), we derive for Λ_1 and Λ_2 as functions of the parameters (a_1, a_2, a_3) the following expressions in the ellipsoidal variables ρ_i:

$$\begin{aligned}\Lambda_1(a_1,a_2,a_3) = & -\frac{4(\rho_3+\rho_2)\sqrt{P(\rho_1)}}{(\rho_3-\rho_1)(\rho_2-\rho_1)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1} - \frac{4(\rho_3+\rho_1)\sqrt{P(\rho_2)}}{(\rho_3-\rho_2)(\rho_1-\rho_2)}\frac{\partial}{\partial\rho_2}\sqrt{P(\rho_2)}\frac{\partial}{\partial\rho_2} \\ & - \frac{4(\rho_2+\rho_1)\sqrt{P(\rho_3)}}{(\rho_2-\rho_3)(\rho_1-\rho_3)}\frac{\partial}{\partial\rho_3}\sqrt{P(\rho_3)}\frac{\partial}{\partial\rho_3} \\ & + a_0^4\frac{(\rho_3^2-\rho_2^2)P(\rho_1)+(\rho_1^2-\rho_3^2)P(\rho_2)+(\rho_2^2-\rho_1^2)P(\rho_3)}{(\rho_3-\rho_1)(\rho_2-\rho_1)(\rho_2-\rho_3)}\end{aligned} \tag{8}$$

$$\begin{aligned}\Lambda_2(a_1,a_2,a_3) = & \frac{4\rho_3\rho_2\sqrt{P(\rho_1)}}{(\rho_3-\rho_1)(\rho_2-\rho_1)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1} + \frac{4\rho_3\rho_1\sqrt{P(\rho_2)}}{(\rho_3-\rho_2)(\rho_1-\rho_2)}\frac{\partial}{\partial\rho_2}\sqrt{P(\rho_2)}\frac{\partial}{\partial\rho_2} \\ & + \frac{4\rho_2\rho_1\sqrt{P(\rho_3)}}{(\rho_2-\rho_3)(\rho_1-\rho_3)}\frac{\partial}{\partial\rho_3}\sqrt{P(\rho_3)}\frac{\partial}{\partial\rho_3} \\ & - a_0^4\frac{\rho_3\rho_2(\rho_2-\rho_3)P(\rho_1)+\rho_3\rho_1(\rho_3-\rho_1)P(\rho_2)+\rho_2\rho_1(\rho_1-\rho_2)P(\rho_3)}{(\rho_3-\rho_1)(\rho_2-\rho_1)(\rho_2-\rho_3)}\end{aligned} \tag{9}$$

Passing in (8) and (9) from the variables ρ_i to the Cartesian ones, after long and tedious calculations we arrive at the following expression for the ellipsoidal integrals of motion:

$$\begin{aligned}\Lambda_1(a_1,a_2,a_3) &= (L_1^2+L_2^2+L_3^2)+(a_2+a_3)S_{33}+(a_1+a_3)S_{22}+(a_1+a_2)S_{11} \\ \Lambda_2(a_1,a_2,a_3) &= -a_1L_1^2-a_2L_2^2-a_3L_3^2-a_2a_3S_{33}-a_1a_3S_{22}-a_1a_2S_{11}\end{aligned} \tag{10}$$

where L_i are the components of the orbital moment operator, and $S_{ik}, (i,k = x,y,z)$ is the symmetric tensor (Yu.N.Demkov [6]) that is an additional integral of motion for the isotropic oscillator:

$$L_i = \frac{1}{2}\epsilon_{ikl}M_{kl}, \quad M_{kl} = -i(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}), \quad S_{ik} = -\frac{\partial^2}{\partial x_i \partial x_k} + a_0^4 x_i x_k$$

$$[L_i, \mathcal{H}] = [S_{ik}, \mathcal{H}] = 0$$

$$[M_{kl}, M_{ij}] = iM_{lj}\delta_{ki} + iM_{ki}\delta_{lj} - iM_{kj}\delta_{li} - iM_{li}\delta_{kj}$$

$$[S_{ik}, S_{jl}] = ia_0^4[M_{ij}\delta_{lk} + M_{il}\delta_{jk} + M_{kj}\delta_{li} + M_{kl}\delta_{ji}]$$

$$[S_{ik}, M_{jl}] = iS_{ij}\delta_{kl} + iS_{kj}\delta_{il} + iS_{il}\delta_{kj} + iS_{kl}\delta_{ij}$$

