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VARIATIONAL PERTURBATION THEORY

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A nonperturbative method — variational perturbation theory (VPT) — is discussed. A quantity we are interested in is represented by a series, a finite number of terms of which not only describe the region of small coupling constant but reproduce well the strong coupling limit. The method is formulated only in terms of the Gaussian quadratures, and diagrams of the conventional perturbation theory are used. Its efficiency is demonstrated for the quantum-mechanical anharmonic oscillator. The properties of convergence are studied for series in VPT for the $\phi^4_{(d)}$ model. It is shown that it is possible to choose variational additions such that they lead to convergent series for any values of the coupling constant. Upper and lower estimates for the quantities under investigation are considered.

The nonperturbative Gaussian effective potential is derived from a more general approach, VPT. Various versions of the variational procedure are explored and the preference for the anharmonic variational procedure in view of convergence of the obtained series is argued.

We investigate the renormalization procedure in the φ^4 model in VPT. The nonperturbative β function is derived in the framework of the proposed approach. The obtained result is in agreement with four-loop approximation and has the asymptotic behavior as $g^{3/2}$ for a large coupling constant.

We construct the VPT series for Yang–Mills theory and study its convergence properties. We introduce coupling to spinor fields and demonstrate that they do not influence the VPT series convergence properties.

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1. Introduction

Approximation of a quantity under consideration by a finite number of terms of a certain series is a standard computational procedure in many problems of physics. In quantum field theory this is conventionally an expansion into a perturbative series. This approach combined with the renormalization procedure is now a basic method for computations. As is well known, perturbative series for many interesting models, including realistic models, are not convergent. Nevertheless, at small values of the coupling constant these series may be considered as asymptotic series and could provide useful information. However, even in the theories with a small coupling constant, for instance in quantum electrodynamics, there exist problems which cannot be solved by perturbative methods. Also, a lot of problems of quantum chromodynamics require nonperturbative approaches. At present, a central problem of quantum field theory is to go beyond the scope of perturbation theory.

A great number of studies have been devoted to the development of nonperturbative methods. Among them is the summation of a perturbative series; see Refs. 1–3. The main difficulty is that the procedure of summation of asymptotic series is not unique, which is generally a functional arbitrariness, and the correct formulation of a problem of summation is ensured by further information on the sum of a series.[4] At present information of that kind is known only for the simplest field-theoretical models.[5]

In Refs. 6–12 approaches are proposed which are not directly based on the perturbative series. Thus, the method of Gaussian effective potential has recently become rather popular.[13–16] Many of nonperturbative approaches make use of a variational procedure for finding the leading contribution. However, in this case there is not always an algorithm for calculating corrections to the value found by a variational procedure, and this makes it difficult to answer the question as to how adequate is the so-called main contribution to the object under investigation and what is the range of applicability of the obtained estimates. Moreover, even if the algorithm for calculating corrections (i.e. terms of a certain approximating series) exists, it is still not sufficient. Here the properties of convergence of a series are of fundamental importance. Indeed, unlike the case where even a divergent perturbative series in the weak coupling constant approximates a given object as an asymptotic series, the approximating series in the absence of a small parameter should obey stricter requirements. Reliable information in this case may be obtained only on the basis of convergent series. It is more reliable to deal not with an arbitrary convergent series but just with the Leibniz series (an alternating series with terms decreasing in absolute value). Then it will become possible to compute upper and lower estimates for a given quantity on the basis of first terms of the series. In the case of additional free parameters influencing the terms of the series, these estimates may be made as close as possible to each other.

In this paper, we consider a method of variational perturbation theory (VPT).[17–20] The mathematical basis for this approach is the functional integral formalism.[11,12] Despite the word "perturbation" being present in the name of the approach, the VPT method does not use the smallness of the coupling constant. The additions in the method are calculable because this method employs only Gaussian functional quadratures. Besides, a VPT series can be written so that its terms are defined by the usual Feynman diagrams. In this case, the VPT series will surely differ in structure from the conventional perturbation theory, and diagrams will contain a modified propagator.

2. Variational Perturbation Theory

Here we will apply the VPT method to Green functions of the φ^4 model in the Euclidean d-dimensional space. To this end we write the 2ν-point function in the form

$$G_{2\nu} = \int D\varphi\{\varphi^{2\nu}\}\exp(-S[\varphi])\,, \tag{2.1}$$

where

$$\{\varphi^{2\nu}\} = \varphi(x_1)\cdots\varphi(x_{2\nu})$$

and the functional of action looks as follows:

$$\begin{aligned} S[\varphi] &= S_0[\varphi] + \frac{m^2}{2}S_2[\varphi] + \lambda S_4[\varphi]\,, \\ S_0[\varphi] &= \frac{1}{2}\int dx(\partial\varphi)^2\,, \\ S_p[\varphi] &= \int dx\varphi^p\,. \end{aligned} \tag{2.2}$$

The measure of integration in (2.1) is normalized so that

$$\int D\varphi \exp\left(-S_0[\varphi] - \frac{m^2}{2}S_2[\varphi]\right) = 1\,. \tag{2.3}$$

We shall construct a VPT series by using the following Gaussian functional quadratures:

$$\int D\varphi \exp\left[-\left(\frac{1}{2}\langle\varphi\hat{K}\varphi\rangle + \langle\varphi J\rangle\right)\right]$$
$$= \left(\det \frac{\hat{K}}{-\partial^2 + m^2}\right)^{-1/2} \exp\left(\frac{1}{2}\langle J\hat{K}^{-1}J\rangle\right). \qquad (2.4)$$

The VPT series for the Green functions (2.1) is constructed in the following way:

$$G_{2\nu} = \sum_{n=0}^{\infty} G_{2\nu,n}\,, \qquad (2.5)$$

$$G_{2\nu,n} = \frac{(-1)^n}{n!}\int D\varphi\{\varphi^{2\nu}\}(\lambda S_4[\varphi] - \tilde{S}[\varphi])^n \exp\left(-S_0[\varphi] - \frac{m^2}{2}S_2[\varphi] - \tilde{S}[\varphi]\right). \qquad (2.6)$$

The variational functional $\tilde{S}[\varphi]$ will be taken to be dependent on certain parameters, but the total sum (2.5) surely will not depend on these parameters. The choice can be such as to provide the expansion (2.5) as being optimal.

The functional $\tilde{S}[\varphi]$ should be defined so that the terms of the VPT series (2.6) are calculable, i.e. the form of $\tilde{S}[\varphi]$ should be such that the functional integral in (2.6) can be reduced to the Gaussian quadratures (2.4). This requirement does not mean that the functional $\tilde{S}[\varphi]$ must be quadratic in fields. We can pass to the Gaussian functional integral by using the Fourier transformation

$$F(A[\varphi]) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} F(p) \exp\left[\pm i(A[\varphi] - p)x\right], \qquad (2.7)$$

where $A[\varphi]$ is the functional quadratic in fields.

We choose here the sum of harmonic and anharmonic functionals being $\tilde{S}[\varphi]$, i.e.

$$\tilde{S}[\varphi] = \frac{M^2}{2} S_2[\varphi] + \theta^2 S_2^2[\varphi]\,, \qquad (2.8)$$

where M and θ are the parameters through which the VPT series is optimized.

The Gaussian quadrature can be obtained by using the following representation for the anharmonic term:

$$\exp(-\theta^2 S_2^2[\varphi]) = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left(-\frac{u^2}{4} \pm iu\theta S_2[\varphi]\right). \qquad (2.9)$$

For the VPT series terms (2.6) we have

$$G_{2\nu,n} = \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{1}{l!k!(n-k-l)!}$$
$$\times \int D\varphi\{\varphi^{2\nu}\}(-\lambda S_4[\varphi])^k \theta^{2l}(M^2 - m^2)^{n-k-l}$$
$$\times \left(\frac{S_2[\varphi]}{2}\right)^{n+l-k} \exp\left\{-\left(S_0[\varphi] + \frac{m^2}{2}S_2[\varphi] + \theta^2 S_2^2[\varphi]\right)\right\}. \qquad (2.10)$$

It is convenient to write the expression $(S_2[\varphi]/2)^{n+l-k}$ in the form of operator differentiation:

$$\left(\frac{S_2[\varphi]}{2}\right)^{n+l-k} \to \left(-\frac{\partial}{\partial M^2}\right)^{n+l-k}. \tag{2.11}$$

Then in (2.10) before the exponential there remain, besides $\{\varphi^{2\nu}\}$, only the powers of the initial action functional $(-\lambda S_4[\varphi])^k$, which leads to the conventional vertices in the Feynman rules. Owing to the change of the quadratic form in the exponent, the propagator is modified in form. So, (2.10) can be calculated by using the standard Feynman graphs for the $\lambda\varphi^4$ theory with the mass parameter in the propagator $\chi^2 = M^2 + iu\theta$. As a result, (2.10) assumes the form

$$G_{2\nu,n} = \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{1}{l!(n-k-l)!}\int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left(-\frac{u^2}{4}\right)$$
$$\times\, \theta^{2l}(M^2 - m^2)^{n-k-l}\left(-\frac{\partial}{\partial M^2}\right)^{n+l-k} \tilde{g}_{2\nu}^{(k)}(\chi^2), \tag{2.12}$$

where

$$\tilde{g}_{2\nu,n}^{(k)}(\chi^2) = \frac{1}{k!}\int D\varphi\{\varphi^{2\nu}\}(-\lambda S_4[\varphi])^k \exp\left\{-\left(S_0[\varphi] + \frac{\chi^2}{2}S_2[\varphi]\right)\right\}. \tag{2.13}$$

The latter expression can be written as

$$\tilde{g}_{2\nu,n}^{(k)}(\chi^2) = \det\left(\frac{-\partial^2 + \chi^2}{-\partial^2 + m^2}\right)^{-1/2} \cdot g_{2\nu,n}^{(k)}(\chi^2), \tag{2.14}$$

where $g_{2\nu,n}^{(k)}(\chi^2)$ are calculated on the basis of diagrams of the kth order of conventional perturbation theory with the propagator

$$\Delta(p,\chi^2) = (p^2 + \chi^2)^{-1}. \tag{2.15}$$

Thus, the Nth order of the VPT expansion (2.5) can be constructed with the same diagrams as the conventional perturbation Nth order.

3. A Simple Example

Let us consider the integral

$$Z[g] = \int d\mathbf{x} \exp(-S[x]), \tag{3.1}$$

where

$$S[x] = S_0[x] + gS_{\text{int}}[x], \qquad S_0[x] = \mathbf{x}^2 = x_1^2 + x_2^2,$$
$$S_{\text{int}}[x] = x_1^4 + x_2^4, \qquad d\mathbf{x} = dx_1 dx_2. \tag{3.2}$$

The quantity (3.2) can be considered as a zero-dimensional analog of the corresponding functional integrals in the φ^4 quantum field model. In the following we shall operate only with Gaussian functional quadratures, and therefore we shall use here, for calculating (3.2), only Gaussian integrals

$$\int dx\, P(x) \exp(-S_0[x])\,, \tag{3.3}$$

where $P(x)$ is a polynomial of x_1 and x_2 variables. The first obvious opportunity is the expansion of the integrand in (3.1) in powers of the coupling constant g. As a result, we derive the ordinary perturbation theory

$$Z[g] = \sum g^n C_n\,, \tag{3.4}$$

$$C_n = \frac{(-1)^n}{n!} \int d\mathrm{x}\, S^n_{\mathrm{int}} \exp(-S_0[x])\,. \tag{3.5}$$

It is well known that the series (3.4) is asymptotical and, therefore, does not give any possibility of judging about the quantity $Z[g]$ in the nonperturbative region without additional information about its sum. However, the standard perturbative expansion (3.4), (3.5) is not unique based on the Gaussian quadratures. We shall here consider two kinds of such expansions differing from the series (3.4). They differ from one another in the manner of introducing variational terms into the action (3.2). Namely, they utilize the "harmonic" and "anharmonic" recipes of introducing the variational terms, respectively. In fact, these recipes are not a single possibility of a variational procedure construction. In particular, a composition of them can be used.

The first method we shall consider is the *harmonic variational procedure*, where the free action $S_0[x]$ will be used as a harmonic variational extra term. The total action is rewritten in the form

$$S[x] = S^h_0[x] + S^h_{\mathrm{int}}[x]\,, \tag{3.6}$$

where

$$S^h_0[x] = S_0[x] + \chi S_0[x]\,, \tag{3.7}$$

$$S^h_{\mathrm{int}}[x] = g S_{\mathrm{int}}[x] - \chi S_0[x]\,, \tag{3.8}$$

and the expansion in powers of a new "functional of interaction" $S^h_{\mathrm{int}}[x]$ is performed. It is easy to see that the task is formulated only in terms of the Gaussian functional quadratures. As a result, the VPT series takes the form

$$Z[g] = \sum_n Z_n[g,\chi]\,, \tag{3.9}$$

$$Z_n[g,\chi] = \frac{(-g)^n}{n!(1+\chi g)^{1+2n}} \int d\mathrm{x}\, [\, S_{\mathrm{int}} - \chi(1+\chi g) S_0]^n \exp(-S_0[x])\,. \tag{3.10}$$

Actually, the original quantity $Z[g]$ does not depend on the variational parameter χ. Therefore, the freedom in choosing χ can be used to improve the properties of the VPT series. Various ways of optimal choice of variational parameters have been considered in Ref. 18.

