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ANALYSIS OF SERIES CONVERGENCE IN VARIATIONAL PERTURBATION THEORY AND GAUSSIAN EFFECTIVE POTENTIAL

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The properties of convergence are studied for the series in the variational perturbation theory for the $\varphi^4(d)$ model. The nonperturbative Gaussian effective potential is derived from a more general approach, the variational perturbation theory. Various versions of the variational procedure are explored and the preference of the anharmonic variational procedure in view of convergence of the obtained series is argued.

1. Introduction

The approximation of a quantity under consideration by a finite number of terms of a certain series is a standard computational procedure in dealing with the many problems of physics. In quantum field theory this is conventionally an expansion into a perturbative series. This approach combined with the renormalization procedure is now a basic method for computations. As is well known, perturbative series for many interesting models including realistic models are not convergent. Nevertheless, at small values of the coupling constant, these series may be considered as asymptotic series and could provide useful information. However, even in theories with a small coupling constant, for instance in quantum electrodynamics, there exist problems which cannot be solved by perturbative methods. Also, a lot of problems of quantum chromodynamics require nonperturbative approaches. At present, a central problem of quantum field theory is to go beyond the scope of perturbation theory.

The creation of effective methods of nonperturbative calculations is a central task in quantum field theory. There are a lot of approaches to solve this problem. The method of the Gaussian effective potential (GEP)[1,2] belongs to the most powerful and constructive of them and has often been used in recent years[3–5] for obtaining a number of results in the nonperturbative region of quantum field theory.

A very important problem that disturbs most of the nonperturbative methods is how to obtain a natural scheme of calculating corrections to the basic contributions, and thus there arises also the question on stability and reliability of the results obtained in the framework of these approaches. The GEP method has an advantage over such approaches in this connection.[6–8] It should be noted, however, that there is no implicit small expansion parameter in nonperturbative tasks. Therefore, it is important not only to have in principle a possibility of expanding the exploring quantity in a series, but also to know certain properties of this series.

In the framework of a nonperturbative scheme called the variational perturbation theory (VPT) in our previous works,[9–11] it is possible to represent the searched quantity in the form of a series from the very beginning, and it is possible to influence the properties of the convergence of this series through certain parameters of the variational type. Moreover in a number of interesting cases the VPT series proves to be an absolutely convergent series. It is important that it is not necessary to introduce new diagrams whose structure differs from that of ordinary perturbative diagrams.

The VPT method offers a wide spectrum of possibilities for choices of additional "variational terms" in action. We will consider two of them here — the "harmonic" and "anharmonic" recipes of the variational procedure. The zero-dimensional analog and one-dimensional case of the anharmonic oscillator will be studied. It will be shown that the "anharmonic" recipe of an addition of "variational terms" is preferred compared with the "harmonic" one in view of the convergence of the VPT series. The Gaussian effective potential arises from VPT as a variational correction of the one-loop contribution and there are various possibilities of introducing "variational terms." The arguments will be that the question of stability of the results derived from GEP should be investigated on the basis of the "anharmonic recipe" of the VPT procedure.

2. Zero-Dimensional Analog

Let us consider the integral

$$Z[g] = \int d\mathbf{x} \exp(-S[x]) , \tag{2.1}$$

where

$$\begin{aligned} S[x] &= S_0[x] + gS_{\text{int}}[x] , \qquad & S_0[x] &= \mathbf{x}^2 = x_1^2 + x_2^2 , \\ S_{\text{int}}[x] &= x_1^4 + x_2^4 , & d\mathbf{x} &= dx_1\, dx_2 . \end{aligned} \tag{2.2}$$

The quantity (2.2) can be considered as a zero-dimensional analog of the corresponding functional integrals in the φ^4 quantum field model. In the following we shall operate only with Gaussian functional quadratures and, therefore, we shall use here, for calculating (2.2), only Gaussian integrals

$$\int dx\, P(x) \exp\{-S_0[x]\}\,, \tag{2.3}$$

where $P(x)$ is a certain polynomial of x_1 and x_2 variables. The first obvious opportunity is the expansion of the integrand in (2.1) in powers of the coupling constant g. As a result, we derive the ordinary perturbation theory

$$Z[g] = \sum g^n C_n\,, \tag{2.4}$$

$$C_n = \frac{(-1)^n}{n!} \int \mathrm{dx}\, S_{\rm int}^n \exp(-S_0[x])\,. \tag{2.5}$$

It is well known that the series (2.4) is asymptotical and, therefore, does not allow one to judge the quantity $Z[g]$ in the nonperturbative region without additional information about its sum. However, the standard perturbative expansion (2.4), (2.5) is not unique based on the Gaussian quadratures. We shall consider here two kinds of such expansions differing from series (2.4). They differ from one another in their manner of introducing variational terms into the action (2.2). Namely, they utilize the "harmonic" and "anharmonic" recipes of introducing the variational terms, respectively. In fact, these recipes are not a single possibility of a variational procedure construction. In particular, a composition of them can be used.

The first method we shall consider is the *harmonic variational procedure*, where the free action $S_0[x]$ will be used as a harmonic variational extra term. The total action is rewritten in the form

$$S[x] = S_0^h[x] + S_{\rm int}^h[x]\,, \tag{2.6}$$

where

$$S_0^h[x] = S_0[x] + \chi S_0[x]\,, \tag{2.7}$$

$$S_{\rm int}^h[x] = g S_{\rm int}[x] - \chi S_0[x]\,, \tag{2.8}$$

and the expansion in powers of a new "functional of interaction" $S_{\rm int}^h[x]$ is performed. It is easy to see that the task is formulated only in terms of the Gaussian functional quadratures. As a result, the VPT series takes the form

$$Z[g] = \sum_n Z_n[g, \chi]\,, \tag{2.9}$$

$$Z_n[g, \chi] = \frac{(-g)^n}{n!(1+\chi g)^{1+2n}} \int \mathrm{dx}[S_{\rm int} - \chi(1+\chi g)S_0]^n \exp\{-S_0[x]\}\,. \tag{2.10}$$

Actually, the original quantity $Z[g]$ does not depend on the variational parameter χ. Therefore, the freedom in choosing χ can be used to improve the properties of the VPT series. Various ways of optimal choice of variational parameters have been considered in Refs. 9 and 10.

In the field theory, we know, as a rule, only a few first terms of the VPT expansion, and maybe the asymptotic behavior of remote terms too. Just based on this information the optimal value of variational parameters can be obtained. In the majority of cases the asymptotic and the first nontrivial order that permit us to obtain an equation for the variational parameter, are used. Then the stability of the results will be achieved only when the contribution of subsequent terms of the series prove to be small in comparison with the basic contribution.