Instead of the system of operators (10) it is more convenient to use new operators $\hat{\lambda}$ and $\hat{\mu}$ that depend only on two parameters R_1^2 and R_2^2 and are connected with the old Λ_1 and Λ_2 according to

$$\hat{\lambda} = \Lambda_1(a_1,a_2,a_3) - a_2\frac{4m}{h^2}\mathcal{H}, \qquad \hat{\mu} = a_2\Lambda_1(a_1,a_2,a_3) + \Lambda_2(a_1,a_2,a_3) - a_2^2\frac{2m}{h^2}\mathcal{H}.$$

Thus, a complete set of commuting operators corresponding to the ellipsoidal basis of the isotropic oscillator is the system of the following three operators:

$$\begin{aligned}\mathcal{H} &= \frac{h^2}{2m}[S_{11}+S_{22}+S_{33}] \\ \hat{\lambda}(R_1^2,R_2^2) &= L^2 + R_1^2S_{33} - R_2^2S_{11} + \frac{2m}{h^2}(R_2^2-R_1^2)\mathcal{H} \\ \hat{\mu}(R_1^2,R_2^2) &= R_1^2L_3^2 - R_2^2L_1^2 + R_1^2R_2^2S_{22}\end{aligned} \tag{11}$$

From the system of operators (11) one can easily derive for particular values of the parameters R_1^2 and R_2^2 all the rest of possible (or equivalent to them) sets of diagonal operators $\{\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2\}$ corresponding to different bases of the isotropic oscillator.

I. The case $R_2^2 \to 0$, $R_1^2 = R^2/4$. **Prolate spheroidal basis.**

$$\mathcal{L}_1 = \hat{\lambda}(R_1^2, 0) = L^2 + \frac{R^2}{4} S_{33} - \frac{mR^2}{2\hbar^2}\mathcal{H}, \quad \mathcal{L}_2 = \frac{\hat{\mu}(R_1^2, 0)}{R_1^2} = L_3^2$$

II. The case $R_1^2 \to 0$, $R_2^2 = R^2/4$. **Oblate spheroidal basis**

$$\mathcal{L}_1 = \hat{\lambda}(0, R_2^2) = L^2 - \frac{R^2}{4} S_{11} + \frac{mR^2}{2\hbar^2}\mathcal{H}, \quad \mathcal{L}_2 = -\frac{\hat{\mu}(0, R_2^2)}{R_2^2} = L_1^2$$

III. The case $R_1^2 \to 0$, $R_2^2 \to 0$ and $R_1^2/(R_1^2 + R_2^2) = k^2$. **Sphero–conical basis**

$$\mathcal{L}_1 = \hat{\lambda}(0,0) = L^2, \quad \mathcal{L}_2 = \lim_{\substack{R_1^2 \to 0 \\ R_2^2 \to 0}} \frac{\hat{\mu}(R_1^2, R_2^2)}{(R_1^2 + R_2^2)} = k^2 L_3^2 - k'^2 L_1$$

IV. The case $R_1^2 \to 0$, $R_2^2 \to 0$ and $R_2^2/R_1^2 = 0$. **Spherical basis**

$$\mathcal{L}_1 = \hat{\lambda}(0,0) = L^2, \quad \mathcal{L}_2 = \lim_{\substack{R_1^2 \to 0 \\ R_2^2 \to 0}} \frac{\hat{\mu}(R_1^2, R_2^2)}{R_1^2} = L_3^2$$

V. The case $R_1^2 \to \infty$, $R_2^2 = R^2/4$. **Circular elliptic basis**

$$\mathcal{L}_1 = \lim_{R_1^2 \to \infty} \frac{\hat{\lambda}(R_1^2, R_2^2)}{R_1^2} = S_{33} - \frac{m}{2\hbar^2}\mathcal{H}, \quad \mathcal{L}_2 = \lim_{R_1^2 \to \infty} \frac{\hat{\mu}(R_1^2, R_2^2)}{R_1^2} = L_3^2 + \frac{R^2}{4} S_{22}$$