In the field theory, we know, as a rule, only a few first terms of the VPT expansion, and maybe the asymptotic behavior of remote terms. Just due to this information the optimal value of variational parameters can be chosen. In the majority of cases the asymptotic and the first nontrivial order which permit us to obtain an equation for the variational parameter, are used. Then the stability of the results will be achieved only when the contribution of subsequent terms of the series proves to be small in comparison with the basic contribution.

The fact that the exact quantity does not depend on variational parameters results in the wonderful possibility of choosing their values so that the considered VPT order would maximally approximate the searched quantity. Indeed, let

$$Z[g] = Z^{(N)}[g,\chi] + \Delta Z^{(N)}[g,\chi], \tag{3.11}$$

where

$$Z^{(N)}[g,\chi] = \sum_{n=0}^{N} Z_n[g,\chi],$$

$$\Delta Z^{(N)}[g,\chi] = \sum_{n=N+1}^{\infty} Z_n[g,\chi].$$

Then

$$\frac{\partial Z^{(N)}[g,\chi]}{\partial \chi} = -\frac{\partial \Delta Z^{(N)}[g,\chi]}{\partial \chi},$$

and thus, if χ_0 is a point of maximum for $Z^{(N)}$, then this point is a point of the minimum for the whole reminder $\Delta Z^{(N)}$, simultaneously. Thus we have to require

$$\frac{\partial Z^{(N)}[g,\chi]}{\partial \chi} = 0. \tag{3.12}$$

Making use Eqs. (3.8) and (3.9) and setting $N = 1$ we find from (3.12) that

$$\chi = \frac{1/\tau - 1}{g}, \qquad \tau = \frac{2}{9g}(\sqrt{1+9g} - 1). \tag{3.13}$$

The results of calculations will be discussed and compared with those obtained in the framework of the anharmonic variational procedure somewhat later.

The *anharmonic method* of introducing the variational addition is based on the following representation of the action:

$$S[x] = S_0^a[x] + S_{\rm int}^a[x], \tag{3.14}$$

where

$$S_0^a[x] = S_0[x] + \theta S_0^2[x]\,, \tag{3.15}$$

$$S_{\rm int}^a[x] = gS_{\rm int}[x] - \theta S_0^2[x]\,. \tag{3.16}$$

In this case the expansion of the integrand in (3.1) is carried out in powers of $S_{\rm int}^a[x]$. The situation here is somewhat more complicated as compared with the previous case as the addend $\theta S_0^2[x]$ is present in the exponential, which leads to the non-Gaussian form of the emerging integral. However, this problem can be easily solved by using the Fourier transformation

$$\exp[-\theta S_0^2(x)] = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left[-\frac{u^2}{4} + iu\sqrt{\theta}S_0(x)\right]. \tag{3.17}$$

As a result the VPT series takes the form

$$Z[g] = \sum Z_n[g,\theta]\,, \tag{3.18}$$

$$Z_n[g,\theta] = \int_0^{\infty} d\alpha(\alpha^2\theta)^n \exp(-\alpha - \alpha^2\theta) \sum_{k=0}^{n} \frac{a_k}{2k} \frac{(-g/\theta)^k}{(n-k)!}\,, \tag{3.19}$$

$$a_k = \sum_{l=0}^{k} \frac{\Gamma(2l+1/2)}{l!} \frac{\Gamma[2(k-l)+1/2]}{(k-l)!}\,. \tag{3.20}$$

The optimization of the first nontrivial approximation yields

$$\theta = \frac{3}{4}g\,. \tag{3.21}$$

The behavior of the Nth order partial sum of the series (3.9) and (3.18), normalized to the exact value $Z[g]$, is represented in Table 1. In Fig. 1 the N dependence of the quantity $Z^{(N)}[g=1]$ in the case of the harmonic variational procedure is plotted. We can see that for $g \geq 1$, when the harmonic variational procedure is performed, even first terms of the VPT series become sensitive to its asymptotic nature (i.e. the partial sum "beats" emerge). For g larger than that shown in Table 1 the situation becomes still more complicated. A relatively stable result for the series (3.9) occurs as $g < 1$. Comparison of the results given by the ordinary perturbation theory (3.4) and by the series (3.9) for $g = 0.1$ is plotted in Fig. 2. As regards the anharmonic variational procedure [Eqs. (3.18)–(3.21)], we derive a stable result in the whole region of the coupling constant.

Introducing $t = 2\theta/g$ we rewrite (3.19) in the form

$$Z_n[g,\theta] = \sqrt{\frac{2}{gt}} \int_0^{\infty} d\alpha\, \alpha^{2n} \exp\left(-\alpha^2 - \alpha\sqrt{\frac{2}{gt}}\right) \sum_{k=0}^{n} \frac{(-2/t)^k}{(2k)!(n-k)!} a_k\,, \tag{3.22}$$

It is interesting that the expressions (3.22) allow immediate determination of the functional dependence of $Z[g]$ when $g \to \infty$. Indeed, in the VPT N order for $g \to \infty$ we get

$$Z^{(N)} = \frac{A^{(N)}}{\sqrt{g}}, \tag{3.23}$$

Table 1. The behavior of Nth partial sums for the harmonic and unharmonic variational procedures.

	$g=1$		$g=10$		$g=100$	
N	$Z^{(N)}_{\rm har}/Z_{\rm ex}$	$Z^{(N)}_{\rm anhar}/Z_{\rm ex}$	$Z^{(N)}_{\rm har}/Z_{\rm ex}$	$Z^{(N)}_{\rm anhar}/Z_{\rm ex}$	$Z^{(N)}_{\rm har}/Z_{\rm ex}$	$Z^{(N)}_{\rm anhar}/Z_{\rm ex}$
0	0.806	0.992	0.701	0.984	0.658	0.981
1	0.945	0.992	0.891	0.984	0.864	0.981
2	1.034	1.000	1.078	0.999	1.099	0.999
3	0.905	1.000	0.650	0.999	0.480	0.999
4	1.310	1.000	2.735	1.000	3.960	1.000
5	−0.265	1.000	−9.909	1.000	−20.41	1.000
6	7.253	1.000	84.24	1.000	189.1	1.000
7	−59.68	1.000	−1223.	1.000	−3170.	1.000
8	−26.51	1.000	−212.7	1.000	−172.4	1.000
9	−34.12	1.000	−574.0	1.000	−1410.	1.000
10	−23.80	1.000	190.7	1.000	1614.	1.000

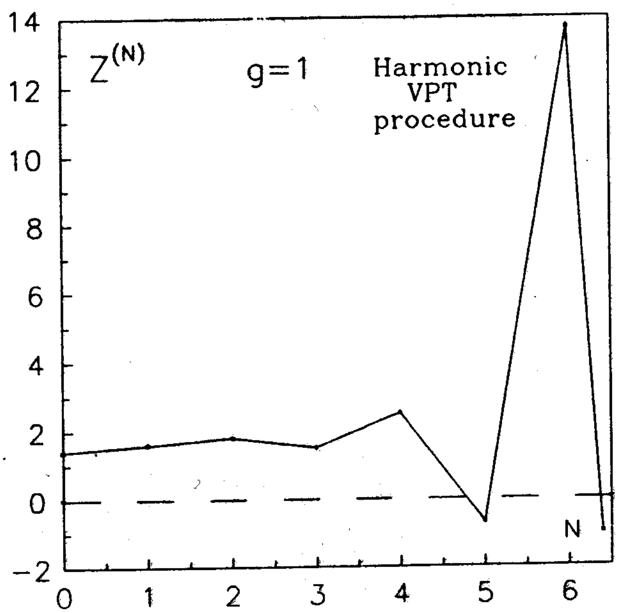

Fig. 1. N dependence of the Nth partial sums $Z^{(N)}[g]$ in the case of the harmonic VPT procedure for $g = 1$.

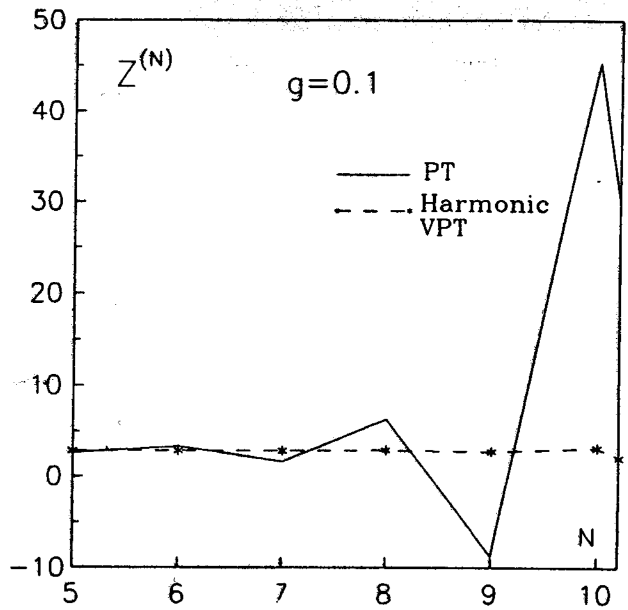

Fig. 2. The behavior of the Nth partial sums $Z^{(N)}[g]$ for the cases of perturbation theory and the harmonic variational procedure.

where

$$A^{(N)} = \sqrt{\frac{2}{t}}\,\Gamma\left(N + \frac{3}{2}\right) \sum_{n=0}^{N} \frac{(-2/t)^k}{(2k+1)!(N-k)!}\, a_k \,. \tag{3.24}$$

Optimization of the first nontrivial order (the condition $\partial Z^{(1)}/\partial t = 0$) gives $t_{\rm opt} = 3/2$ and thus $A^{(1)} = 3.212$, whereas the exact value $A = \Gamma^2(1/4)/4 = 3.286$. The series (3.18) can easily be verified to be convergent at $t > 1/2$, this being also valid for any positive g. An analog of the Sobolev inequality in the case under consideration is the relation

$$\frac{S_4[x,y]}{S_2^2[x,y]} \leq 1\,, \tag{3.25}$$

from which it follows that for $t > 1$ the VPT series (3.18) is of positive sign, and for $t = 1$ the regime is changed, and for $1/2 < t < 1$ the series becomes the Leibniz series. Note that the value $t = 1$ of the variational parameter at which the alternating series turns into a series of fixed sign corresponds to value t, found from the criterion of asymptotic optimization of a VPT series according to which the contribution of next order terms is minimized.[17–20]

For the Leibniz series regime the exact value $Z_{\rm exact}[g]$ obeys the following estimate of upper and lower bounds:

$$Z^{2N+1} < Z_{\rm exact} < Z^{2N}\,, \tag{3.26}$$

where Z^{2N+1} and Z^{2N} are, respectively, odd and even partial sums of the VPT series. In Fig. 3 we draw the corridor of the estimates of upper and lower bounds defined by the functions $Z^{(N)}/Z_{\rm exact}$ for $N = 0, 1, 2, 3$ and the parameter $t = 1$. It is seen that even the first partial sums provide an acceptable accuracy in the whole range of variation of the coupling constant.

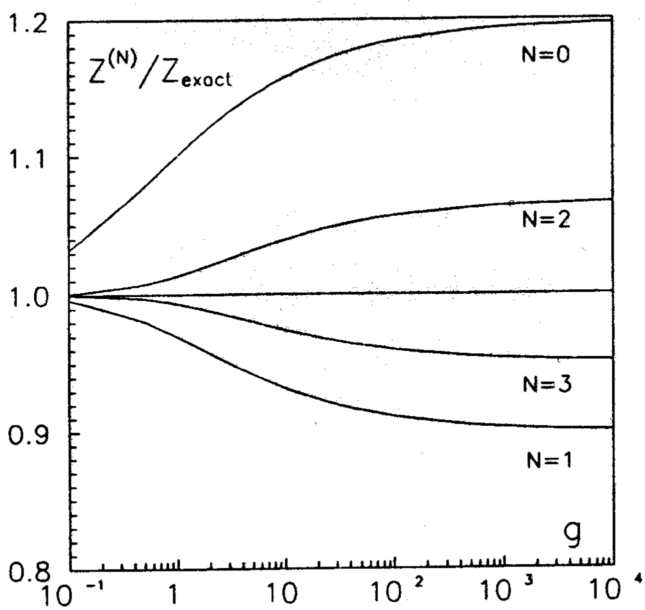

Fig. 3. A corridor of estimates of upper and lower bounds determined by the functions $Z^{(N)}[g]/Z_{\rm ex}[g]$.