The fact that the exact quantity does not depend on variational parameters results in the wonderful possibility of choosing their values so that the considered VPT order would maximally approximate the searched quantity. Indeed, let

$$Z[g] = Z^{(N)}[g,\chi] + \Delta Z^{(N)}[g,\chi]\,, \tag{2.11}$$

where

$$Z^{(N)}[g,\chi] = \sum_{n=0}^{N} Z_n[g,\chi]\,,$$

$$\Delta Z^{(N)}[g,\chi] = \sum_{n=N+1}^{\infty} Z_n[g,\chi]\,.$$

Then

$$\frac{\partial Z^{(N)}[g,\chi]}{\partial\chi} = -\frac{\partial \Delta Z^{(N)}[g,\chi]}{\partial\chi}\,,$$

and thus, if χ_0 is a point of maximum for $Z^{(N)}$, then this point is a point of minimum for the whole reminder $\Delta Z^{(N)}$, simultaneously. Thus we require that

$$\frac{\partial Z^{(N)}[g,\chi]}{\partial\chi} = 0\,. \tag{2.12}$$

Making use of Eqs. (2.8) and (2.9) and setting $N = 1$ we find from (2.12) that

$$\begin{aligned} \chi &= (1/\tau - 1)/g\,,\\ \tau &= \frac{2}{9g}\left[\sqrt{1+9g} - 1\right]. \end{aligned} \tag{2.13}$$

The results of the calculations will be discussed and compared with the ones obtained in the framework of the anharmonic variational procedure somewhat later.

The *anharmonic method* of introducing the variational addition is based on the representation of the action

$$S[x] = S_0^a[x] + S_{\text{int}}^a[x]\,, \tag{2.14}$$

where

$$S_0^a[x] = S_0[x] + \theta S_0^2[x]\,, \tag{2.15}$$

$$S_{\text{int}}^a[x] = g S_{\text{int}}[x] - \theta S_0^2[x]\,. \tag{2.16}$$

In this case the expansion of the integrand in (2.1) is carried out in powers of $S_{\text{int}}^a[x]$. The situation here is somewhat more complicated when compared with the previous case as the addend $\theta S_0^2[x]$ is present in the exponential, which leads to the non-Gaussian form of the emerging integral. However, this problem can be easily solved by using the Fourier transformation

$$\exp(-\theta S_0^2(x)) = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left\{-\frac{u^2}{4} + iu\sqrt{\theta} S_0(x)\right\}. \tag{2.17}$$

As a result the VPT series takes the form[10]

$$Z[g] = \sum Z_n[g,\theta]\,, \tag{2.18}$$

$$Z_n[g,\theta] = \int_0^{\infty} d\alpha (\alpha^2\theta)^n \exp(-\alpha - \alpha^2\theta) \sum_{k=0}^{n} \frac{a_k}{2k} \frac{(-g/\theta)^k}{(n-k)!}\,, \tag{2.19}$$

$$a_k = \sum_{l=0}^{k} \frac{\Gamma(2l+1/2)}{l!} \frac{\Gamma(2(k-l)+1/2)}{(k-l)!}\,. \tag{2.20}$$

The optimization of the first nontrivial approximation yields

$$\theta = \frac{3}{4} g\,. \tag{2.21}$$

The behavior of the Nth order partial sum of the series (2.9) and (2.18), normalized to the exact value $Z[g]$, is represented in Table 1. In Fig. 1 the N dependence of the quantity $Z^{(N)}[g=1]$ in the case of the anharmonic variational procedure is plotted. We can see that for $g \geq 1$, when the harmonic variational procedure is performed, even first terms of the VPT series become sensitive to its asymptotic nature (i.e. the partial sum "beats" emerge). For g larger than that shown in Table 1 the situation becomes even more complicated. A relatively stable result for the series (2.9) occurs as $g < 1$. The comparison of the results given by the ordinary perturbation theory (2.4) and by the series (2.9) for $g = 0.1$ is plotted in Fig. 2. As regards the anharmonic variational procedure [Eqs. (2.18)–(2.21)] we derive a stable result in the whole region of the coupling constant.

Table 1. The behavior of Nth partial sums for the harmonic and anharmonic variational procedure.

	$g=1$		$g=10$		$g=100$	
N	$Z^{(N)}_{\text{harm}}/Z_{\text{ex}}$	$Z^{(N)}_{\text{anharm}}/Z_{\text{ex}}$	$Z^{(N)}_{\text{harm}}/Z_{\text{ex}}$	$Z^{(N)}_{\text{anharm}}/Z_{\text{ex}}$	$Z^{(N)}_{\text{harm}}/Z_{\text{ex}}$	$Z^{(N)}_{\text{anharm}}/Z_{\text{ex}}$
0	0.806	0.992	0.701	0.984	0.658	0.981
1	0.945	0.992	0.891	0.984	0.864	0.981
2	1.034	1.000	1.078	0.999	1.099	0.999
3	0.905	1.000	0.650	0.999	0.480	0.999
4	1.310	1.000	2.735	1.000	3.960	1.000
5	−0.265	1.000	−9.909	1.000	−20.41	1.000
6	7.253	1.000	84.24	1.000	189.1	1.000
7	−59.68	1.000	−1223.	1.000	−3170.	1.000
8	−26.51	1.000	−212.7	1.000	−172.4	1.000
9	−34.12	1.000	−574.0	1.000	−1410.	1.000
10	−23.80	1.000	190.7	1.000	1614.	1.000

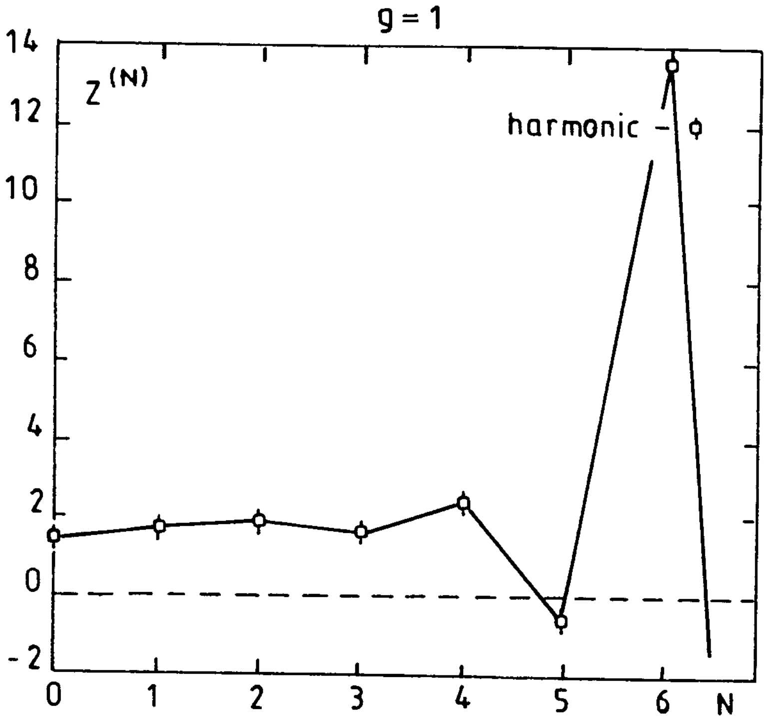

Fig. 1. N-dependence of the Nth partial sum of the series (2.9) for $g=1$.