VI. The case $R_1^2 \to \infty$, $R_2^2 \to 0$. **Circular polar basis**

$$\mathcal{L}_1 = \lim_{R_1^2 \to \infty} \frac{\hat{\lambda}(R_1^2, 0)}{R_1^2} = S_{33} - \frac{m}{2\hbar^2}\mathcal{H}, \quad \mathcal{L}_2 = \lim_{R_1^2 \to \infty} \frac{\hat{\mu}(R_1^2, 0)}{R_1^2} = L_3^2$$

VII. The case $R_1^2 \to \infty$, $R_2^2 \to \infty$ and $R_2^2/R_1^2 = 0$. **Cartesian basis**

$$\mathcal{L}_1 = \lim_{R_2^2 \to \infty} \lim_{R_1^2 \to \infty} \frac{\hat{\lambda}(R_1^2, R_2^2)}{R_1^2} = S_{33} - \frac{m}{2\hbar^2}\mathcal{H}, \quad \mathcal{L}_2 = \lim_{R_2^2 \to \infty} \lim_{R_1^2 \to \infty} \frac{\hat{\mu}(R_1^2, R_2^2)}{R_1^2 R_2^2} = S_{22}$$

Note that other particular cases, for instance $R_1^2 \to 0$ and $R_2^2 \to \infty$, do not lead to new sets of diagonal operators as these cases either directly or by renaming the axes reduce to the cases mentioned above.

Thus, by means of different limiting transitions of the parameters R_1^2 and R_2^2 we have obtained all seven nonequivalent sets of operators corresponding to separation of the variables in the Schrödinger equation for the isotropic oscillator in simpler coordinate systems.

5. The solution of the ellipsoidal equation

Choosing a wave function $\psi(\rho; a_1, a_2, a_3)$ in the form

$$\psi(\rho; a_1, a_2, a_3) = \exp\{-\frac{{a_0}^2}{2}\rho\} Z(\rho; a_1, a_2, a_3)$$

after substition into (7) we arrive at the equation for the function $Z(\rho;a_1,a_2,a_3)$:

$$\frac{d^2Z}{d\rho^2}+\frac{1}{2}\left\{\sum_{i=1}^{3}\frac{1}{(\rho-a_i)}-2a_0^2\right\}\frac{dZ}{d\rho}+\frac{1}{4}\left\{\frac{2a_0^2\varepsilon\rho^2-\lambda_1\rho-\lambda_2}{(\rho-a_1)(\rho-a_2)(\rho-a_3)}-\sum_{i=1}^{3}\frac{a_0^4}{(\rho-a_i)}\right\}Z=0 \quad (12)$$

The solution of equation (12) is sought for as a power series around one of the singularities

$$Z^{(\alpha_1,\alpha_2,\alpha_3)}(\rho;a_1,a_2,a_3)=(\rho-a_1)^{\alpha_1/2}(\rho-a_2)^{\alpha_2/2}(\rho-a_3)^{\alpha_3/2}\sum_{t=0}^{\infty}b_t^{(\alpha_1,\alpha_2,\alpha_3)}(\rho-a_2)^t \quad (13)$$

where α_i $(i=1,2,3)$ may take one of the values (0.1). Substituting series (13) into equation (12) we derive the four-term recurrence relations that are to be satisfied by the coefficients b_t

$$\begin{aligned} R_1^2R_2^2(2t+2)(2t+2\alpha_2+1)b_{t+1} &+ (\gamma_t+2a_0^2\varepsilon a_2^2-a_2\lambda_1-\lambda_2)b_t+(\delta_t+4a_0^2\varepsilon a_2-\lambda_1)b_{t-1} \\ &+ 2a_0^2\{\varepsilon-2(t-2)-\sum_{i=1}^{3}\alpha_i-3/2\}b_{t-2}=0 \end{aligned} \quad (14)$$