It is interesting to generalize our results to the φ^{2k} interaction. Now we consider the VPT method for the simple numerical integral which can be considered as a zero-dimensional analog of the two-component scalar model in the field theory with $\lambda\varphi^{2k}$ interaction:

$$Z[g] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \exp\left[-\left(S_0 + gS_1\right)\right], \tag{3.27}$$

where $S_0 = x_1^2 + x_2^2 = \mathbf{x}^2$ is an analog of the free action and $S_1 = x_1^{2k} + x_2^{2k}$ is the action of interaction. If we rewrite the total action in the form $S = S_0 + \theta {S_0}^k + gS_1 - \theta {S_0}^k$, we construct a new expansion of $Z[g]$:

$$Z[g] = \sum_{n=0}^{\infty} Z_n[g, \theta], \tag{3.28}$$

$$Z_n[g, \theta] = \frac{1}{n!} \int d\mathbf{x}(\theta S_0^k - gS_1)^n \exp[-(S_0 + \theta {S_0}^k)], \tag{3.29}$$

where θ is the arbitrary parameter so far. Since $Z[g]$ is independent of the parameter θ, we can choose its value so that a finite number of series terms in (3.28) would provide the best approximation value, (3.27). One can propose different versions of the optimization procedure. First, one can determine the variational parameter θ from the minimality requirement for the absolute value of the sum of the last series terms is VPT being minimal:

$$\min \left| \sum_{i=k}^{N} Z_i[g,\theta] \right| , \qquad 1 \le k \le N .$$

Second, since the exact value of $Z[g]$ is independent of the parameter θ, the optimization procedure can be

$$\frac{\partial Z^{(N)}[g,\theta]}{\partial \theta} = 0 .$$

And third, one can require the contribution of the distant terms in the VPT series to be minimal (the so-called asymptotic optimization). The asymptotic behavior of the coefficients $Z_n[g,\theta]$ with large n is

$$Z_n[g,\theta] \underset{n\to\infty}{\sim} \frac{\sqrt{2\pi}}{n\theta^{1/k}} \sqrt{\frac{t-1}{k^3(k-1)}} \left(\frac{t-1}{t}\right)^n \exp\left(-\frac{\sqrt{n}}{\theta^{1/k}}\right) , \tag{3.30}$$

where

$$t = \frac{2^{k-1}\theta}{g} . \tag{3.31}$$

Hence, we can see that the VPT series has the finite region of convergence for $g \le 2^{k-1}\theta$. In the case of $t = 1$, which corresponds to the asymptotic optimization, the VPT series becomes an alternating sign convergent series of Leibniz, and there is a possibility of carrying out upper and lower bilateral estimates of the sum of series proceeding from the first terms.

We obtain the next expression for the VPT series terms:

$$Z_n[g,\theta] = \int_0^\infty d\alpha\, \alpha^{kn} \exp(-\alpha - \theta\alpha^k) \sum_{j=0}^{n} \frac{\theta^{n-j}}{(n-j)!\Gamma(kj+1)} Z_j[g] , \tag{3.32}$$

where

$$Z_j[g] = \frac{1}{j!} \int d\mathbf{x}\, \left[-g(x_1^{2k} + x_2^{2k})\right]^j \exp(-\mathbf{x}^2) . \tag{3.33}$$

Here $Z_j[g]$ is the ordinary coefficient of perturbation theory. Then, in the first nontrivial order we find that

$$Z_0[g,\theta] = \pi \int_0^\infty d\alpha \exp(-\alpha - \alpha^k\theta) , \tag{3.34}$$

$$Z_1[g,\theta] = \pi \int_0^\infty d\alpha\, \alpha^k\theta \exp(-\alpha - \alpha^k\theta) \left[1 - \frac{g}{k!\theta} \frac{(2k-1)!!}{2^{k-1}}\right] . \tag{3.35}$$

Using the optimization procedure in conformity with versions 1 and 2, we obtain

$$t = t_1 = t_2 = \frac{(2k-1)!!}{k!}. \tag{3.36}$$

The results for different k are shown in Table 2. Note that the interval $(Z^{(1)}[g,\theta], Z^{(0)}[g,\theta])$ ($\theta = \theta_3$ is the variational parameter for asymptotic optimization) determines the upper and lower estimates for $Z[g]$, which corresponds to the Leibniz series.

Table 2. The results of calculation of $Z[g,\theta]$ for different k in the first order VPT, where $Z^{(1)}[g,\theta] = Z_0[g,\theta] + Z_1[g,\theta]$, $Z^{(0)}[g,\theta] = Z_0[g,\theta]$ and $\theta_1, \theta_2, \theta_3$ are the variational parameters for different optimization procedures.

k	g	Z^{exact}	$Z^{(1)}[g,\theta]$ $\theta=\theta_1=\theta_2$	$Z^{(1)}[g,\theta]$ $\theta=\theta_3$	$Z^{(1)}[g,\theta]$ $\theta=\theta_3$
2	0.1	2.8025	2.7994	2.7902	2.8929
	1.0	1.8726	1.8585	1.8153	2.0599
	10.0	0.8500	0.8369	0.7920	0.9841
	100.0	0.3076	0.3016	0.2801	0.3642
	10000.0	0.0326	0.0320	0.0294	0.0391
3	0.1	2.7046	2.6881	2.6222	2.8813
	1.0	1.9919	1.9556	1.7716	2.2763
	10.0	1.2138	1.1769	0.9642	1.4680
	100.0	0.6496	0.6247	0.4711	0.8126
	10000.0	0.1552	0.1482	0.1033	0.1992
4	0.1	2.6220	2.5873	2.3596	2.8620
	1.0	2.0535	1.9974	1.5435	2.3914
	10.0	1.4391	1.3806	0.8439	1.7692
	100.0	0.9276	0.8806	0.4178	1.1837
	10000.0	0.3334	0.3131	0.0986	0.4417

In the strong coupling limit for $g \to \infty$ we find from (3.32) the expression for $Z[g]$ in the Nth order of VPT:

$$Z^{(N)}[g] = x^{1/k} \frac{\Gamma(N+1+\frac{1}{k})}{g^{1/k}} \sum_{j=0}^{N} \frac{(-x)^j}{(kj+1)!(N-j)!} a_j , \tag{3.37}$$

where

$$x = \frac{2^{k-1}}{t}, \qquad a_j = \sum_{m=0}^{j} \frac{\Gamma(km+\frac{1}{2})\Gamma[k(j-m)+\frac{1}{2}]}{m!(j-m)!}. \tag{3.38}$$

The same results for $Z[g]$ in the first order of VPT ($g \to \infty$) for different k are shown in Table 3.

Table 3. The value $Z^{(1)}[g]$ in the first order of VPT ($g \to \infty$).

k	Z^{exact}	$Z^{(1)}[g]$	Error (%)
2	$3.28626g^{-1/2}$	$3.21488g^{-1/2}$	2.172
3	$3.44265g^{-1/3}$	$3.28119g^{-1/3}$	4.689
4	$3.54752g^{-1/4}$	$3.31130g^{-1/4}$	6.658

4. Variational Perturbation Theory for an Anharmonic Oscillator

4.1. *Green functions*

Let us consider a quantum-mechanical anharmonic oscillator (AO) case as an example of exploiting the VPT method. The AO from the point of view of the continual integral formalism is a one-dimensional φ^4 model. The Euclidean action looks as follows:

$$S[\varphi] = S_0[\varphi] + \frac{m^2}{2} S_2[\varphi] + g S_4[\varphi] , \tag{4.1}$$

where

$$S_0[\varphi] = \frac{1}{2} \int dx (\partial\varphi)^2 , \tag{4.2}$$

$$S_2[\varphi] = \int dx \varphi^2 , \tag{4.3}$$

$$S_4[\varphi] = \int dx \varphi^4 . \tag{4.4}$$

The Green functions are expressed in terms of the functional integral:

$$G_{2\nu} = \int D\varphi \{\varphi^{2\nu}\} \exp(-S[\varphi]) , \tag{4.5}$$

where $\{\varphi^{2\nu}\}$ denotes the product of fields $\varphi(x_1) \cdots \varphi(x_n)$. We pass to dimensionless variables: $\varphi \to g^{-1/6}\varphi, x \to g^{-1/3}x$. Then the functional of action reads

$$S[\varphi] = S_0[\varphi] + \frac{\omega^2}{2} S_2[\varphi] + S_4[\varphi] , \tag{4.6}$$

where

$$\omega^2 = m^2 g^{-2/3} . \tag{4.7}$$

The dimensionless Green functions $G_{2\nu}$ will be represented through (4.5) with the action (4.6).

Like the ordinary perturbation theory, the VPT method uses only Gaussian quadratures:

$$\int D\varphi \exp\left[-\left(\frac{1}{2}\langle\varphi K\varphi\rangle + \langle\varphi J\rangle\right)\right]$$
$$= \left(\det \frac{K}{-\partial^2 + m^2}\right)^{-1/2} \exp\left(\frac{1}{2}\langle JK^{-1}J\rangle\right). \quad (4.8)$$

At the same time any possible polynomial in the integrand of (4.8) can be obtained by the corresponding number of differentiations of the exponential with respect to the source $J(x)$. The variational addition to the action will be constructed on the basis of the functional

$$A[\varphi] = \theta S_0[\varphi] + \frac{\chi}{2} S_2[\varphi], \quad (4.9)$$

where θ and χ are variational type parameters.

We shall first consider the *harmonic variational procedure*. In this case the action functional splits as follows:

$$S[\varphi] = S_0^h[\varphi] + S_{\rm int}^h[\varphi], \quad (4.10)$$

where

$$S_0^h[\varphi] = S_0[\varphi] + \frac{\omega^2}{2} S_2[\varphi] + A[\varphi], \quad (4.11)$$

$$S_{\rm int}^h[\varphi] = S_{\rm int}[\varphi] - A[\varphi]. \quad (4.12)$$

The expansion in the VPT series reads

$$G_{2\nu} = \sum G_{2\nu,n}(\theta, \chi), \quad (4.13)$$

$$G_{2\nu,n}[\theta, \chi] = \frac{(-1)^n}{n!} \int D\varphi \{\varphi^{2\nu}\} (S_{\rm int}^h[\varphi])^n \exp(-S_0^h[\varphi]). \quad (4.14)$$

Obviously the functional integral (4.14) is Gaussian. It is convenient to use the ordinary coefficients of a perturbative series when it is calculated. To do this, let us rewrite (4.14) as

$$G_{2\nu,n} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \left(-\frac{\partial}{\partial\alpha}\right)^{n-k}$$
$$\times \int D\varphi \{\varphi^{2\nu}\} (-S_4)^k \exp\left[-\left(S_0 + \frac{\omega^2}{2} S_2 + \alpha A\right)\right], \quad (4.15)$$

where the parameter α is to be set to 1 after differentiation. Having in mind the intermediate dimensional regularization and making the change $\varphi \to \varphi/\sqrt{1+\alpha\theta}$, we obtain

$$G_{2\nu,n} = \sum_{k=0}^{n} \frac{1}{(n-k)!} \left(-\frac{\partial}{\partial\alpha}\right)^{n-k} \frac{g_{2\nu,k}(z^2)}{(1+\alpha\theta)^{\nu+2k}}, \quad (4.16)$$

where

$$g_{2\nu,k}(z^2) = \frac{1}{k!}\int D\varphi\{\varphi^{2\nu}\}(-S_4)^k \exp\left[-\left(S_0 + \frac{z^2}{2}S_2\right)\right] \tag{4.17}$$

are the ordinary perturbative expansion coefficients for the Green functions (4.5). To calculate them, the standard Feynman diagrams, for example, can be used. The quantity z^2 in (4.17) looks as follows:

$$z^2 = \frac{\omega^2 + \alpha\chi}{1 + \alpha\theta}\,. \tag{4.18}$$

The properties of the series (4.13) are determined by the asymptotics of the functional integral

$$\frac{(-1)^n}{n!}\int D\varphi(S_4[\varphi] - A[\varphi])^n \exp\left\{-\left(S_0[\varphi] + \frac{\omega^2}{2}S_2[\varphi] + A[\varphi]\right)\right\} \tag{4.19}$$

at large n.

It is easy to see that the investigation of the asymptotic behavior of the expression (4.19) in the leading order in n is, in fact, equivalent to finding the ordinary perturbative series coefficients. The series (4.13) turns out to be asymptotic like the ordinary one. Actually, its behavior may be influenced by the θ and χ parameters, to attain the greater stability of results as compared with standard perturbation theory. However, one is compelled to remain in the region of the weak coupling constant, mainly, as it turns out to be impossible for arbitrary values of the dimensionless coupling constant g/m^3 to gain, within the harmonic variational procedure, the stable results with respect to corrections. The latter is explained by the fact that at large n a sensible contribution to (4.19) comes from such field configurations at which the quantity $|\varphi(x)|$ is large. In this case the compensation by the harmonic addition $A[\varphi]$ of large $S_{\rm int}[\varphi]$ containing the fourth power of the field proves to be not sufficient.

Under the *anharmonic variational procedure* the action functional is represented as follows:

$$S[\varphi] = S_0^a[\varphi] + S_{\rm int}^a[\varphi]\,, \tag{4.20}$$

$$S_0^a[\varphi] = S_0[\varphi] + \frac{\omega^2}{2}S_2[\varphi] + A^2[\varphi]\,, \tag{4.21}$$

$$S_{\rm int}^a[\varphi] = S_{\rm int}[\varphi] - A^2[\varphi]\,. \tag{4.22}$$

Now, the field power in the compensating addition is the same as in the interaction action $S_{\rm int}[\varphi]$. Keeping in mind that we have also the variational parameters at our disposal, we may anticipate that the convergence of the VPT series will be good enough.

As a concrete example, we shall consider the ground state energy for the anharmonic oscillator connected with the four-point Green function $G_4(0,0,0,0)$.

4.2. *Ground state energy*

We will proceed from the partition function represented by the path integral

$$\exp(-TE) = \int D\varphi \exp(-S[\varphi]), \tag{4.23}$$

where the integration in (4.23) runs over $\varphi(t)$ with the condition $\varphi(-T/2) = \varphi(T/2)$, and the functional of action is given by (4.1) but integration runs over t from $-T/2$ to $T/2$.