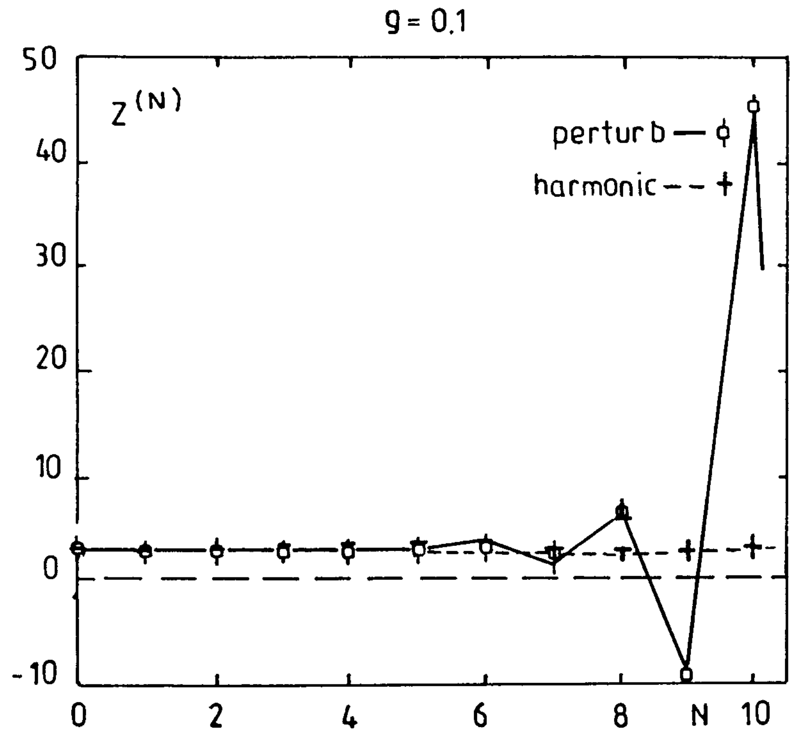

Fig. 2. The behavior of Nth partial sum for the cases of perturbation theory (2.4) and the harmonic variational procedure (2.9).

3. Variational Perturbation Theory for Anharmonic Oscillator

Let us consider a quantum-mechanical anharmonic oscillator (AO) case as an example of exploiting the VPT method. The AO, from the point of view of the continual integral formalism, is a one-dimensional φ^4 model. The Euclidean action looks as follows:

$$S[\varphi] = S_0[\varphi] + \frac{m^2}{2} S_2[\varphi] + gS_4[\varphi] , \tag{3.1}$$

where

$$S_0[\varphi] = \frac{1}{2}\int dx(\partial\varphi)^2 , \tag{3.2}$$

$$S_2[\varphi] = \int dx\, \varphi^2 , \tag{3.3}$$

$$S_4[\varphi] = \int dx\, \varphi^4 . \tag{3.4}$$

The Green functions are expressed in terms of the functional integral

$$G_{2\nu} = \int D\varphi\{\varphi^{2\nu}\}\exp(-S[\varphi]), \tag{3.5}$$

where $\{\varphi^{2\nu}\}$ denotes the product of fields $\varphi(x_1)\cdots\varphi(x_n)$. We pass to dimensionless variables: $\varphi \to g^{-1/6}\varphi$, $x \to g^{-1/3}x$. Then the functional of action reads

$$S[\varphi] = S_0[\varphi] + \frac{\omega^2}{2}S_2[\varphi] + S_4[\varphi], \tag{3.6}$$

where

$$\omega^2 = m^2 g^{-2/3}. \tag{3.7}$$

The dimensionless Green functions $G_{2\nu}$ will be represented through (3.5) with the action (3.6).

Like the ordinary perturbation theory, the VPT method uses only Gaussian quadratures:

$$\int D\varphi \exp\left\{-\left[\frac{1}{2}\langle\varphi K\varphi\rangle + \langle\varphi J\rangle\right]\right\}$$
$$= \left(\det\frac{K}{-\partial^2+m^2}\right)^{-1/2}\exp\left[\frac{1}{2}\langle JK^{-1}J\rangle\right]. \tag{3.8}$$

At the same time any possible polynomial in the integrand of (3.8) can be obtained by the corresponding number of differentiations of the exponential with respect to the source $J(x)$. The variational addition to the action will be constructed on the basis of the functional

$$A[\varphi] = \theta S_0[\varphi] + \frac{\chi}{2}S_2[\varphi], \tag{3.9}$$

where θ and χ are variational type parameters.

We shall first consider the *harmonic variational procedure*. In this case the action functional splits as follows:

$$S[\varphi] = S_0^h[\varphi] + S_{\text{int}}^h[\varphi], \tag{3.10}$$

where

$$S_0^h[\varphi] = S_0[\varphi] + \frac{\omega^2}{2}S_2[\varphi] + A[\varphi], \tag{3.11}$$

$$S_{\text{int}}^h[\varphi] = S_{\text{int}}[\varphi] - A[\varphi]. \tag{3.12}$$

The expansion in the VPT series reads

$$G_{2\nu} = \sum G_{2\nu,n}(\theta,\chi), \tag{3.13}$$

$$G_{2\nu,n}[\theta,\chi] = \frac{(-1)^n}{n!}\int D\varphi\{\varphi^{2\nu}\}(S_{\text{int}}^h[\varphi])^n\exp(-S_0^h[\varphi]). \tag{3.14}$$

Obviously the functional integral (3.14) is Gaussian. It is convenient to use the ordinary coefficients of a perturbative series when it is calculated. To do this, let us rewrite (3.14) as

$$G_{2\nu,n} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\left(-\frac{\partial}{\partial\alpha}\right)^{n-k} \times \int D\varphi\{\varphi^{2\nu}\}(-S_4)^k \exp\left[-\left(S_0 + \frac{\omega^2}{2}S_2 + \alpha A\right)\right], \tag{3.15}$$

where the parameter α is to be set to 1 after differentiation. Having in mind the intermediate dimensional regularization and making the change $\varphi \rightarrow \varphi/\sqrt{1+\alpha\theta}$ we obtain

$$G_{2\nu,n} = \sum_{k=0}^{n} \frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\alpha}\right)^{n-k} \frac{g_{2\nu,k}(z^2)}{(1+\alpha\theta)^{\nu+2k}}, \tag{3.16}$$

where

$$g_{2\nu,k}(z^2) = \frac{1}{k!}\int D\varphi\{\varphi^{2\nu}\}(-S_4)^k \exp\left[-\left(S_0 + \frac{z^2}{2}S_2\right)\right] \tag{3.17}$$

are the ordinary perturbative expansion coefficients for the Green functions (3.5). To calculate them, the standard Feynman diagrams, for example, can be used. The quantity z^2 in (3.17) looks as follows:

$$z^2 = \frac{\omega^2 + \alpha\chi}{1+\alpha\theta}. \tag{3.18}$$

The properties of series (3.13) are determined by the asymptotics of the functional integral

$$\frac{(-1)^n}{n!}\int D\varphi(S_4[\varphi] - A[\varphi])^n \exp\left[-\left(S_0[\varphi] + \frac{\omega^2}{2}S_2[\varphi] + A[\varphi]\right)\right] \tag{3.19}$$

at large n.