where

$$\begin{aligned} \gamma_t &= R_1^2(2t+\alpha_2+\alpha_3)^2-R_2^2(2t+\alpha_1+\alpha_2)^2+a_0^2R_1^2R_2^2(4t+2\alpha_2+1) \\ \delta_t &= \left(2t+\sum_{i=1}^{3}\alpha_i\right)\left(2t-3+\sum_{i=1}^{3}\alpha_i\right)+2-2a_0^2\{R_1^2(2t+\alpha_2+\alpha_3-1) \\ &\quad -2a_0^2\{R_1^2(2t+\alpha_2+\alpha_3-1)-R_2^2(2t+\alpha_1+\alpha_2-1)\} \end{aligned}$$

Using the procedure of studying series cvonvergence, suggested in ref. [8], one can show that series (13) will be finite in the whole region of the variable ρ only if they are truncated.

Let all the coefficients of the four–term recurrence relation (14) starting from b_{N+1} be zero at any integer N, i.e., the following condition is fulfilled:

$$b_{N+1}=b_{N+2}=b_{N+3}=\dots\dots\dots=0$$

Substituting $t=N+2$ into the recurrence relation (14) and taking into account that $b_N\neq0$, we have

$$\varepsilon=2N+\sum_{i=1}^{3}\alpha_i+3/2, \qquad N=0,1,2,\cdots$$

and, consequently, arrive at the formula for the energy spectrum of the isotropic oscillator which is well known in the literature [7]

$$E=\hbar\omega\left(2N+\sum_{i=1}^{3}\alpha_i+3/2\right)=\hbar\omega(n+3/2),$$

where $n=2N+\sum_{i=1}^{3}\alpha_i$ is the principal quantum number.

Thus, we finally get that the expansion coefficients b_t satisfy the four-term recurrence relations

$$R_1^2R_2^2(2t+2)(2t+2\alpha_2+1)b_{t+1}+(\gamma_t-\mu)b_t+(\delta_t-\lambda)b_{t-1}+4a_0^2(N-t+2)b_{t-2}=0 \quad (15)$$

which are to be added by the conditions $b_{-2}=b_{-1}=0$ and $b_0=1$. The constants μ and λ are determined by the relations

$$\lambda=\lambda_1-4a_0^2\varepsilon a_2, \qquad \mu=a_2\lambda_1+\lambda_2-2a_0^2\varepsilon a_2^2$$

and are eigenvalues of the operators in formula (11).

If series (13) are truncated, the four-term recurrence relation (15) turns into the system of $(N+2)$ homogeneous equations with respect to $(N+1)$ coefficients $(b_0, b_1, \ldots, b_N)$:

$$\begin{array}{ccccccccc}
(\gamma_0-\mu)b_0 & + & \beta_0 b_1 & & & & & & = 0 \\
(\delta_1-\lambda)b_0 & + & (\gamma_1-\mu)b_1 & + & \beta_1 b_2 & & & & = 0 \\
4a_0^2 N b_0 & + & (\delta_2-\lambda)b_1 & + & (\gamma_2-\mu)b_2 & + & \beta_2 b_3 & & = 0 \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
4a_0^2\cdot 3b_{N-3} & + & (\delta_{N-1}-\lambda)b_{N-2} & + & (\gamma_{N-1}-\mu)b_{N-1} & + & \beta_{N-1}b_N & & = 0 \\
 & & 4a_0^2\cdot 2b_{N-2} & + & (\delta_N-\lambda)b_{N-1} & + & (\gamma_N-\mu)b_N & & = 0 \\
 & & & & 4a_0^2 b_{N-1} & + & (\delta_{N+1}-\lambda)b_N & & = 0
\end{array}$$

Consequently, the latter is overdetermined, and the corresponding matrix is rectangular:

$$A = \begin{pmatrix}
\gamma_0-\mu & \beta_0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\delta_1-\lambda & \gamma_1-\mu & \beta_1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
4a_0^2\cdot N & \delta_2-\lambda & \gamma_2-\mu & \beta_2 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 4a_0^2\cdot 3 & \delta_{N-1}-\lambda & \gamma_{N-1}-\mu & \beta_{N-1} \\
0 & 0 & 0 & 0 & \ldots & 0 & 4a_0^2\cdot 2 & \delta_N-\lambda & \gamma_N-\mu \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 4a_0^2 & \delta_{N+1}-\lambda
\end{pmatrix}$$