The ground state energy E_0 follows from (4.23) in the limit $T \to \infty$. It is convenient to pass from the functional integrals typical of statistical mechanics to the functional integrals of the Euclidean field theory. To this end consider the quantity dE_0/dg, which is expressed in terms of the four-point Euclidean Green function. So, passing to the dimensionless variables from (4.23) we obtain

$$\frac{dE_0}{dg} = g^{-2/3} G_4(0), \tag{4.24}$$

where

$$G_4(0) = N^{-1} \int D\varphi \varphi^4(0) \exp\left[-\left(S_0 + \frac{\omega^2}{2}\tilde{S}_2 + S_4\right)\right], \tag{4.25}$$

$$N = \int D\varphi \exp[-(S_0 + \omega^2 \tilde{S} + S_I)]. \tag{4.26}$$

In what follows we will be interested in the strong coupling limit

$$\frac{g}{m^3} \to \infty \quad (\omega^2 \to 0).$$

Defining the functional $A[\varphi]$ [see Eq. (4.9)], we rewrite (4.25) in the form of a VPT series:

$$G_4(0) = N^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int D\varphi \varphi^4(0)(A^2 - S_I)^n \exp[-(S_0 + \omega^2 \tilde{S} + S_I)]. \tag{4.27}$$

Next we will find the asymptotics of the path integral

$$\int D\varphi (A^2 - S_I)^n \exp[-(S_0 - A^2)] \tag{4.28}$$

at large n. Changing the variables, $\varphi \to n^{1/4}\varphi$, we represent (4.28) as follows:

$$n^n \int D\varphi \exp(-nS_{\text{eff}}[\varphi] - n^{1/2} S_0[\varphi]), \tag{4.29}$$

where

$$S_{\text{eff}} = A^2 - \ln(A^2 - S_I). \tag{4.30}$$

The functional integral (4.29) contains a large parameter n and can be calculated by the functional saddle point method. [1-3,21] The saddle point function φ_0 is determined from the condition $\delta S_{\rm eff}/\delta\varphi = 0$, which leads to the equation

$$-\ddot{\varphi}_0 + a\varphi_0 - b\varphi_0^3 = 0\,, \tag{4.31}$$

where

$$a = \frac{\chi}{\theta}\,, \quad b = \{\theta A[\varphi_0](1 - D[\varphi_0])\}^{-1}\,, \quad D[\varphi_0] = A^2[\varphi_0] - S_I[\varphi_0]\,. \tag{4.32}$$

The solution to (4.31) decreasing at infinity, corresponding to a finite action, and given a major contribution to the functional integral (4.29) at large n, is of the form

$$\varphi_0 = \pm\sqrt{\frac{2a}{b}}\,[\cosh\sqrt{a}(t - t_0)]^{-1}\,, \tag{4.33}$$

where t_0 is an arbitrary parameter showing the theory to be translationality-invariant. It is not difficult to compute the functional (4.30) for the function (4.33):

$$S_{\rm eff}[\varphi_0] = 1 - \ln D[\varphi_0]\,, \tag{4.34}$$

where

$$D[\varphi_0] = 1 - \frac{3}{4}(\theta\chi^3)^{-1/2}\,. \tag{4.35}$$

Here we may take advantage of version 3 (see Ref. 18) of the optimization procedure requiring the contribution of higher order terms being minimal, which means the condition $D[\varphi_0] = 0$. Therefore the variational parameters θ and χ are related as follows:

$$\chi = \left(\frac{9}{16}\theta\right)^{1/3}. \tag{4.36}$$

The remaining variational parameter θ is fixed on the basis of a finite number of the VPT expansion terms; we now will restrict our consideration to the first order.

Further transformations with (4.27) will proceed as follows. Since any power of A^2 in front of the exponential of (4.27) can be obtained by differentiation, we do not introduce new diagrams but those of conventional perturbation theory. Performing intermediate dimensional regularization and reducing the functional integral with the use of (2.7) to the Gaussian form, we get

$$G_4(0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n-m)!}\left(\frac{d}{d\alpha}\right)^{n-m}$$
$$\times\,\langle g_m(z^2)(1 + iu\theta\sqrt{1-\alpha})^{-2-2m}\rangle\,, \tag{4.37}$$

where

$$g_m(z^2) = \frac{(-1)^m}{m!} \int D\varphi\varphi^4(0) \exp\left[-\left(S_0 + \frac{z^2}{2} S_2\right)\right],$$
$$z^2 = \omega^2 + iu\chi\sqrt{1-\alpha}\left(1 + iu\theta\sqrt{1-\alpha}\right)^{-1}, \tag{4.38}$$

Upon differentiation of (4.37) with respect to α we should put $\alpha = 0$. The functions $g_m(z^2)$ are standard expansion coefficients of $G_4(0)$ into a perturbative series and they can be determined by the standard diagram technique. From (4.37) it is seen that the Nth VPT order requires only those diagrams that are present in the Nth order of conventional perturbation theory.

In this case it is not difficult to connect the expressions (4.38) with the know expansion coefficients A_n of the ground state energy E_0 in the perturbative series

$$E_0(g) = \frac{m}{2} + m \sum_{n=1}^{\infty} A_n \left(\frac{g}{m^3}\right), \tag{4.39}$$

and this connection looks as follows:

$$g_m(z^2) = \frac{(1+m)A_{1+m}}{z^{2+3m}}. \tag{4.40}$$

The numerical value of the coefficients A_n may be taken from Ref. 22 (the first VPT order requires the values $A_1 = 3/4$ and $A_2 = -21/8$).

Then making use of the expression

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int d\alpha\alpha^{\nu-1} \exp(-a\alpha)$$

we obtain

$$G_4(0) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(1+m)A_{1+m}}{(n-m)!} \left[\Gamma\left(1 + \frac{m}{2}\right)\Gamma\left(1 + \frac{3m}{2}\right)\right]^{-1}$$
$$\times \left(\frac{d}{d\alpha}\right)^{n-m} F_m(\theta, \chi, \alpha), \tag{4.41}$$

where

$$F_m(\theta, \chi, \alpha) = \int_0^{\infty} dx\, x^{m/2} \exp(-x) \int_0^{\infty} dy\, y^{3m/2}$$
$$\times \exp[-\omega^2 y - (1-\alpha)\cdot(x\theta + y\chi)]. \tag{4.42}$$

Note that we are interested in the strong coupling limit and therefore we set $\omega^2 = 0$ in (4.42). However, it is to be noticed that by expanding $\exp(-\omega^2 y)$ in powers of ω^2 we can determine corrections to the main contribution.

From (4.41) and (4.42) we obtain, for the ground state energy in the Nth VPT order in the strong coupling limit,

$$E_0^N = 3g^{1/3}\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{(1+m)A_{1+m}}{(n-m)!}\left(\frac{16}{9}\theta\right)^{1/3+m/2} \times\left[\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(1+\frac{3m}{2}\right)\right]^{-1}R_{n,m}(\theta)\,, \tag{4.43}$$

where

$$R_{n,m}(\theta) = \int_0^\infty dx\, x^{m/2}\exp(-x) \times\int_0^\infty dy\, y^{3m/2}(\theta x+y)^{2(n-m)}\exp[-(\theta x+y)^2]\,. \tag{4.44}$$

The optimal value of the parameter θ in both the first and the second version (see Ref. 18), $\theta_{1,2} \ll 1$; therefore, in the first VPT order we get from (4.43) and (4.44)

$$E_0^{(1)} = g^{1/3}(\varepsilon_0+\varepsilon_1)\,, \tag{4.45}$$

where

$$\varepsilon_0 = \frac{3}{2}A_1\sqrt{\pi}x^2\,, \tag{4.46}$$

$$\varepsilon_1 = \frac{3}{4}A_1\sqrt{\pi}x^2 + \frac{4\Gamma(5/4)}{\sqrt{\pi}}A_2x^5\,, \tag{4.47}$$

$$x = \left(\frac{16}{9}\theta\right)^{1/6}. \tag{4.48}$$

Upon optimization of version 1 we obtain $x_1 = 0.5705$ and the ground state energy

$$E_0^{(1)}(x_1) = 0.649g^{1/3}\,, \tag{4.49}$$

and upon optimization of version 2 we find that $x_2 = 0.6062$ and

$$E_0^{(1)}(x_2) = 0.660g^{1/3}\,. \tag{4.50}$$

It is easy to verify that the second VPT order contributes only several percent.

We have to compare the obtained results with the exact value:[23]

$$E_{\rm exact} = 0.668g^{1/3}\,. \tag{4.51}$$

The behavior of the relative energies $E_0^{(N)}/E_{\rm exact}$ is plotted in Fig. 4. The extremal values correspond to the optimal θ. The stability of the VPT series for values of θ close to the optimal one is indicated in Table 4.

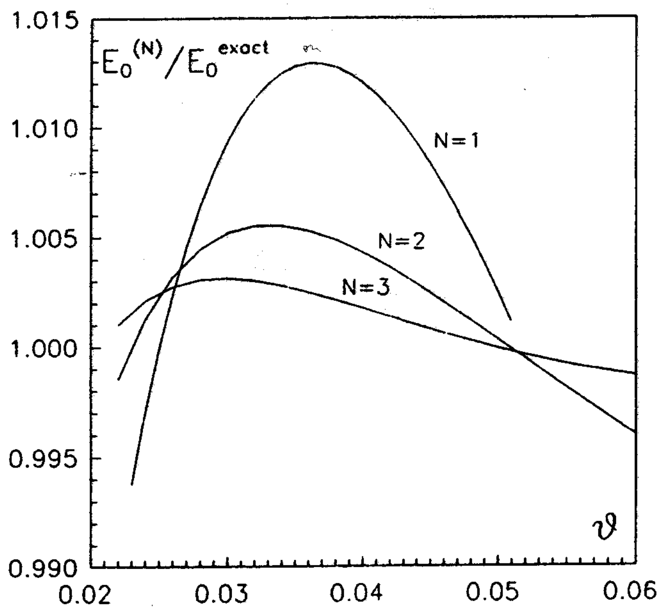

Fig. 4. The behavior of the functions $E_0^{(N)}/E_{\rm ex}$ for $N = 1, 2, 3$ versus the parameter θ.

Table 4. The behavior of $E_0^{(N)}/E_{\rm ex}$ in dependence of N for various values of θ.

	$E_0^{(N)}/E_{\rm ex}$			
N	$\theta = 0.020$	$\theta = 0.028$	$\theta = 0.032$	$\theta = 0.036$
0	0.956	1.063	1.107	1.144
1	0.981	1.006	1.011	1.013
2	0.995	1.005	1.006	1.006
3	1.000	1.004	1.004	1.004
4	1.001	1.003	1.002	1.002
5	1.001	1.002	1.001	1.001
6	1.000	1.000	1.000	1.000

4.3. *Propagator*

We will here also calculate the mass parameter μ^2 connected with the two-point Green function, $\mu^{-2} = G_2(p = 0)$, where

$$G_2(p=0) = \int dt \int D\varphi\, \varphi\Big(\frac{t}{2}\Big)\varphi\Big(-\frac{t}{2}\Big) \exp(-S[\varphi])\,. \tag{4.52}$$

Numerically, this parameter was computed in Ref. 23 in the strong coupling limit:

$$\mu^2_{\rm exact} = 3.009 g^{2/3}\,. \tag{4.53}$$

The VPT series for the function $G_2(0)$ is

$$G_2(0) = g^{-2/3}\,\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\Gamma(n+1/2-m/4)}{(n-m)!}\cdot\frac{B_m}{\Gamma(1+3m/2)}\,x^{2+3m}\,, \tag{4.54}$$

where B_m are dimensionless coefficients of the standard perturbation theory. For the considered first nontrivial VPT order we need the two values $B_0 = 1$ and $B_1 = -6$. In the first VPT order we get from (4.54)

$$G_2^{(1)} = g^{-2/3}(G_{20} + G_{21})\,,$$

where

$$G_{20} = \frac{\sqrt{\pi}}{2}\,x^2\,, \tag{4.55}$$

$$G_{21} = \frac{\sqrt{\pi}}{4}\,x^2 - 4\cdot\Gamma\left(\frac{5}{4}\right)\cdot x^5\,. \tag{4.56}$$

Upon optimization of version 1 ($G_{21} = 0$) we find that

$$\mu^2 = 3.128 g^{2/3}\,, \tag{4.57}$$

and upon optimization of version 2 ($\partial G_2^{(1)}/\partial x = 0$)

$$\mu^2 = 3.078 g^{2/3}\,. \tag{4.58}$$

We can compare these results with the exact value (3.33) and get satisfaction.

With the use of the propagator $G_2(p)$ we may compute the vacuum energy by the relation[24]

$$E_0 = \frac{3}{4}\int\frac{dp}{2\pi}\left[1 - G_{20}^{-1}(p)\cdot G_2(p)\right]\,, \tag{4.59}$$

where $G_{20}(p)$ is the free propagator.