It is easy to see that the investigation of the asymptotic behavior of expression (3.19) in the leading order in n is, in fact, equivalent to finding the ordinary perturbative series coefficients. The series (3.13) turns out to be asymptotic like the ordinary one. Actually, its behavior may be influenced by the θ and χ parameters, to attain the greater stability of results as compared with standard perturbation theory. However, one is compelled to remain in the region of the weak coupling constant, mainly, as it turns out to be impossible for arbitrary values of the dimensionless coupling constant g/m^3 to gain, within the harmonic variational procedure, the stable results with respect to corrections. The latter is explained by the fact that at large n a sensible contribution to (3.19) comes from such field configurations at which the quantity $|\varphi(x)|$ is large. In this case the compensation by the harmonic addition $A[\varphi]$ of large $S_{\rm int}[\varphi]$ containing the fourth power of the field proves to be insufficient.

Under the *anharmonic variational procedure* the action functional is represented as

$$S[\varphi] = S_0^a[\varphi] + S_{\text{int}}^a[\varphi]\,, \tag{3.20}$$

$$S_0^a[\varphi] = S_0[\varphi] + \frac{\omega^2}{2} S_2[\varphi] + A^2[\varphi]\,, \tag{3.21}$$

$$S_{\text{int}}^a[\varphi] = S_{\text{int}}[\varphi] - A^2[\varphi]\,. \tag{3.22}$$

Now, the field power in the compensating addition is the same as in the interaction action $S_{\text{int}}[\varphi]$. Keeping in mind that we also have the variational parameters at our disposal, we may anticipate that the convergence of the VPT series will be good enough.

As a concrete example, we shall consider the ground state energy for the anharmonic oscillator connected with the four-point Green function $G_4(0,0,0,0)$ by the relation

$$\frac{\partial E_0}{\partial g} = g^{-2/3} G_4\,. \tag{3.23}$$

For the dimensionless energy $\varepsilon_0 = E_0/g^{1/3}$ in the strong coupling limit $\omega^2 = 0$, we get

$$\varepsilon_0 = 3G_4\,. \tag{3.24}$$

The exact value of ε_0 can be found in Ref. 12: $\varepsilon_0 = 0.668$. To evaluate ε_0 within our approach, we shall expand the integrand exponential in (3.5) in powers of the new interaction action (3.22). A subsequent transformation to the Gaussian functional quadrature is performed by using the Fourier transformation of type (2.17). The application of the asymptotic optimization that requires the contribution of the remove terms in the VPT series to be minimal allows one to find the relation between the parameters θ and χ:[9]

$$\chi^3 = 9/16\theta\,. \tag{3.25}$$

The remaining parameter θ is fixed from the optimization condition $\partial\varepsilon_0^{(N)}/\partial\theta = 0$, where

$$\varepsilon_0^{(N)}(\theta) = \sum_{n=0}^{N} \varepsilon_n(\theta)\,, \tag{3.26}$$

$$\varepsilon_n(\theta) = 3\sum_{m=0}^{n} \frac{(1+m)A_{1+m}}{(n-m)!}\left(\frac{16}{9}\theta\right)^{1/3+m/2} \frac{R_{n,m}(\theta)}{\Gamma(1+m/2)\Gamma(1+3m/2)}\,, \tag{3.27}$$

$$R_{n,m}(\theta) = \int_0^\infty dx\, x^{m/2} e^{-x} \int_0^\infty dy\, y^{3m/2} (\theta x + y)^{2(n-m)} \exp[-(\theta x + y)^2]\,. \tag{3.28}$$

Here A_k are ordinary perturbative expansion coefficients for the ground state energy.[13]

The behavior of the relative energies $E_0^{(N)}/E_{ex}$ is plotted in Fig. 3. The extremal values correspond to the optimal θ. The stability of the VPT series for values of θ close to the optimal one is indicated in Table 2.

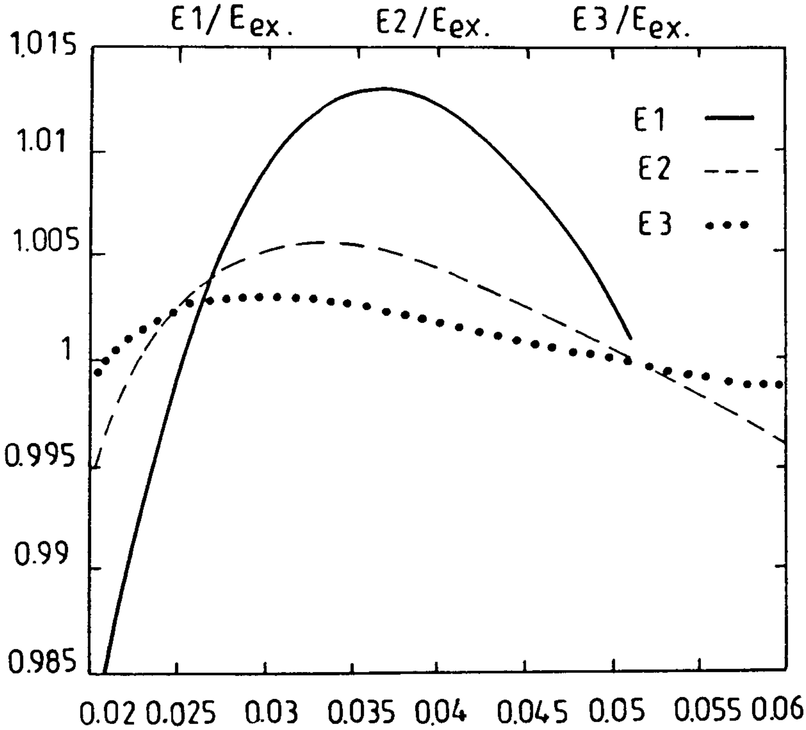

Fig. 3. The behavior of the functions $E_0^{(N)}/E_{ex}$ for $N = 1, 2, 3$ versus the parameter θ.

Table 2. The behavior of $E_0^{(N)}/E_{ex}$ in dependence of N for various values of θ.