As concerns a homogeneous system of equations of that type, it is known that a necessary and sufficient condition for the existence of a nontrivial solution [9] is equality to zero of all determinants of order $(N+1)$. As a result of this procedure, we arrive at the system of $(N+2)$ algebraic equations of the $(N+1)$th degree from which eigenvalues of two separation constants λ and μ are determined. Using the exclusion theory [10] for the system of algebraic equations with many unknowns, the solution of this system of algebraic equations with two unknowns λ and μ can be reduced to the solution of an algebraic equation of degree $(N+1)(N+2)/2$ for one of the variables and one coupling equation for λ and μ. As a result we obtain $(N+1)(N+2)/2$ pairs of different solutions $\{\lambda, \mu\}$. Now let q_1, q_2, q_3 be integers equal to the number of zeroes of the ellipsoidal wave function (13) in the intervals (a_1, a_2), (a_2, a_3) and (a_3, ∞). As the general number of zeroes of the polynomial (13) in these intervals equals N, the ellipsoidal quantum numbers q_1, q_2, q_3 are connected with each other by a simple relation

$$q_1+q_2+q_3 = N, \quad q_i = 0, 1, \ldots N, \quad (i = 1, 2, 3)$$

and can be chosen to enumerate the ellipsoidal separation constants $\{\lambda^N_{q_1,q_2,q_3}, \mu^N_{q_1,q_2,q_3}\}$.

Thus, we have obtained eight types of polynomial solutions of the ellipsoidal equation of the isotropic oscillator:

$$\begin{aligned}
uE^{2N}_{q_1q_2q_3}(\rho; a_1, a_2, a_3) &= \sum_{t=0}^{N} b_t^{(0,0,0)}(\rho-a_2)^t, & n &= 2N \\
cE^{2N+1}_{q_1q_2q_3}(\rho; a_1, a_2, a_3) &= \sum_{t=0}^{N} b_t^{(0,1,0)}(\rho-a_2)^{t+1/2}, & n &= 2N+1 \\
sE^{2N+1}_{q_1q_2q_3}(\rho; a_1, a_2, a_3) &= \sqrt{\rho-a_1}\sum_{t=0}^{N} b_t^{(1,0,0)}(\rho-a_2)^t, & n &= 2N+1 \\
dE^{2N+1}_{q_1q_2q_3}(\rho; a_1, a_2, a_3) &= \sqrt{\rho-a_3}\sum_{t=0}^{N} b_t^{(0,0,1)}(\rho-a_2)^t, & n &= 2N+1
\end{aligned}$$

$$csE^{2N+2}_{q_1q_2q_3}(\rho;a_1,a_2,a_3) = \sqrt{\rho-a_1}\sum_{t=0}^{N} b_t^{(1,1,0)}(\rho-a_2)^{t+1/2}, \qquad n=2N+2$$

$$cdE^{2N+2}_{q_1q_2q_3}(\rho;a_1,a_2,a_3) = \sqrt{\rho-a_3}\sum_{t=0}^{N} b_t^{(0,1,1)}(\rho-a_2)^{t+1/2}, \qquad n=2N+2$$

$$sdE^{2N+2}_{q_1q_2q_3}(\rho;a_1,a_2,a_3) = \sqrt{(\rho-a_1)(\rho-a_3)}\sum_{t=0}^{N} b_t^{(1,0,1)}(\rho-a_2)^{t}, \qquad n=2N+2$$

$$csdE^{2N+3}_{q_1q_2q_3}(\rho;a_1,a_2,a_3) = \sqrt{(\rho-a_1)(\rho-a_3)}\sum_{t=0}^{N} b_t^{(1,1,1)}(\rho-a_2)^{t+1/2}, \qquad n=2N+3$$