It is of interest to employ a more simple version of VPT with one variational parameter, say, χ, and θ is put zero. Just this one-parameter VPT will be used in the next section for constructing the effective potential. In the first order for the two-point Green function we obtain

$$G_2^{(1)}(p) = \int_{-\infty}^{\infty}\frac{du}{2\sqrt{\pi}}\exp\left(-\frac{u^2}{4}\right)\left\{\Delta_0(p,z^2) + \left[\frac{d}{d\alpha}\,\Delta_0(p,z^2) + \Delta_1(p,z^2)\right]\right\}, \tag{4.60}$$

where

$$\Delta_0(p,z^2) = (p^2+z^2)^{-1}\,,$$

$$\Delta_1(p,z^2) = -\frac{6g}{z}\,(p^2+z^2)^{-2}\,,$$

$$z^2 = \omega^2 + iu\chi\sqrt{1-\alpha}\,.$$

Inserting (4.60) into (4.59) we get

$$E_0^{(1)}(\chi) = \frac{3}{8}\left\{\frac{\Gamma(3/4)}{\sqrt{\pi}}\sqrt{\chi} + \left[-\frac{1}{4}\frac{\Gamma(3/4)}{\sqrt{\pi}} + \frac{3}{2}\frac{g}{\chi}\sqrt{\pi}\right]\right\}, \tag{4.61}$$

and upon optimization of version 1 [the expression in brackets in (4.61) is put to zero] we find that

$$E_0^{(1)}(\chi_1) = 0.645 g^{1/3}, \tag{4.62}$$

whereas version 2 gives

$$E_0^{(1)}(\chi_2) = 0.634 g^{1/3}. \tag{4.63}$$

And, finally, we shall estimate the energy of the first excited level, E_1; to do this, we define the energy shift

$$\mu_1 = E_1 - E_0. \tag{4.64}$$

Then, using the spectral representation for the propagator

$$G_2(p) = 2\sum_{n=0}^{\infty} \frac{\mu_n}{p^2 + \mu_n^2} \,|\langle 0|\hat{x}|n\rangle|^2, \tag{4.65}$$

where matrix elements of the coordinate operator are calculated for eigenstates of the total Hamiltonian, we arrive at the following estimate for the energy shift (4.64):

$$\mu_1 \le \mu_1^{(+)}, \qquad \mu_1^{(+)} = \frac{2G_2(t=0)}{G_2(p=0)}. \tag{4.66}$$

By analogy with the sum rules,[25] we may expect a sufficiently rapid saturation of the spectral representation (4.65), which brings μ_1 and $\mu_1^{(+)}$ closer to each other. In the first order of the one-parameter VPT we get

$$\mu_1^{(+)} = 1.763 g^{1/3}, \tag{4.67}$$

whereas the exact value is[23]

$$\mu_1^{\text{exact}} = 1.726 g^{1/3}. \tag{4.68}$$

4.4. *Effective potential*

Consider the generating functional for the Green function (we employ the pseudo-Euclidean signature in the n-dimensional space, keeping in mind applications in field theory)

$$W[J] = \int D\varphi \exp\left\{i\left(S[\varphi] + \langle J\varphi\rangle\right)\right\}, \tag{4.69}$$

where

$$\langle J\varphi\rangle = \int dt\, J(t)\cdot\varphi(t), \tag{4.70}$$

$$S[\varphi] = S_0 - m^2\tilde{S} - gS_I. \tag{4.71}$$

The effective potential is usually constructed in the quasiclassical approximation based on the expansion in powers of the number of loops.[26] In our case this method gives the one-loop potential

$$V_{\text{eff}}^{1\,\text{loop}} = \frac{1}{2}\sqrt{m^2 + 12\,g\,\varphi_0^2}\,, \tag{4.72}$$

which is completely unfit for the description of the nonperturbative region.

In this section we will compute the effective potential by the VPT method. To this end, we introduce variational parameter a^2, rewriting the action in the form

$$S[\varphi] = \left(S_0 - m^2\tilde{S} - \frac{a^2}{T}\tilde{S}^2\right) - \left(gS_I - \frac{a^2}{T}\tilde{S}^2\right). \tag{4.73}$$

The effective potential is obtained from the effective action when the field configurations are constant, $\varphi_0 = \text{const}$, and in this case the variational parameter introduced in the form a^2/T will be independent of the "volume" T of x space.

Expanding the exponential of (4.69) in powers of $gS_I - a^2\tilde{S}^2/T$ and using the above-expounded procedure, we get

$$\begin{aligned} W[J] = \exp\left(-i\frac{\pi}{4}\right)T^{1/2}\cdot\int_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}\exp\left(-i\frac{Tv^2}{4}\right) \\ \times\sum_{n=0}^{\infty} i^n \sum_{k=0}^{n}\frac{1}{(n-k)!}\left(i\frac{d}{d\varepsilon}\right)^{n-k}\frac{1}{k!} \\ \times\int D\varphi(-gS_I)^k\exp[i(S_0 - M^2\tilde{S} + \langle J\varphi\rangle)]\,, \end{aligned} \tag{4.74}$$

where

$$M^2 = m^2 + \sqrt{\varepsilon}\cdot a\cdot v\,, \tag{4.75}$$

and upon differentiation with respect to ε we should set $\varepsilon = 1$. Denoting the perturbative expansion coefficients for the functional $W[J]$ by $\omega_k[J, M^2]$:

$$\omega_k[J, M^2] = \frac{(-i\,g)^k}{k!}\left[\int dt\,\frac{\delta^4}{\delta J^4(t)}\right]^k\cdot\exp\left(\frac{i}{2}\langle J\Delta J\rangle\right), \tag{4.76}$$

where

$$\Delta(p) = (p^2 - M^2 + i\,0)^{-1}\,, \tag{4.77}$$

we obtain from (4.74) in the Nth VPT order

$$\begin{aligned} W^{(N)}[J] = \exp\left(-i\frac{\pi}{4}\right)T^{1/2}\cdot\int_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}\exp\left(-i\frac{Tv^2}{4}\right) \\ \times\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\left(\frac{d}{d\varepsilon}\right)^{n-k}\cdot\left(\det\frac{\partial^2 + M^2}{\partial^2 + m^2}\right)^{-1/2}\omega_k[J, M^2]\,. \end{aligned} \tag{4.78}$$

The functional determinant in (4.78) is calculated by the relation $\det(\cdots) = \exp[\mathrm{Sp}\ln(\cdots)]$, and the result is

$$\left(\det\frac{\partial^2+M^2}{\partial^2+m^2}\right)^{-1/2} = \exp\left\{-i\frac{T}{2}[(M^2)^{1/2}-(m^2)^{1/2}]\right\}. \tag{4.79}$$

In the first VPT order we get

$$W^{(1)}[J] = \exp\left(-i\frac{\pi}{4}\right)T^{1/2}\cdot\int_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}\exp\left(-i\frac{Tv^2}{4}\right) \times\left[1+\left(\frac{\omega_1}{\omega_0}-\frac{d}{d\varepsilon}\ln\tilde{\omega}_0\right)\right], \tag{4.80}$$

where

$$S(v) = \frac{1}{2}\frac{J^2}{M^2}+\frac{v^2}{4}-\frac{1}{2}[(M^2)^{1/2}-(m^2)^{1/2}], \tag{4.81}$$

$$\tilde{\omega}_0 = \exp\left(iT\left\{\frac{J^2}{2M^2}-\frac{1}{2}[(M^2)^{1/2}-(m^2)^{1/2}]\right\}\right), \tag{4.82}$$

$$\frac{\omega_1}{\omega_0} = -igT\left[\frac{3}{4}\frac{1}{M^2}+3\frac{J^2}{(M^2)^{5/2}}+\frac{J^4}{(M^2)^4}\right]. \tag{4.83}$$

In the expressions (4.81)–(4.83) we take constant sources, $J = \mathrm{const}$, which is required for constructing the effective potential.

Introducing the generating functional of the connected Green functions

$$Z[J] = (iT)^{-1}\ln W[J], \tag{4.84}$$

we obtain for the effective potential the standard expression

$$V_{\mathrm{eff}}[\varphi_0] = J\varphi_0 - Z[J], \tag{4.85}$$

where J is derived from the equation

$$\varphi_0 = \frac{dZ[J]}{dJ}. \tag{4.86}$$

The intregrand of (4.80) contains a large parameter, T, in the exponential and thus that integral may be computed by the asymptotic method of a stationary phase. Then in the first VPT order in the strong coupling limit ($m^2 = 0$) we get

$$Z^{(1)}[J] = Z_0[J] + Z_1[J], \tag{4.87}$$

$$Z_0[J] = \frac{3}{4}\frac{J^2}{M^2}-\frac{3}{8}(M^2)^{1/2}, \tag{4.88}$$

$$Z_1[J] = \frac{1}{4}\frac{M^2}{J^2}+\frac{1}{8}(M^2)^{1/2}-g\left[\frac{3}{4}\frac{1}{M^2}+3\frac{J^2}{(M^2)^{5/2}}+\frac{J^4}{(M^2)^4}\right], \tag{4.89}$$

where M^2 is a new variational parameter computed by the optimization procedure. The effective potentials obtained from (4.86)–(4.89) and corresponding to the first and second versions of optimization almost coincide with each other. The corresponding graphs are shown in Fig. 5.

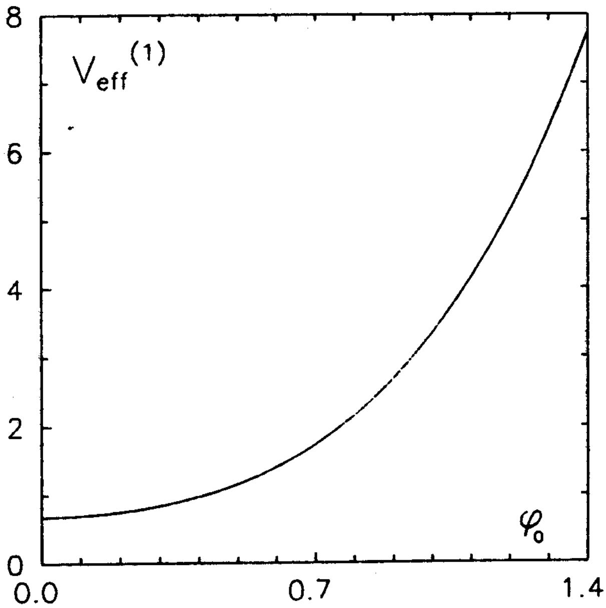

Fig. 5. The effective potential $V^{(1)}_{\rm eff}$.

To compare with numerical results for E_0 and μ^2, we should know the expansion of $V_{\rm eff}(\varphi_0)$ about the extremum. Solving the equation of optimization, $Z_1 = 0$ (version 1), we get

$$M^2 = M_0^2\left[1 + \frac{3}{4}\,\frac{J^2}{(M_0^2)^{3/2}} + O(J^4)\right], \tag{4.90}$$

where

$$M_0^2 = (6g)^{2/3}. \tag{4.91}$$

and then the effective potential reads

$$V^{(1)}_{\rm eff}(\varphi_0) = E_0^{(1)} + \frac{\mu^2_{(1)}}{2}\,\varphi_0^2 + O(\varphi_0^4), \tag{4.92}$$

where

$$E_0^{(1)} = \frac{3}{8}\,(6\,g)^{1/3} = 0.681\cdot g^{1/3}, \tag{4.93}$$

$$\mu^2_{(1)} = M_0^2 = 3.302\cdot g^{2/3}, \tag{4.94}$$

to be compared with the exact values given by (4.51) and (4.53). The second version of optimization leads to the same values for E_0 and μ^2.

As for the behavior of $V_{\rm eff}(\varphi_0)$ for large fields φ_0, it may be found [(4.86)–(4.89)] that

$$V^{(1)}_{\rm eff}(\varphi_0) \sim g\,\varphi_0^4\,, \qquad \varphi_0 \to \infty\,. \tag{4.95}$$

Note that the equality $\mu^2 = M_0^2$ from a field-theoretical point of view means that the variational parameter M^2 is nothing else than the renormalized mass of the field φ. This connection will also hold true for spaces of larger dimensions.