	$E_0^{(N)}/E_{ex}$			
N	$\theta = 0.020$	$\theta = 0.028$	$\theta = 0.032$	$\theta = 0.036$
0	0.956	1.063	1.107	1.144
1	0.981	1.006	1.011	1.013
2	0.995	1.005	1.006	1.006
3	1.000	1.004	1.004	1.004
4	1.001	1.003	1.002	1.002
5	1.001	1.002	1.001	1.001
6	1.000	1.000	1.000	1.000

4. Asymptotic Behavior of Remote Terms of the VPT Series

Now we will consider the massless φ^4 theory in the four-dimensional Euclidean space with the action

$$S[\varphi] = S_0[\varphi] + \lambda S_4[\varphi] , \tag{4.1}$$

where the functionals $S_0[\varphi]$ and $S_4[\varphi]$ are defined by (2.2).

We shall construct the VPT series for the vacuum functional

$$W[0] = \int D\varphi \exp\{-S[\varphi]\} . \tag{4.2}$$

Generalization of the method to the Green functions presents no problems. As a variational addition, we will employ a functional of the *anharmonic* form

$$\tilde{S}[\varphi] = \theta^2 S_0^2[\varphi] . \tag{4.3}$$

Then the VPT series for the functional (4.2) is written as

$$\begin{gathered} W[0] = \sum_{n=0}^{\infty} W_n[0,\theta] , \\ W_n[0,\theta] = \frac{(-1)^n}{n!} \int D\varphi \exp\{-S_{\rm eff}[\varphi, n]\} , \end{gathered} \tag{4.4}$$

where

$$S_{\rm eff}[\varphi, n] = S_0[\varphi] + \theta^2 S_0^2[\varphi] - n \ln\{\lambda S_4[\varphi] - \theta^2 S_0^2[\varphi]\} . \tag{4.5}$$

The basic contribution to the asymptotics of higher order terms of the series (4.4) comes from the configurations of fields that obey the equation

$$\frac{\delta S_{\rm eff}[\varphi_0, n]}{\delta \varphi_0(x)} = 0 \tag{4.6}$$

and leave the functional of effective action to be invariant.[1-3] Varying (4.5) we obtain

$$-\partial^2 \varphi_0 + \frac{a}{3!} \varphi_0^3 = 0 , \tag{4.7}$$

where

$$a = \frac{4! \lambda n}{D[\varphi_0] + 2\theta^2 S_0[\varphi_0](n + D[\varphi_0])} , \tag{4.8}$$

$$D[\varphi] = \lambda S_4[\varphi] - \theta^2 S_0^2[\varphi] . \tag{4.9}$$

Solution of Eq. (4.7) is of the form

$$\varphi_0(x) = \pm \sqrt{\frac{48}{a}} \frac{\mu}{(x - x_0)^2 + \mu^2} . \tag{4.10}$$

Arbitrary parameters x_0 and μ stand for translational and scale invariance of the model under consideration.

Next, it is convenient to define the new variables

$$g = 4C_s\lambda\,, \quad \theta^2 = g\chi\,. \tag{4.11}$$

Here $C_s = 4!/(16\pi)^2$ is a constant entering into the Sobolev inequality (see, for instance, Refs. 18 and 19):

$$S_4[\varphi] \leq 4C_s S_0^2[\varphi]\,. \tag{4.12}$$

For the functional (4.9) of functions (4.10) we obtain

$$D[\varphi_0] = 4\frac{(16\pi^2)^2}{a^2}g(1-\chi)\,. \tag{4.13}$$

Inserting $S_0[\varphi]$ and (4.13) into (4.8) we get an equation for the parameter a whose solution is of the form

$$\begin{aligned} a &= \left\{\sqrt{b^2/4+nb} - b/2\right\}^{-1}, \\ b &= [(32/\pi^2)^2 g\chi]^{-1}\,. \end{aligned} \tag{4.14}$$

In the limit of large n we have

$$D[\varphi_0] \sim n(1-\chi)/\chi \tag{4.15}$$

and in the leading order in n,

$$W_n[0,\theta] \sim \frac{(-1)^n}{n!}n^n\left(\frac{1-\chi}{\chi}\right)^n \exp\{-n\}\,. \tag{4.16}$$

When the next orders in n including the functional determinant are taken into account, a multiplicative factor dependent on n appears in (4.16). However, it is not dominating and does not influence the convergence properties of the series.

From expression (4.16) it is seen that the series absolutely converges for $\chi > 1/2$ irrespective of the values of the coupling constant g; and, as follows from the Sobolev inequality (4.12), the VPT series for $\chi > 1$ is of positive sign. When $1/2 < \chi < 1$, the terms of the series (4.4) at large n form the Leibniz series. Here again the value $\chi = 1$ corresponds both to the change of the regime of the VPT series and to its asymptotic optimization.

The asymptotic behavior of remote terms of the VPT series, when an *anharmonic variational procedure* is performed, is determined by the behavior of the functional integral

$$J_k = \frac{1}{k!}\int D\varphi(A^2[\varphi] - S_4[\varphi])^k \exp[-(S_0[\varphi] + A^2[\varphi])]\,, \tag{4.17}$$

where functional $A[\varphi]$ is defined by (3.9). Making the change $\varphi \to k^{1/4}\varphi$ we derive

$$J_k = \frac{k^k}{k!} I_k, \tag{4.18}$$

$$I_k = \int D\varphi \exp(-kS_{\rm eff}[\varphi] - k^{1/2}S_0[\varphi]), \tag{4.19}$$

where

$$S_{\rm eff}[\varphi] = A^2[\varphi] - \ln D[\varphi], \tag{4.20}$$

$$D[\varphi] = A^2[\varphi] - S_4[\varphi]. \tag{4.21}$$

This integral (4.19) contains a large parameter k in the exponential and therefore its asymptotics can be found by the Laplace functional method.[14,17–19] The main contribution to the integral (4.19) comes from the configurations $\varphi_0(x)$, which minimize the effective action (4.20). The corresponding equation looks as follows:

$$-\partial^2\varphi_0 + a\varphi_0 - b\varphi_0^3 = 0, \tag{4.22}$$

where

$$a = \chi/\theta, \tag{4.23}$$

$$b = 2[\theta A[\varphi_0](1 - D[\varphi_0])]^{-1}. \tag{4.24}$$

It is convenient to pass to the function $f(x)$ which satisfies a differential equation

$$[-\partial^2 + 1]f(x) - f^3(x) = 0 \tag{4.25}$$

and is connected with the function $\varphi_0(x)$ by the relation

$$\varphi_0(x) = \sqrt{\frac{a}{b}}\, f(\sqrt{a}x). \tag{4.26}$$

We define the constant

$$C = \int dx\, f^4(x), \tag{4.27}$$

which depends on the space dimension and can be evaluated following, for example, Ref. 18. In the case under consideration the exact value of C is not important. The functionals $S_4[\varphi_0]$ and $A^2[\varphi_0]$ are expressed via (4.27) as

$$S_4[\varphi_0] = \alpha/b^2\, z, \tag{4.28}$$

$$A^2[\varphi_0] = \alpha^2\tau/b^2, \tag{4.29}$$

where we define the parameters

$$\alpha = Ca^{2-n/2}, \tag{4.30}$$

$$\tau = \theta^2/4. \tag{4.31}$$

Three parameters α, b and τ, as follows from (4.24), are connected by the relation