6. The ellipsoidal basis

According to the afore-said in sect.5. the ellipsoidal basis of the isotropic oscillator is divided into eight classes:

$$\Psi^{(0,0,0)}_{N,q_1q_2q_3} = C^{(0,0,0)}uHe^{2N}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)uHe^{2N}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)uHe^{2N}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(n+2)(n+4)}{8}$$

$$\Psi^{(1,0,0)}_{N,q_1q_2q_3} = C^{(1,0,0)}cHe^{2N+1}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)cHe^{2N+1}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)cHe^{2N+1}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(n+1)(n+3)}{8}$$

$$\Psi^{(0,1,0)}_{N,q_1q_2q_3} = C^{(0,1,0)}sHe^{2N+1}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)sHe^{2N+1}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)sHe^{2N+1}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(n+1)(n+3)}{8}$$

$$\Psi^{(0,0,1)}_{N,q_1q_2q_3} = C^{(0,0,1)}dHe^{2N+1}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)dHe^{2N+1}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)dHe^{2N+1}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(n+1)(n+3)}{8}$$

$$\Psi^{(1,1,0)}_{N,q_1q_2q_3} = C^{(1,1,0)}csHe^{2N+2}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)csHe^{2N+2}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)csHe^{2N+2}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{n(n+2)}{8}$$

$$\Psi^{(1,0,1)}_{N,q_1q_2q_3} = C^{(1,0,1)}\,cdHe^{2N+2}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)\,cdHe^{2N+2}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)\,cdHe^{2N+2}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{n(n+2)}{8}$$

$$\Psi^{(0,1,1)}_{N,q_1q_2q_3} = C^{(0,1,1)}\,sdHe^{2N+2}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)\,sdHe^{2N+2}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)\,sdHe^{2N+2}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3).$$
$$n=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{n(n+2)}{8}$$

$$\Psi^{(1,1,1)}_{N,q_1q_2q_3} = C^{(1,1,1)}\,csdHe^{2N+3}_{q_1q_2q_3}(\rho_1;a_1,a_2,a_3)\,csdHe^{2N+3}_{q_1q_2q_3}(\rho_2;a_1,a_2,a_3)\,csdHe^{2N+3}_{q_1q_2q_3}(\rho_3;a_1,a_2,a_3)$$
$$n=2N+3, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(n-1)(n+1)}{8}$$

Here by He we denote polynomials E multiplied by a factor $\exp\{-\frac{a\omega^2}{2}\rho\}$, and D is the number of states of a given principal quantum number n. The multiplicity of degeneracy of energy levels of the isotropic oscillator is determined by a sum of all states of even or odd fixed n and is correspondingly equal to $(n+1)(n+2)/2$.

The coefficients $C^{(i,j,k)}$ where $i,j,k=0,1$, are determined from the normalisation condition of the ellipsoidal basis of the isotropic oscillator

$$\frac{1}{8}\int_{a_1}^{a_2}\int_{a_2}^{a_3}\int_{a_3}^{\infty}\left[\Psi^{(i,j,k)}_{Nq_1q_2q_3}(\rho_1,\rho_2,\rho_3;R_1^2,R_2^2)\right]^2\frac{(\rho_2-\rho_1)(\rho_3-\rho_2)(\rho_3-\rho_1)}{\sqrt{-P(\rho_1)P(\rho_2)P(\rho_3)}}\,d\rho_1\,d\rho_2\,d\rho_3=1$$

The complex form of the ellipsoidal basis of the isotropic oscillator finally depends on the degree of algebraic equations from which eigenvalues for two separation constants $\lambda^N_{q_1q_2q_3}(R_1^2,R_2^2)$ and $\mu^N_{q_1q_2q_3}(R_1^2,R_2^2)$ are determined.

7. Particular cases

Let us write down eigenvalues of the ellipsoidal separation constants and the ellipsoidal basis of the isotropic oscillator for the lowest quantum numbers $n=0,1,2$.