4.5. φ^{2k} *oscillator*

Now we formulate the VPT method for the φ^{2k} anharmonic oscillator (as one-dimensional model of field theory with interaction of $\lambda\varphi^{2k}$). In this case we also consider the $\partial E_0/\partial g$ quantity, which is connected with the $2k$-point Euclidean Green function by the expression

$$\frac{\partial E_0}{\partial g} = g^{-\frac{k}{k+1}} G_{2k}(0)\,, \tag{4.96}$$

where the dimensionless $2k$-point Green function takes the form

$$G_{2k}(0) = N^{-1}\int D\varphi\,\varphi^{2k}(0)\exp\left[-\left(S_0 + \frac{\omega^2}{2}S_2 + S_{2k}\right)\right]\,, \tag{4.97}$$

$$N = \int D\varphi\exp\left[-\left(S_0 + \frac{\omega^2}{2}S_2 + S_{2k}\right)\right]\,, \tag{4.98}$$

$$\omega^2 = m^2 g^{-\frac{2}{k+1}}\,. \tag{4.99}$$

We introduce an auxiliary functional in the form

$$A = \theta\,S_0 + \frac{\nu}{2}S_2\,,$$

with arbitrary parameters θ and ν, for constructing a new expansion, and rewrite the action as

$$S = S_0' + S_{\rm int}'\,,$$

where

$$S_0' = S_0 + \frac{\omega^2}{2}S_2 + A^k\,, \qquad S_{\rm int}' = S_{2k} - A^k\,.$$

We carry out the expansion in powers of the new action of the interaction $S_{\rm int}'$. Then, the VPT series is written as

$$G_{2k}(0) = N^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int D\varphi\,\varphi^{2k}(0)[-S_{\rm int}']^n\exp(-S_0')\,. \tag{4.100}$$

Again, using the fact that the exact value of $G_{2k}(0)$ is independent of θ and ν, we can use any optimization condition. Let us use the asymptotic optimization. For that we must find the asymptotic form of the functional integral

$$\int D\varphi (A^k - S_{2k})^n \exp[-(S_0 + A^k)] \tag{4.101}$$

for large n. We use substitution $\varphi \to n^{1/2k}\varphi$ and the functional method of steepest descent to determine the saddle point function φ_0 which gives the basic contribution to the functional integral (4.101):

$$\varphi_0(t) = \pm\left(\sqrt{\frac{ka}{b}}\{\cosh[(k-1)\sqrt{a}(t-t_0)]\}^{-1}\right)^{\frac{1}{k-1}},$$

$$a = \frac{\nu}{\theta}, \qquad b = \frac{2}{\theta(1-D[\varphi_0])A^{k-1}[\varphi_0]}, \qquad D[\varphi_0] = A^k[\varphi_0] - S_1[\varphi_0],$$

where the parameter t_0 reflects the translational invariance of the theory. The contribution of the distant terms of the VPT series will be minimal when $D[\varphi_0] = 0$. This requirement leads to the relation between the parameters θ and ν:

$$\nu_{\rm opt}[\theta] = \left\{\frac{2^{k-2}}{k}\left[\frac{(k^2-1)\Gamma\left(\frac{2}{k-1}\right)}{k\sqrt{\theta}\Gamma\left(\frac{1}{k-1}\right)}\right]^{k-1}\right\}^{\frac{2}{k+1}}. \tag{4.102}$$

The is a limit: $\lim_{k\to\infty}\nu(\theta) = 1/\theta$. The remaining variational parameter θ will be fixed proceeding from the finite number of expansion terms of VPT. Using again our technical trick and having in mind the intermediate dimensional regularization, as well as introducing differentiation with respect to the parameter α in order to achieve any power of A, we find in the strong coupling limit

$$G_{2k}(0) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{1}{(n-j)!}\left(\frac{d}{d\alpha}\right)^{n-j}\int_{-\infty}^{\infty} du\, F(u)\,\frac{g_j(z^2)}{[1+iu\theta(1-\alpha)^{1/k}]^{k(j+1)}}\Bigg|_{\alpha=0}, \tag{4.103}$$

where

$$z^2 = \frac{\omega^2 + iu\nu(1-\alpha)^{1/k}}{1+iu\theta(1-\alpha)^{1/k}},$$
$$g_j(z^2) = \frac{(-1)^j}{j!}\int D\varphi\,\varphi^{2k}(0)S_1^j \exp\left[-\left(S_0 + \frac{z^2}{2}\tilde{S}\right)\right]. \tag{4.104}$$

$g_j(z^2)$ are ordinary coefficients for perturbation theory series. We can establish their connection with the A_n coefficients for the expansion $E_0(g)$ in the perturbation theory series:

$$E_0(g) = \frac{m}{2} + m\sum_{n=1}^{\infty} A_n\left(\frac{g}{m^{k+1}}\right)^n.$$

The corresponding expression has the form

$$g_j(z^2) = \frac{(1+j)A_{1+j}}{z^{(k+1)j+k}}\,.$$

Then, in the Nth order of our approximation we obtain (if $\nu = \nu_{\rm opt}$ and $\omega^2 = 0$)

$$E_0^{(N)} = (k+1)g^{\frac{1}{k+1}} \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{(1+j)A_{1+j}}{(n-j)!} \left(\frac{1}{\nu}\right)^{\frac{k(j+1)+j}{2}}$$

$$\times \left\{\Gamma\left[\frac{k(j+1)-j}{2}\right]\Gamma\left[\frac{k(j+1)+j}{2}\right]\right\}^{-1} R_{n,j}(\theta)\,, \qquad (4.105)$$

where

$$R_{n,j}(\theta) = \int_0^\infty dx \exp(-x) x^{\frac{k(j+1)-j}{2}-1}$$

$$\times \int_0^\infty dy\, y^{\frac{k(j+1)+j}{2}-1} (\theta x + y)^{k(n-j)} \exp[-(\theta x + y)^k]\,. \qquad (4.106)$$

The calculational results for different k and for various optimization procedures are shown in Table 5. The exact numerical results for E_0 were taken from Ref. 23. The ground state energy $E_0^{(5)}$ ($k = 2$) for different $g(m^2 = 1)$ is shown in Table 6.

Table 5. The ground state energy $E_0^{(1)}[g]$ for different k ($g \to \infty$).

k	θ	$E_0^{\rm exact}[g]$	$E_0^{(1)}[g]$
2	0.027926	$0.668g^{1/3}$	$0.663g^{1/3}$
3	0.038009	$0.680g^{1/4}$	$0.698g^{1/4}$
4	0.040149	$0.704g^{1/5}$	$0.709g^{1/5}$

Table 6. The ground state energy $E_0^{(5)}(k = 2)$ for different g ($m^2 = 1$).

g	$E_0^{\rm exact}$	$E_0^{(5)}$	$\theta_{\rm opt}$	Error (%)
0.1	0.559	0.56407	0.0255	0.906
0.5	0.696	0.69793	0.0246	0.277
1.0	0.804	0.80557	0.0241	0.220
2.0	0.952	0.95334	0.0218	0.141
50	2.499	2.50322	0.0215	0.141
200	3.931	3.93627	0.0215	0.134
1000	6.694	6.70317	0.0215	0.137
8000	13.367	13.38603	0.0229	0.142
20000	18.137	18.16315	0.0229	0.144

We use the ratio connecting the ground energy level with the propagator:

$$E_0 = \frac{k+1}{2k} \int_{-\infty}^{\infty} \frac{dp}{2\pi} [1 - G_0^{-1}(p) G(p)] . \tag{4.107}$$

Assuming that $\theta = 0$ we can rewrite the expression (4.103) in the form

$$G_{2k}(p) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{(n-j)!} \left(\frac{d}{d\alpha} \right)^{n-j} \int_{-\infty}^{\infty} du\, F(u)\, G_j(p, z^2) \bigg|_{\alpha=0} , \tag{4.108}$$
$$z^2 = \omega^2 + iu\nu(1-\alpha)^{1/k} ,$$

where

$$G(p) = \frac{1}{p^2+z^2} - g(k) \frac{1}{z^{k-1}(p^2+z^2)^2} + \cdots ; \tag{4.109}$$

$g(k)$ and the results of calculation are shown in Table 7.

Table 7. The results of calculation of $E_0^{(1)}[g]$ in the case $\theta \doteq 0$.

k	$g(k)$	$E^{\text{exact}}[g]$	$E_0^{(1)}[g]$	Error (%)
2	12	$0.668g^{1/3}$	$0.645g^{1/3}$	3.41
3	22.5	$0.680g^{1/4}$	$0.602g^{1/4}$	11.49
4	105	$0.704g^{1/5}$	$0.602g^{1/5}$	14.45

Finally, we consider the construction of a nonperturbative effective potential using the proposed method. We introduce a variational parameter, by analogy with the previous case, as follows:

$$S[\varphi] = \left(S_0 - \frac{m^2}{2} S_2 - \frac{a^k}{\Omega^{k-1}} {S_2}^k \right) - \left(g S_{2k} - \frac{a^k}{\Omega^{k-1}} {S_2}^k \right) .$$

Further, expanding the integrand exponent in powers of the new interaction action and making some transformations (in particular, the forward and inverse Fourier transform), we obtain

$$W[J] = \frac{\Omega}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dC \exp\left[i\Omega\left(vC - C^k \right) \right]$$
$$\times \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} \left(\frac{d}{d\epsilon} \right)^{n-j} w_j[J, M^2] ,$$

$$w_j[J, M^2] = \frac{(-ig)^j}{j!} \left[\int dx \frac{\partial^{2k}}{\partial J^{2k}(x)} \right]^j \exp\left(-\frac{i}{2} \langle J \Delta J \rangle \right) ,$$

$$\Delta(p) = (p^2 - M^2 + i0)^{-1} , \qquad M^2 = m^2 + \epsilon^{1/k} a v .$$

In the first order of VPT (J = const) we find that

$$W[J] = \frac{\Omega}{2\pi}\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dC \exp\left[i\Omega S(v,C)\right]\left[1 + i\Omega\Delta S(v,C)\right], \qquad (4.110)$$

where

$$S = Cv - C^k - \frac{1}{2}(M^2)^{\frac{1}{2}}, \qquad (4.111)$$

$$\Delta S(v,C) = \frac{1}{4k}(M^2)^{\frac{1}{2}} - g(2k-1)!!\left[-\frac{1}{2}(M^2)^{-\frac{1}{2}}\right]^k. \qquad (4.112)$$

We require the optimum value of the parameter M^2 to correspond to the minimum of the absolute value of $\Delta S(v,C)$:

$$M^2 : \min|\Delta S(v,C)|.$$

In the case of k being even, the optiinization condition is $\Delta S(v,C) = 0$, and since

$$V_{\rm eff}^{(1)}(\varphi_0) = E_0^{(1)} + O(\varphi_0^2), \qquad (4.113)$$

the ground state energy is found from the stationarity condition for the function $S(v,C)$:

$$E_0^{(1)} = -S(v_0, C_0), \qquad (4.114)$$

where

$$\left.\frac{\partial S}{\partial C}\right|_{\substack{C=C_0\\ v=v_0}} = 0, \qquad \left.\frac{\partial S}{\partial v}\right|_{\substack{C=C_0\\ v=v_0}} = 0.$$

Then we have for even k

$$\begin{aligned}(M^2)^{1/2} &= \frac{1}{2}[8kg(2k-1)!!]^{\frac{1}{(k+1)}},\\ E_0^{(1)} &= \frac{k+1}{4k}(M^2)^{1/2}.\end{aligned} \qquad (4.115)$$

For $k = 4$ we find that $E_0^{(1)} = 0.792g^{1/5}$. For case of $k = 3$ the optimization will consist in choosing such a real positive value of M^2 at which $|\Delta S| = \min$. Since

$$\Delta S = \frac{(M^2)^{1/2}}{12} + g\,\frac{15}{8}\,(M^2)^{-3/2},$$

the parameter $(M^2)^{1/2}$ is $2.866\,\lambda^{1/4}$ and for the ground state energy we obtain $E_0^{(1)} = 0.6396g^{1/4}$. The corresponding exact value can be find in Table 7.

5. Asymptotic Behavior of Remote Terms of the Variational Perturbation Theory Series

Now we will consider the massless φ^4 theory in the four-dimensional Euclidean space with the action

$$S[\varphi] = S_0[\varphi] + \lambda S_4[\varphi]\,, \tag{5.1}$$

where the functionals $S_0[\varphi]$ and $S_4[\varphi]$ are as defined by (2.2).

We shall construct the VPT series for the vacuum functional

$$W[0] = \int D\varphi \exp(-S[\varphi])\,. \tag{5.2}$$

Generalization of the method to the Green functions presents no problems. As a variational addition, we will employ a functional of the *anharmonic* form

$$\tilde{S}[\varphi] = \theta^2 S_0^2[\varphi]\,. \tag{5.3}$$

Then the VPT series for the functional (5.2) is written as

$$\begin{aligned} W[0] &= \sum_{n=0}^{\infty} W_n[0,\theta]\,, \\ W_n[0,\theta] &= \frac{(-1)^n}{n!} \int D\varphi \exp(-S_{\text{eff}}[\varphi, n])\,, \end{aligned} \tag{5.4}$$

where

$$S_{\text{eff}}[\varphi, n] = S_0[\varphi] + \theta^2 S_0^2[\varphi] - n \ln(\lambda S_4[\varphi] - \theta^2 S_0^2[\varphi])\,. \tag{5.5}$$

The basic contribution to the asymptotics of higher order terms of the series (5.4) comes from the configurations of fields that obey the equation

$$\frac{\delta S_{\text{eff}}[\varphi_0, n]}{\delta \varphi_0(x)} = 0\,. \tag{5.6}$$

Varying (5.5) we obtain

$$-\,\partial^2 \varphi_0 + \frac{a}{3!}\varphi_0^3 = 0\,, \tag{5.7}$$

where

$$a = \frac{4!\lambda\, n}{D[\varphi_0] + 2\theta^2 S_0[\varphi_0](n + D[\varphi_0])}\,, \tag{5.8}$$

$$D[\varphi] = \lambda S_4[\varphi] - \theta^2 S_0^2[\varphi]\,. \tag{5.9}$$

The solution of Eq. (5.7) is of the form

$$\varphi_0(x) = \pm\sqrt{\frac{48}{a}}\,\frac{\mu}{(x - x_0)^2 + \mu^2}\,. \tag{5.10}$$

The arbitrary parameters x_0 and μ stand for translational and scale invariance of the model under consideration.