$$\alpha\tau(1 - D[\varphi_0]) = 1, \tag{4.32}$$

where

$$D[\varphi_0] = \alpha(\alpha\tau - 1)/b^2. \tag{4.33}$$

Thus, as before, only two parameters are independent. In the leading order in k we obtain for the integral (4.17)

$$J_k \sim k^{-1/2} D^k[\varphi_0] \exp\{-k[A^2[\varphi_0] - 1]\}. \tag{4.34}$$

The region of the values of parameters at which the VPT series is convergent is determined by the inequality

$$|D[\varphi_0]| < \exp\{A^2[\varphi_0] - 1\}. \tag{4.35}$$

The best choice of the parameters at which the contribution of remote terms of the VPT series is minimal (the so-called *asymptotic optimization*[9,10]) implies the condition

$$D[\varphi_0] = 0, \tag{4.36}$$

leading to the connection of the parameters

$$\alpha\tau = 1. \tag{4.37}$$

Thus the single independent parameter remains which can also be fixed by the optimization of the first terms of the VPT series. The asymptotic optimization condition for original parameters θ and χ is written as

$$\chi = \left(\frac{16}{\theta^n C^2}\right)^{\frac{1}{4-n}}. \tag{4.38}$$

In particular, in the one-dimensional case, $C = 16/3$ and the condition (4.38) transform into (3.26).

5. Gaussian Effective Potential in Variational Perturbation Theory

In this section the Gaussian effective potential will be derived on the basis of the VPT under various choices of the variational addition. In the $\lambda\varphi^4$ theory in the n-dimensional space the GEP has the form[3]

$$V_{\rm GEP} = V_c + \Delta V_{\rm GEP}\,, \tag{5.1}$$

$$\Delta V_{\rm GEP} = \frac{(z^2)^{n/2}}{n} A_n + \frac{1}{2}(m^2 - z^2)(z^2)^{n/2-1} A_n + 3\lambda(z^2)^{n-2}A_n^2 + 6\lambda(z^2)^{n/2-1}A_n\varphi^2\,, \tag{5.2}$$

$$A_n = \mu^{2\varepsilon}\Gamma(1-n/2)/(4\pi)^{n/2}\,, \quad n = d - 2\varepsilon\,, \quad d = 1, 2, \ldots\,,$$

and z^2 satisfies the equation

$$z^2 = m^2 + 12\lambda\varphi^2 + 12\lambda A_n (z^2)^{n/2-1}\,. \tag{5.3}$$

We will consider the φ^4 theory in the n-dimensional space with the pseudo-Euclidean signature. The action functional looks as follows:

$$S[\varphi] = \int dx \left[\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \lambda\varphi^4\right]. \tag{5.4}$$

The generating functional of Green functions reads

$$W[J] = \int D\varphi \exp\{i[S[\varphi] + \langle J\varphi\rangle]\} = \exp\{i[S[\varphi_c] + \langle J\varphi_c\rangle]\} D[J]\,, \tag{5.5}$$

where

$$D[J] = \int D\varphi \exp\{-iA[\varphi]\}\,, \tag{5.6}$$

$$A[\varphi] = \int dx \left[\frac{1}{2}\varphi(\partial^2 + m^2 + 12\lambda\varphi_c^2)\varphi + 4\lambda\varphi_c\varphi^3 + \lambda\varphi^4\right], \tag{5.7}$$

and the function φ_c satisfies a classical equation of motion

$$(\partial^2 + m^2)\varphi_c + 4\lambda\varphi_c^3 = J\,. \tag{5.8}$$

In the standard one-loop approximation only the terms quadric in fields φ are retained in expression (5.7) for $A[\varphi]$. In this case the functional integral for $D[J]$ becomes Gaussian.

We shall evaluate the quantity $D[J]$ by means of VPT. Let us first consider the *harmonic variational procedure*. We rewrite the functional $A[\varphi]$ as

$$A[\varphi] = \int dx \left[\frac{1}{2}\varphi(\partial^2 + z^2)\varphi + \lambda\left(4\varphi_c\varphi^3 + \varphi^4 - \frac{\chi^2}{2}\varphi^2\right)\right], \tag{5.9}$$

where

$$z^2 = m^2 + 12\lambda\varphi_c^2 + \lambda\chi^2\,.$$

As a result, the VPT series for the quantity $D[J]$ is by the harmonic variational type transformed to

$$D = \left[\det \frac{\partial^2 + z^2}{\partial^2}\right]^{-1/2} \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!}$$

$$\times \left[\int dx \left(4\varphi_c\hat{\varphi}^3 + \hat{\varphi}^4 - \frac{\chi^2}{2}\hat{\varphi}^2\right)\right]^n \exp\left[-\frac{i}{2}\langle j\Delta j\rangle\right]_{j=0}, \qquad (5.10)$$

where

$$\Delta(p) = (p^2 - z^2 + i0)^{-1}, \quad \hat{\varphi}(x) = i\,\frac{\delta}{\delta j(x)}.$$

Considering (5.10), let us restrict ourselves to the first two addends in the sum that give rise to the first nontrivial approximation. The contributions to the effective potential, corresponding to these addends, equal

$$V_0 = \frac{1}{n}\, z^2\Delta_0(z^2), \qquad (5.11)$$

$$V_1 = \lambda\left[3\Delta_0^2(z^2) - \frac{\chi^2}{2}\,\Delta_0(z^2)\right], \qquad (5.12)$$

where

$$\Delta_0(z^2) = \mu^{2\epsilon}\,\frac{\Gamma(1-n/2)}{(4\pi)^{n/2}}\,(z^2)^{n/2-1}. \qquad (5.13)$$

The optimization condition

$$\frac{d(V_0 + V_1)}{dz^2} = 0 \qquad (5.14)$$

gives the equation for the variational parameter z^2:

$$z^2 = m^2 + 12\lambda\varphi^2 + 12\lambda\Delta_0(z^2). \qquad (5.15)$$

With the help of (5.15) in the considered order of VPT we find for the effective potential the expression

$$V_{\text{eff}}(\varphi) = V_{\text{cl}} + V_0 + V_1 = \frac{1}{2}\,m^2\varphi^2 + \lambda\varphi^4 + \frac{1}{n}\,z^2\Delta_0(z^2)$$

$$+ \frac{1}{2}\,(m^2 - z^2)\Delta_0(z^2) + \lambda[3\Delta_0^2(z^2) + 6\varphi^2\Delta_0(z^2)]. \qquad (5.16)$$

When comparing (5.1)–(5.3) with (5.13), (5.15) and (5.16), it is easy to see that both the functions (5.2) and (5.16) and Eqs. (5.3) and (5.15) for the massive parameter coincide with one another.