I. $n=0,\quad N=0;\quad \alpha_1=\alpha_2=\alpha_3=0$

$$\Psi^{(0,0,0)}_{0,000}=\left(\frac{2a_0}{\sqrt{\pi}}\right)^{3/2} exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2,R_2^2)=2a_0^2(R_2^2-R_2^2),\quad \mu(R_1^2,R_2^2)=a_0^2R_1^2R_2^2$$

II. $n=1,\quad N=0;\quad \alpha_1+\alpha_2+\alpha_3=1$

$$\Psi^{(0,0,1)}_{0,000}=\frac{\sqrt{8}a_0^2}{\sqrt{\pi}}\frac{\sqrt{(a_3-\rho_1)(a_3-\rho_2)(\rho_3-a_3)}}{(a_3-a_1)(a_3-a_2)}exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2,R_2^2)=2+2a_0^2(R_2^2-2R_1^2),\quad \mu(R_1^2,R_2^2)=R_1^2+a_0^2R_1^2R_2^2$$

$$\Psi^{(0,1,0)}_{0,000}=\frac{\sqrt{8}a_0^2}{\sqrt{\pi}}\frac{\sqrt{(a_2-\rho_1)(\rho_2-a_2)(\rho_3-a_2)}}{(a_3-a_2)(a_2-a_1)}exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2,R_2^2)=2+4a_0^2(R_2^2-R_1^2),\quad \mu(R_1^2,R_2^2)=R_1^2-R_2^2+3a_0^2R_1^2R_2^2$$

$$\Psi^{(1,0,0)}_{0,000}=\frac{\sqrt{8}a_0^2}{\sqrt{\pi}}\frac{\sqrt{(\rho_1-a_1)(\rho_2-a_1)(\rho_3-a_1)}}{(a_3-a_1)(a_2-a_1)}exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2,R_2^2)=2+2a_0^2(2R_2^2-R_1^2),\quad \mu(R_1^2,R_2^2)=-R_2^2+a_0^2R_1^2R_2^2$$

III. $n=2,\quad N=0;\quad \alpha_1+\alpha_2+\alpha_3=2$

$$\Psi^{(1,1,0)}_{0,000}=\left(\frac{8a_0^5}{\sqrt{\pi}}\right)\frac{\sqrt{(\rho_1-a_1)(\rho_2-a_1)(\rho_3-a_1)(a_2-\rho_1)(\rho_2-a_2)(\rho_3-a_2)}}{(a_3-a_1)(a_2-a_1)^2(a_3-a_2)}$$
$$\cdot\ exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2,R_2^2)=6+2a_0^2(3R_2^2-2R_1^2),\quad \mu(R_1^2,R_2^2)=R_1^2-4R_2^2+3a_0^2R_1^2R_2^2$$

$$\Psi^{(1,0,1)}_{0,000}=\left(\frac{8a_0^5}{\sqrt{\pi}}\right)\frac{\sqrt{(\rho_1-a_1)(\rho_2-a_1)(\rho_3-a_1)(a_3-\rho_1)(a_3-\rho_2)(\rho_3-a_3)}}{(a_3-a_1)^2(a_2-a_1)(a_3-a_2)}$$
$$\cdot\ exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2,R_2^2)=6+4a_0^2(R_2^2-R_1^2),\quad \mu(R_1^2,R_2^2)=R_1^2-R_2^2+a_0^2R_1^2R_2^2$$

$$\Psi^{(0,1,1)}_{0,000}=\left(\frac{8a_0^5}{\sqrt{\pi}}\right)\frac{\sqrt{(a_2-\rho_1)(\rho_2-a_2)(\rho_3-a_2)(a_3-\rho_1)(a_3-\rho_2)(\rho_3-a_3)}}{(a_3-a_1)^2(a_2-a_1)(a_3-a_2)^2}$$
$$\cdot\ exp\{-\frac{a_0^2}{2}[(\rho_1+\rho_2+\rho_3)-(a_1+a_2+a_3)]\}$$

$$\lambda(R_1^2, R_2^2) = 6 + 2a_0^2(2R_2^2 - 3R_1^2), \quad \mu(R_1^2, R_2^2) = 4R_1^2 - R_2^2 + 3a_0^2 R_1^2 R_2^2$$

In conclusion, we wish to express our sincere gratitude to R.Airapetyan, L.S. Davtyan, V.M. Ter-Antonyan and D.I.Zaslavsky for numerous discussions and useful remarks.

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