Next, it is convenient to define new variables

$$g = 4C_s\lambda\,, \qquad \theta^2 = g\chi\,. \tag{5.11}$$

Here $C_s = 4!/(16\pi)^2$ is a constant entering into the Sobolev inequality (see, for instance, Refs. 27 and 28):

$$S_4[\varphi] \leq 4C_s S_0^2[\varphi]\,. \tag{5.12}$$

For the functional (5.9) of the functions (5.10) we obtain

$$D[\varphi_0] = 4\,\frac{(16\pi^2)^2}{a^2}\,g(1-\chi)\,. \tag{5.13}$$

Inserting $S_0[\varphi]$ and (5.13) into (5.8) we get an equation for the parameter a whose solution is of the form

$$a = \left(\sqrt{\frac{b^2}{4} + nb} - \frac{b}{2}\right)^{-1}, \qquad b = \left[\left(\frac{32}{\pi^2}\right)^2 g\chi\right]^{-1}. \tag{5.14}$$

In the limit of large n we have

$$D[\varphi_0] \sim \frac{n(1-\chi)}{\chi} \tag{5.15}$$

and in the leading order in n,

$$W_n[0,\theta] \sim \frac{(-1)^n}{n!}\,n^n\left(\frac{1-\chi}{\chi}\right)^n \exp(-n)\,. \tag{5.16}$$

When the next orders in n including the functional determinant are taken into account, a multiplicative factor dependent on n appears in (5.16). However, it is not dominating and does not influence the convergence properties of the series.

From the expression (5.16) it is seen that the series absolutely converges for $\chi > 1/2$ irrespective of the values of the coupling constant g; and, as follows from the Sobolev inequality (5.12), the VPT series for $\chi > 1$ is of positive sign. When $1/2 < \chi < 1$, the terms of the series (5.4) at large n form the Leibniz series. Here again the value $\chi = 1$ corresponds both to the change of the regime of the VPT series and to its asymptotic optimization.

The asymptotic behavior of remote terms of the VPT series, when an *anharmonic variational procedure* is performed, is determined by the behavior of the functional integral

$$J_k = \frac{1}{k!}\int D\varphi (A^2[\varphi] - S_4[\varphi])^k \exp\{-(S_0[\varphi] + A^2[\varphi])\}\,, \tag{5.17}$$

where the functional $A[\varphi]$ is as defined by (4.9). Making the change $\varphi \to k^{1/4}\varphi$ we derive

$$J_k = \frac{k^k}{k!} I_k \,, \tag{5.18}$$

$$I_k = \int D\varphi \exp(-kS_{\text{eff}}[\varphi] - k^{1/2}S_0[\varphi]) \,, \tag{5.19}$$

where

$$S_{\text{eff}}[\varphi] = A^2[\varphi] - \ln D[\varphi] \,, \tag{5.20}$$

$$D[\varphi] = A^2[\varphi] - S_4[\varphi] \,. \tag{5.21}$$

The integral (5.19) contains a large parameter k in the exponential and, therefore, its asymptotics can be found by the Laplace functional method. The main contribution to the integral (5.19) comes from the configurations $\varphi_0(x)$, which minimize the effective action (5.20). The corresponding equation is

$$-\partial^2\varphi_0 + a\varphi_0 - b\varphi_0^3 = 0 \,, \tag{5.22}$$

where

$$a = \frac{\chi}{\theta} \,, \tag{5.23}$$

$$b = 2\{\theta A[\varphi_0](1 - D[\varphi_0])\}^{-1} \,. \tag{5.24}$$

It is convenient to pass to the function $f(x)$, which satisfies a differential equation,

$$(-\partial^2 + 1)f(x) - f^3(x) = 0 \,, \tag{5.25}$$

and is connected with the function $\varphi_0(x)$ by the relation

$$\varphi_0(x) = \sqrt{\frac{a}{b}} f(\sqrt{a}x) \,. \tag{5.26}$$

We define the constant

$$C = \int dx\, f^4(x) \,. \tag{5.27}$$

As was proved in Ref. 1, the spherically symmetrical solution of a motion equation provides the absolute minimum of total action. By using this fact and Eq. (5.22) it is easy to show that spherically symmetrical solutions provide a minimum of S_{eff} as well. The constant (5.27) depends on the space dimension and can be evaluated following, for example, Ref. 18. In the case under consideration the exact value of C is not important. The functionals $S_4[\varphi_0]$ and $A^2[\varphi_0]$ are expressed via (5.27) as follows:

$$S_4[\varphi_0] = \frac{\alpha}{b^2} \,, \tag{5.28}$$

$$A^2[\varphi_0] = \frac{\alpha^2 \tau}{b^2} \,, \tag{5.29}$$

where we define the parameters

$$\alpha = Ca^{2-n/2}, \tag{5.30}$$

$$\tau = \frac{\theta^2}{4}. \tag{5.31}$$

Three parameters α, b and τ, as follows from (5.24), are connected by the relation

$$\alpha\tau(1 - D[\varphi_0]) = 1, \tag{5.32}$$

where

$$D[\varphi_0] = \frac{\alpha(\alpha\tau - 1)}{b^2}. \tag{5.33}$$

Thus, as before, only two parameters are independent. In the leading order in k we obtain for the integral (5.17)

$$J_k \sim k^{-1/2} D^k[\varphi_0] \exp\{-k(A^2[\varphi_0] - 1)\}. \tag{5.34}$$

By using the equation of motion (5.22) it is easy to see that $A[\varphi_0] = 1$. The region of the values of parameters at which the VPT series is convergent is determined by the inequality

$$\left|D[\varphi_0]\right| < 1. \tag{5.35}$$

The best choice of the parameters at which the contribution of remote terms of the VPT series is minimal (asymptotic optimization[18,19]) implies the condition

$$D[\varphi_0] = 0, \tag{5.36}$$

leading to the following connection of the parameters:

$$\alpha\tau = 1. \tag{5.37}$$

Thus, the single independent parameter remains which also can be fixed by the optimization of first terms of the VPT series. The asymptotic optimization condition for original parameters θ and χ is written as

$$\chi = \left(\frac{16}{\theta^n C^2}\right)^{\frac{1}{4-n}}. \tag{5.38}$$

In particular, in the one-dimensional case $C = 16/3$ and the condition (5.38) transforms into (4.36).

6. Gaussian Effective Potential in Variational Perturbation Theory

In this section the Gaussian effective potential (GEP) will be derived on the basis of the VPT under various choices of the variational addition. In the $\lambda\varphi^4$ theory in the n-dimensional space the GEP has the form[29]

$$V_{\rm GEP} = V_c + \Delta V_{\rm GEP}\,, \tag{6.1}$$

$$\Delta V_{\rm GEP} = \frac{(z^2)^{n/2}}{n} A_n + \frac{1}{2}(m^2 - z^2)(z^2)^{n/2-1} A_n + 3\lambda (z^2)^{n-2} A_n^2 + 6\lambda (z^2)^{n/2-1} A_n \varphi^2\,, \tag{6.2}$$

$$A_n = \mu^{2\varepsilon}\Gamma(1-n/2)/(4\pi)^{n/2}\,, \qquad n = d - 2\varepsilon\,, \qquad d = 1, 2, \ldots,$$

and z^2 satisfies the equation

$$z^2 = m^2 + 12\lambda\varphi^2 + 12\lambda A_n (z^2)^{n/2-1}\,. \tag{6.3}$$

We will consider the φ^4 theory in the n-dimensional space with the pseudo-Euclidean signature. The action functional looks as follows:

$$S[\varphi] = \int dx \left[\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \lambda\varphi^4\right]. \tag{6.4}$$

The generating functional of Green functions reads

$$W[J] = \int D\varphi \exp[i(S[\varphi] + \langle J\varphi\rangle)] = \exp[i(S[\varphi_c] + \langle J\varphi_c\rangle)\} D[J]\,, \tag{6.5}$$

where

$$D[J] = \int D\varphi \exp(-iA[\varphi])\,, \tag{6.6}$$

$$A[\varphi] = \int dx \left[\frac{1}{2}\varphi(\partial^2 + m^2 + 12\lambda\varphi_c^2)\varphi + 4\lambda\varphi_c\varphi^3 + \lambda\varphi^4\right] \tag{6.7}$$

and the function φ_c satisfies a classical equation of motion

$$(\partial^2 + m^2)\varphi_c + 4\lambda\varphi_c^3 = J\,. \tag{6.8}$$

In the standard one-loop approximation only the terms quadratic in the fields φ are retained in the expression (6.7) for $A[\varphi]$. In this case the functional integral for $D[J]$ becomes Gaussian.

We shall evaluate the quantity $D[J]$ by means of VPT. Let us first consider the *harmonic variational procedure*. We rewrite the functional $A[\varphi]$ as

$$A[\varphi] = \int dx \left[\frac{1}{2}\varphi(\partial^2 + z^2)\varphi + \lambda\left(4\varphi_c\varphi^3 + \varphi^4 - \frac{\chi^2}{2}\varphi^2\right)\right], \tag{6.9}$$

where

$$z^2 = m^2 + 12\lambda\varphi_c^2 + \lambda\chi^2\,.$$

As a result, the VPT series for the quantity $D[J]$ is by the harmonic variational procedure transformed to

$$D = \left(\det \frac{\partial^2 + z^2}{\partial^2}\right)^{-1/2} \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!}$$
$$\times \left[\int dx \left(4\varphi_c \hat{\varphi}^3 + \hat{\varphi}^4 - \frac{\chi^2}{2}\hat{\varphi}^2\right)\right]^n \exp\left(-\frac{i}{2}\langle j\Delta j\rangle\right)_{j=0}, \tag{6.10}$$

where

$$\Delta(p) = (p^2 - z^2 + i0)^{-1},$$
$$\hat{\varphi}(x) = i\frac{\delta}{\delta j(x)}.$$

Considering (6.10), let us restrict ourselves to the first two addends in the sum which give rise to the first nontrivial approximation. The contributions to the effective potential, corresponding to these addends, equal

$$V_0 = \frac{1}{n} z^2 \Delta_0(z^2), \tag{6.11}$$

$$V_1 = \lambda\left[3\Delta_0^2(z^2) - \frac{\chi^2}{2}\Delta_0(z^2)\right], \tag{6.12}$$

where

$$\Delta_0(z^2) = \mu^{2\epsilon}\frac{\Gamma(1-n/2)}{(4\pi)^{n/2}}(z^2)^{n/2-1}. \tag{6.13}$$

The optimization condition

$$\frac{d(V_0 + V_1)}{dz^2} = 0 \tag{6.14}$$

gives the equation for the variational parameter z^2:

$$z^2 = m^2 + 12\lambda\varphi^2 + 12\lambda\Delta_0(z^2). \tag{6.15}$$

With the help of (6.15) in the considered order of VPT we find for the effective potential the expression

$$V_{\text{eff}}(\varphi) = V_{cl} + V_0 + V_1 = \frac{1}{2}m^2\varphi^2 + \lambda\varphi^4 + \frac{1}{n}z^2\Delta_0(z^2)$$
$$+ \frac{1}{2}(m^2 - z^2)\Delta_0(z^2) + \lambda\left[3\Delta_0^2(z^2) + 6\varphi^2\Delta_0(z^2)\right]. \tag{6.16}$$

When comparing (6.1)–(6.3) with (6.13), (6.15), (6.16), we easily see that both of the functions (6.2) and (6.16) and Eqs. (6.3) and (6.15) for the massive parameter coincide with one another.

Let us now calculate the quantity $D[J]$ by using for (6.5) the *anharmonic variation* of the action functional. We choose the anharmonic addition in the form $\tilde{S}^2[\varphi]$, where

$$\tilde{S}[\varphi] = \frac{\chi}{2\Omega^{1/2}} \int dx\, \varphi^2(x) . \tag{6.17}$$

The coordinate space volume Ω in (6.17) appears because the derivation of V_{eff} from the effective action requires us to consider a constant field configuration. Then the parameter optimizing the VPT series χ in (6.17) does not depend on Ω. As a result, we find that

$$\begin{aligned} D[J] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int D\varphi \{\lambda(\varphi^4 + 4\varphi_c\varphi^3) - \tilde{S}^2[\varphi]\}^n \\ \times \exp\left(- i\left\{ \frac{1}{2}\varphi\left(\partial^2 + m^2 + 12\lambda\varphi_c^2\right)\varphi + \tilde{S}^2[\varphi]\right\}\right) . \end{aligned} \tag{6.18}$$

Any power of $\tilde{S}^2[\varphi]$ in (6.18) can be obtained by the corresponding number of differentiations of the expression $\exp(-i\varepsilon\tilde{S}^2[\varphi])$ with respect to parameter ε with putting $\varepsilon = 1$ at the end. As to the addend $\tilde{S}^2[\varphi]$ in the exponential in (6.18) which makes the functional integral non-Gaussian, the problem is easy to solve by using the transformation

$$\exp(-i\varepsilon\, \tilde{S}^2[\varphi]) = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp\left\{ i\left(\frac{u^2}{4} \pm \sqrt{\varepsilon}\tilde{S}[\varphi]\right) - i\frac{\pi}{4}\right\} . \tag{6.19}$$

As a result, the VPT series takes the form

$$\begin{aligned} D[J] = \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} \frac{(-i)^{n-k}}{(n-k)!n!}\left[\frac{d}{d\varepsilon}\right]^{n-k} \\ \times \sqrt{\Omega}\int_{-\infty}^{\infty} \frac{dv}{2\sqrt{\pi}} \exp\left(i\Omega\frac{v^2}{4} - i\frac{\pi}{4}\right)\left(\det\frac{\partial^2 + M^2}{\partial^2}\right)^{-1/2} \\ \times \left[\lambda\int dx(4\varphi_c\hat{\varphi}^3 + \hat{\varphi}^4)\right]^k \exp\left(-\frac{i}{2}(j\Delta j)\right)_{j=0} , \end{aligned} \tag{6.20}$$

where

$$\begin{aligned} M^2 &= m^2 + 12\lambda\varphi_c^2 + \sqrt{\varepsilon}\chi v , \\ \Delta(p) &= (p^2 - M^2 + i0)^{-1} . \end{aligned} \tag{6.21}$$

The integral over v in (6.20) contains the large parameter Ω and therefore can be evaluated, for example, by using the method of a stationary phase. As a result, the effective potential in the first nontrivial VPT order reads

$$V_{\text{eff}} = V_c + \Delta V_{\text{eff}} , \qquad \Delta V_{\text{eff}} = V_0 + V_1 ,$$

where

$$V_0 = \frac{1}{n} M^2 \Delta_0 - \frac{\chi^2}{4} \Delta_0^2 ,$$
$$V_1 = -\frac{\chi^2}{4} \Delta_0^2 + 3\lambda \Delta_0^2 . \tag{6.22}$$

Here M^2 is the massive parameter derived from (6.21) at $\varepsilon = 1$ and $v = v_0$, where v_0 is the stationary phase point in the integral (6.20). The corresponding equation has the form

$$M^2 = m^2 + 12\lambda\varphi^2 + \chi^2 \Delta_0 . \tag{6.23}$$

The quantity $\Delta_0 = \Delta_0(M^2)$ is determined by the expression (6.13) and represents the Euclidean propagator $\Delta(x = 0, M^2)$ written with the help of the dimensional regularization.