Let us now calculate the quantity $D[J]$ by using for (5.5) the *anharmonic variation* of the action functional. We choose the anharmonic addition in the form $\tilde{S}^2[\varphi]$, where

$$\tilde{S}[\varphi] = \frac{\chi}{2\Omega^{1/2}} \int dx\, \varphi^2(x). \qquad (5.17)$$

The coordinate space volume Ω in (5.17) appears because the derivation of $V_{\rm eff}$ from the effective action requires one to consider a constant field configuration. Then the parameter optimizing VPT series χ in (5.17) does not depend on Ω. As a result, we find

$$D[J] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int D\varphi [\lambda(\varphi^4 + 4\varphi_c\varphi^3) - \tilde{S}^2[\varphi]]^n$$
$$\times \exp\left\{ - i\left[\frac{1}{2}\varphi(\partial^2 + m^2 + 12\lambda\varphi_c^2)\varphi + \tilde{S}^2[\varphi]\right]\right\}. \tag{5.18}$$

Any power of $\tilde{S}^2[\varphi]$ in (5.18) can be obtained by the corresponding number of differentiation of the expression $\exp(-i\varepsilon\tilde{S}^2[\varphi])$ with respect to parameter ε by putting $\varepsilon = 1$ at the end. As to the addend $\tilde{S}^2[\varphi]$ in the exponential in (5.18) which makes the functional integral non-Gaussian, the problem is easy to solve by using the transformation

$$\exp\{-i\varepsilon\tilde{S}^2[\varphi]\} = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left\{ i\left(\frac{u^2}{4} \pm \sqrt{\varepsilon}\tilde{S}[\varphi]\right) - i\,\frac{\pi}{4}\right\}. \tag{5.19}$$

As a result, the VPT series takes the form

$$D[J] = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \frac{(-i)^{n-k}}{(n-k)!n!} \left[\frac{d}{d\varepsilon}\right]^{n-k}$$
$$\times \sqrt{\Omega} \int_{-\infty}^{\infty} \frac{dv}{2\sqrt{\pi}} \exp\left\{ i\Omega\,\frac{v^2}{4} - i\,\frac{\pi}{4}\right\} \left[\det \frac{\partial^2 + M^2}{\partial^2}\right]^{-1/2}$$
$$\times \left[\lambda \int dx(4\varphi_c\hat{\varphi}^3 + \hat{\varphi}^4)\right]^k \exp\left[-\frac{i}{2}\langle j\Delta j\rangle\right]_{j=0}, \tag{5.20}$$

where

$$M^2 = m^2 + 12\lambda\varphi_c^2 + \sqrt{\varepsilon}\chi v\,,$$
$$\Delta(p) = (p^2 - M^2 + i0)^{-1}. \tag{5.21}$$

The integral over v in (5.20) contains the large parameter Ω and therefore can be evaluated, for example, by using the method of stationary phase. As a result, the effective potential in the first nontrivial VPT order reads

$$V_{\rm eff} = V_c + \Delta V_{\rm eff}\,, \quad \Delta V_{\rm eff} = V_0 + V_1\,,$$

where

$$V_0 = \frac{1}{n} M^2\Delta_0 - \frac{\chi^2}{4}\Delta_0^2\,,$$
$$V_1 = -\frac{\chi^2}{4}\Delta_0^2 + 3\lambda\Delta_0^2. \tag{5.22}$$

Here M^2 is the massive parameter derived from (5.21) at $\varepsilon = 1$ and $v = v_0$, where v_0 is the stationary phase point in the integral (5.20). The corresponding equation has the form

$$M^2 = m^2 + 12\lambda\varphi^2 + \chi^2\Delta_0 \,. \tag{5.23}$$

The quantity $\Delta_0 = \Delta_0(M^2)$ is determined by the expression (5.13) and represents the Euclidean propagator $\Delta(x = 0, M^2)$ written with the help of the dimensional regularization.

Then, let us consider two optimization schemes of the VPT series (see Refs. 9 and 10). In accordance with the ***first optimization version*** the variational parameter is determined by the condition for the contribution of "nonleading" terms of the series being minimal. In the present case we require that $\min|V_1|$. It is easy to see that the function V_1 is such that the equation V_1 admits a solution. This situation is, obviously, the most preferred as the considered optimization version is performed. From (5.22) we find the optimal value of χ^2:

$$\chi^2 = 12\lambda \,. \tag{5.24}$$

Equation (5.23) for this choice of χ^2, up to the change of M^2 to z^2, transforms into the GEP method Eq. (5.3) for the massive parameter z^2. Thus, in the considered optimization procedure $\Delta V_{\text{eff}} = V_0$ and now, by using Eqs. (5.22)–(5.24), it is easy to show that

$$\Delta V_{\text{eff}} = \Delta V_{\text{GEP}} \,.$$

To implement the ***second optimization version***, we should keep the parameter M^2 variational. Making use of Eqs. (5.22) and (5.23) we obtain

$$\Delta V_{\text{eff}} = \left(\frac{1}{n} - \frac{1}{2}\right) M^2\Delta_0 + \frac{1}{2}(m^2 + 12\lambda\varphi^2)\Delta_0 + 3\lambda\Delta_0 \,. \tag{5.25}$$

The optimization condition has the form

$$\frac{\partial \Delta V_{\text{eff}}}{\partial M^2} = 0 \tag{5.26}$$

and gives rise to the equation for M^2:

$$M^2 = m^2 + 12\lambda\varphi^2 + 12\lambda\Delta_0 \,. \tag{5.27}$$

Comparing (5.27) with (5.23) and (5.24) we conclude that the two optimization versions lead to the same result:

$$V_{\text{eff}}^{(1)} = V_{\text{GEP}} \,.$$

Within the previous consideration we have obtained the GEP by building the VPT series for a variational correction to the one-loop approximation. Let us derive the GEP by another approach that does not use the loop expansion and directly operates with the original functional $W[j]$. We will consider the two-parameter *anharmonic type* addition to the action:

$$\tilde{S}[\varphi] = \frac{a^2}{\Omega} S_2^2[\varphi] + \frac{b^4}{\Omega^3} S_1^4[\varphi] , \tag{5.28}$$

where

$$S_1[\varphi] = \int dx \, \varphi(x) ,$$

$$S_2[\varphi] = \int dx \, \varphi^2(x) .$$

The VPT series for the generating functional of Green functions looks as follows:

$$W[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int D\varphi [\tilde{S} - S_{\text{int}}]^n$$
$$\times \exp\left\{ i \left[S_0 - m^2 S_2 - \varepsilon \frac{a^2}{\Omega} S_2^2 - \theta \frac{b^4}{\Omega^3} S_1^4 + \langle j\varphi \rangle \right] \right\} . \tag{5.29}$$

The parameters ε and θ are introduced here to give one possibility of obtaining in the integrand the terms connected with S_1 and S_2 by differentiating with respect to ε and θ. Then only the interaction action S_{int} remains in a factor in front of the exponential. The expression in the exponential in (5.29) is reduced to the form quadric in the fields by using the Fourier transformation

$$F(A[\varphi]) = \int_{-\infty}^{\infty} dx \frac{dp}{2\pi} F(p) \exp[\pm i(A[\varphi] - p)x] ,$$

where $A[\varphi]$ is the functional quadric in fields. Then (5.29) is rewritten as

$$W[j] = \Omega^2 \int_{-\infty}^{\infty} dx \frac{dp}{2\pi} \int_{-\infty}^{\infty} dy \frac{dq}{2\pi} \exp\{ i\Omega[px - qy - p^2 - q^4] \}$$
$$\times \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{i^{n-k}}{m!(n-k-m)!} \left(i \frac{\partial}{\partial \varepsilon} \right)^m \left(i \frac{\partial}{\partial \theta} \right)^{n-k-m}$$
$$\times \left[\det \frac{\partial^2 + M^2}{\partial^2} \right]^{-1/2} w_k[J, M^2] , \tag{5.30}$$

where

$$M^2 = m^2 + \sqrt{\varepsilon} a x ,$$
$$J = j + \theta^{1/4} b y , \tag{5.31}$$

and $w_k[J, M^2]$ are the ordinary perturbative expansion coefficients for the generating functional of Green functions $W[j]$:

$$w_k[J, M^2] = \frac{(-i\lambda)^k}{k!}\left[\int dx \frac{\delta}{\delta J^4(x)}\right]^k \exp\left\{-\frac{i}{2}\langle J\Delta J\rangle\right\}. \tag{5.32}$$

In the first nontrivial order for the generating functional of connected Green functions

$$Z[j] = (i\Omega)^{-1}\ln W[j],$$

we find

$$\begin{aligned} Z_0[j] &= \frac{1}{2}\frac{J^2}{M^2} + \frac{1}{4}(M^2 - m^2)\left[\Delta_0 + \frac{J^2}{(M^2)^2}\right] \\ &\quad - \frac{3}{4}\frac{J}{M^2}(J - j) - \frac{1}{n}M^2\Delta_0, \\ Z_1[j] &= \frac{1}{4}(M^2 - m^2)\left[\Delta_0 + \frac{J^2}{(M^2)^2}\right] - \frac{1}{4}\frac{J}{M^2}(J - j) \\ &\quad - \lambda\left[3\Delta_0^2 + 6\Delta_0\frac{J^2}{(M^2)^2} + \frac{J^4}{(M^2)^4}\right]. \end{aligned} \tag{5.33}$$

Here, as before, the method of a stationary phase has been applied to the numerical integrals. In (5.33), instead of original a^2, b^4, the more transparent variational parameters J and M^2 have been used. The optimization conditions in this case read

$$\frac{\partial Z^{(1)}}{\partial J} = 0, \quad \frac{\partial Z^{(1)}}{\partial M^2} = 0,$$

where

$$Z^{(1)} = Z_0 + Z_1.$$

However, it is more convenient here to define the new variables

$$x = J/M^2, \quad y = M^2/m^2. \tag{5.34}$$

From (5.33) we get

$$\begin{aligned} Z^{(1)}[j] &= jx - \frac{1}{2}m^2x^2 + \left(\frac{1}{2} - \frac{1}{n}\right)m^2y\Delta_0(m^2y) \\ &\quad - \frac{1}{2}m^2\Delta_0(m^2y) - \lambda[3\Delta_0^2(m^2y) + 6\Delta_0(m^2y)x^2 + x^4]. \end{aligned} \tag{5.35}$$

The optimization condition $\partial Z^{(1)}/\partial x = 0$ yields the equation

$$m^2x + 4\lambda x(3\Delta_0 + x^2) = j. \tag{5.36}$$

By analogy, requiring $\partial Z^{(1)}/\partial y = 0$ we get the equation

$$m^2(y-1) = 12\lambda(\Delta_0 + x^2)\,. \tag{5.37}$$

Making use of (5.36) and (5.37) we easily find

$$\varphi = \frac{dZ^{(1)}}{dj} = \frac{\partial Z^{(1)}}{\partial j} = x\,. \tag{5.38}$$

For the effective potential we obtain

$$\begin{aligned} V_{\text{eff}} = \frac{1}{2}\,m^2\varphi^2 + \left(\frac{1}{n} - \frac{1}{2}\right)M^2\Delta_0(M^2) + \frac{1}{2}\,m^2\Delta_0(M^2) \\ + \lambda[3\Delta_0^2(M^2) + 6\Delta_0(M^2)\varphi^2 + \varphi^4]\,. \end{aligned} \tag{5.39}$$

As follows from (5.37) and (5.38), the parameter M^2 satisfies Eq. (5.27), by means of which it is easy to show that (5.39) coincides with V_{GEP}.

6. Conclusion

In this paper the nonperturbative method of the Gaussian effective potential (GEP) is analyzed from the point of view of a more general approach, the variational perturbation theory (VPT). In the VPT method, the initial quantity, such as the Green function, is represented in the form of some series, whose convergence can be governed by choosing certain values of variational type parameters. An important technical peculiarity is that the VPT series can be constructed by using only the standard Feynman rules.

We have shown here how the GEP emerges in the framework of the VPT in the first nontrivial order. It is important that from the very beginning we deal with a series that, in principle, allows one to calculate the corrections and thus, to explore the question about the stability of the results obtained by using the "main contribution." The possibility of calculating corrections advantageously distinguishes the VPT method from other nonperturbative approaches, where the question about the stability of the results obtained, for example, by using the variational method, turns into a serious problem because of the absence of a simple algorithm of calculating corrections. Moreover, the VPT method allows one to construct a series whose convergence properties can be influenced through special parameters. It is particularly important in the essentially nonperturbative tasks, where, despite the absence of a small initial parameter, the reliable results can be obtained on the basis of a series whose convergence is fast enough.

In this paper the GEP as a first nontrivial VPT order has been derived by using one or another variational addition to the action. In other words, we have shown that VPT series possessing different structures may give rise to the same result when only the leading contribution is retained. Certainly, it is doubtful to hope that just the first nontrivial order would permit us to make sure the good enough degree of the approximation to the considered quantity. So, for example, in the two-dimensional case, the GEP gives rise to the first order phase transition[2] (see also Refs. 5, 8, 20 and 21), which is in contradiction with the known rigorous results.[22,23]

In this work we argue that the harmonic recipe of introducing a variational addition leads to a divergent series, which nevertheless can be used as the asymptotical series if a small parameter is present. In particular, by means of this procedure, one may improve the results of perturbation theory and penetrate into the region of larger values of the coupling constant. The small parameter may emerge effectively also, as it, for example, occurs in the anharmonic oscillator case. If the small parameter is unknown, however, it is problematic to obtain reliable results by using the harmonic variational procedure.

In this connection, the VPT method with the anharmonic variational additions appears to be more advantageous. Under its consideration it turns out that it is not at all always reasonable to reduce the "main contribution" to the GEP. These questions will be considered in subsequent publications.

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