Then, let us consider two optimization schemes of the VPT series (see Refs. 9, 10). In accordance with the *first optimization version* the variational parameter is determined by the condition for the contribution of "nonleading" terms of the series being minimal. In the present case we have to require min $|V_1|$. It is easy to see that the function V_1 is such that the equation V_1 admits a solution. This situation is, obviously, the most preferable as the considered optimization version is performed. From (6.22) we find the optimal value of χ^2:

$$\chi^2 = 12\lambda . \tag{6.24}$$

Equation (6.23) for this choice of χ^2, up to the change of M^2 to z^2, transforms into the GEP method equation (6.3) for the massive parameter z^2. Thus, in the considered optimization procedure $\Delta V_{\text{eff}} = V_0$ and now, by using (6.22)–(6.24), it is easy to show that

$$\Delta V_{\text{eff}} = \Delta V_{\text{GEP}} .$$

To implement the *second optimization version*, we should keep the parameter M^2 being variational. Making use of Eqs. (6.22) and (6.23) we obtain

$$\Delta V_{\text{eff}} = \left(\frac{1}{n} - \frac{1}{2}\right) M^2 \Delta_0 + \frac{1}{2}(m^2 + 12\lambda\varphi^2)\Delta_0 + 3\lambda\Delta_0 . \tag{6.25}$$

The optimization condition has the form

$$\frac{\partial \Delta V_{\text{eff}}}{\partial M^2} = 0 \tag{6.26}$$

and gives rise to the equation for M^2:

$$M^2 = m^2 + 12\lambda\varphi^2 + 12\lambda\Delta_0 . \tag{6.27}$$

Comparing (6.27) with (6.23) and (6.24) we conclude that the two optimization versions lead to the same result:

$$V_{\text{eff}}^{(1)} = V_{\text{GEP}} .$$

Within the previous consideration we have obtained the GEP by building the VPT series for a variational correction to the one-loop approximation. Let us derive the GEP by another approach that does not use the loop expansion and directly operates with the original functional $W[j]$. We will consider the two-parameter *anharmonic type* addition to the action:

$$\tilde{S}[\varphi] = \frac{a^2}{\Omega} S_2^2[\varphi] + \frac{b^4}{\Omega^3} S_1^4[\varphi] , \tag{6.28}$$

where

$$S_1[\varphi] = \int dx \, \varphi(x) ,$$

$$S_2[\varphi] = \int dx \, \varphi^2(x) .$$

The VPT series for the generating functional of Green functions looks as follows:

$$W[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int D\varphi (\tilde{S} - S_{\rm int})^n \times \exp\left[i \left(S_0 - m^2 S_2 - \varepsilon \frac{a^2}{\Omega} S_2^2 - \theta \frac{b^4}{\Omega^3} S_1^4 + \langle j\varphi \rangle \right) \right] . \tag{6.29}$$

The parameters ε and θ are introduced here to give one possibility of obtaining in the integrand the terms connected with S_1 and S_2, by differentiating with respect to ε and θ. Then only the interaction action $S_{\rm int}$ remains in a factor in front of the exponential. The expression in the exponential in (6.29) is reduced to the form quadric in the fields by using the Fourier transformation. Then (6.29) is rewritten as

$$W[j] = \Omega^2 \int_{-\infty}^{\infty} dx \, \frac{dp}{2\pi} \int_{-\infty}^{\infty} dy \, \frac{dq}{2\pi} \exp[i\Omega(px - qy - p^2 - q^4)] \times \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{i^{n-k}}{m!(n-k-m)!} \left(i\frac{\partial}{\partial \varepsilon} \right)^m \left(i\frac{\partial}{\partial \theta} \right)^{n-k-m} \times \left(\det \frac{\partial^2 + M^2}{\partial^2} \right)^{-1/2} w_k[J, M^2] , \tag{6.30}$$

where

$$M^2 = m^2 + \sqrt{\varepsilon} a x , \quad J = j + \theta^{1/4} b y , \tag{6.31}$$

and $w_k[J, M^2]$ are the ordinary perturbative expansion coefficients for the generating functional of Green functions $W[j]$:

$$w_k[J, M^2] = \frac{(-i\lambda)^k}{k!} \left[\int dx \, \frac{\delta}{\delta J^4(x)} \right]^k \exp\left(-\frac{i}{2} \langle J \Delta J \rangle \right) . \tag{6.32}$$

In the first nontrivial order for the generating functional of connected Green functions

$$Z[j] = (i\Omega)^{-1} \ln W[j] ,$$

we find that

$$\begin{aligned} Z_0[j] &= \frac{1}{2}\frac{J^2}{M^2} + \frac{1}{4}(M^2 - m^2)\left[\Delta_0 + \frac{J^2}{(M^2)^2}\right] \\ &\quad - \frac{3}{4}\frac{J}{M^2}(J - j) - \frac{1}{n}M^2\Delta_0 , \\ Z_1[j] &= \frac{1}{4}(M^2 - m^2)\left[\Delta_0 + \frac{J^2}{(M^2)^2}\right] - \frac{1}{4}\frac{J}{M^2}(J - j) \\ &\quad - \lambda\left[3\Delta_0^2 + 6\Delta_0\frac{J^2}{(M^2)^2} + \frac{J^4}{(M^2)^4}\right] . \end{aligned} \tag{6.33}$$

Here, as before, the method of a stationary phase has been applied to the numerical integrals. In (6.33), instead of the original a^2, b^4, the more transparent variational parameters J and M^2 have been used. The optimization conditions in this case read

$$\frac{\partial Z^{(1)}}{\partial J} = 0 , \qquad \frac{\partial Z^{(1)}}{\partial M^2} = 0 ,$$

where

$$Z^{(1)} = Z_0 + Z_1 .$$

However, it is more convenient here to define the new variables

$$x = \frac{J}{M^2} , \qquad y = \frac{M^2}{m^2} . \tag{6.34}$$

From (6.33) we get

$$\begin{aligned} Z^{(1)}[j] &= jx - \frac{1}{2}m^2x^2 + \left(\frac{1}{2} - \frac{1}{n}\right)m^2y\Delta_0(m^2y) \\ &\quad - \frac{1}{2}m^2\Delta_0(m^2y) - \lambda[3\Delta_0^2(m^2y) + 6\Delta_0(m^2y)x^2 + x^4] . \end{aligned} \tag{6.35}$$

The optimization condition $\partial Z^{(1)}/\partial x = 0$ yields the equation

$$m^2x + 4\lambda x(3\Delta_0 + x^2) = j . \tag{6.36}$$

By analogy, requiring $\partial Z^{(1)}/\partial y = 0$ we get the equation

$$m^2(y - 1) = 12\lambda(\Delta_0 + x^2) . \tag{6.37}$$

Making use of (6.36) and (6.37) we easily find that

$$\varphi = \frac{d\,Z^{(1)}}{dj} = \frac{\partial Z^{(1)}}{\partial j} = x . \tag{6.38}$$

For the effective potential we obtain

$$V_{\text{eff}} = \frac{1}{2}m^2\varphi^2 + \left(\frac{1}{n} - \frac{1}{2}\right)M^2\Delta_0(M^2) + \frac{1}{2}m^2\Delta_0(M^2) + \lambda[3\Delta_0^2(M^2) + 6\Delta_0(M^2)\varphi^2 + \varphi^4]. \tag{6.39}$$

As follows from (6.37) and (6.38), the parameter M^2 satisfies Eq. (6.27), by means of which it is easy to show that (6.39) coincides with V_{GEP}.

7. Renormalization and the Nonperturbative β Function for the φ^4 Model

The massless φ^4 model in four dimensions has the Euclidean action

$$S[\varphi] = S_0[\varphi] + S_I[\varphi], \tag{7.1}$$

where

$$S_0[\varphi] = \frac{1}{2}\int dx\,\varphi(-\partial^2)\varphi, \tag{7.2}$$

$$S_I[\varphi] = \frac{(4\pi)^2}{4!}\,g\int dx\,\varphi^4. \tag{7.3}$$

As is well known, the series of perturbation theory for the generating functional of the Green functions

$$W[J] = \int D\varphi\,\exp\left(-S[\varphi] + \int dx\,J\cdot\varphi\right) \tag{7.4}$$

diverges. A formal argument consists in a meaningless functional integral for a negative coupling constant. The function $W[J]$ as a function of g does not appear as the analytic function for $g = 0$. The concrete asymptotic behavior of higher order terms can be determined by the functional saddle point method. The large parameter is the number of the order term. The main contribution to the functional integral (7.4) comes from the configurations of the fields φ proportional to the positive power of the large saddle point parameter. In this case, the functional interaction (7.3) cannot be considered as the perturbative term in the comparison with the expression (7.2). Consequently, it appears in divergence of the perturbation series.

The idea of the VPT method consists in the organization of a new effective functional interaction, S_I'. We expect that this functional can be considered as a small value when compared with a new functional S_0'. For the realization of this idea we must be careful about the possibility of making the calculation. Practically, we must use only the Gaussian functional integrals, i.e. the form of $\tilde{S}[\varphi]$ should be such that the functional integral in (7.4) can be reduced to Gaussian quadratures.

Let us consider the VPT functional

$$\tilde{S}[\varphi] = \theta^2 S_0^2[\varphi] \tag{7.5}$$

and rewrite the total action (7.1) as

$$S[\varphi] = S_0'[\varphi] + \eta S_I'[\varphi], \tag{7.6}$$

where

$$S_0'[\varphi] = S_0[\varphi] + \tilde{S}[\varphi], \tag{7.7}$$

$$S_I'[\varphi] = S_I[\varphi] - \tilde{S}[\varphi]. \tag{7.8}$$

In this case, the expansion of the expression (7.4) is carried out in powers of η. After all calculations we should put $\eta = 1$. The parameter θ^2 in Eq. (7.5) is a parameter of the variational type. The initial functional (7.4) certainly does not depend on this parameter. We may take θ^2 so as to provide the best approximation with a finite number of VPT series terms.

It is convenient to define the new parameter t by the relation

$$\theta^2 = 4C_s \frac{(4\pi)^2}{4!} g \cdot t, \tag{7.9}$$

where $C_s = 4!/(16\pi)^2$ is a constant entering into the Sobolev inequality:

$$\int dx\, \varphi^4 \leq C_s \left[\int dx\, \varphi(-\partial^2)\varphi \right]^2. \tag{7.10}$$

The parameter t is fixed if we require the contribution of higher order terms of the VPT series to be minimal. This way of determining a variational parameter is called the asymptotic optimization of VPT series, and gives the value $t = 1$ (see Sec. 5 and Refs. 19, 30).

After expansion in the powers of η we find that the rest contains the $\tilde{S}[\varphi]$ in the exponent and, consequently, we have a non-Gaussian form of the functional integral. However, the problem is easily solved by implementing the Fourier transformation. As a result, Green-function $G_{2\nu}$ in the Nth order of VPT takes the form

$$G_{2\nu}^{(N)} = \int_0^\infty d\alpha\, \alpha^{\nu-1} \exp(-\alpha - \theta^2\alpha^2) \sum_{n=0}^{N} \eta^n \alpha^{2n} \sum_{k=0}^{n} \frac{(\theta^2)^{n-k}}{(n-k)!} \frac{g_{2\nu}^k}{\Gamma(2k+\nu)}. \tag{7.11}$$

Here the functions $g_{2\nu}^k$ are ordinary perturbative coefficients for the Green function $G_{2\nu}$. To calculate them, the standard Feynman diagrams can be used.

It should be stressed that expansion of the expression (7.11) in powers of the coupling constant g contains all powers of g. The first N terms of this expansion coincide with N terms of perturbative series.

Let us consider the procedure of renormalization. Instead of the field φ and the coupling constant g we introduce the bare field φ_0 and the bare coupling constant g_0. The field φ_0 is connected with the renormalized field by the relation $\varphi_0 = Z^{1/2}\varphi$. The divergence constants Z and g_0 are obtained from VPT expansion. The constant Z can be calculated by the propagator G_2. We will be employing the constant Z in the first order of VPT series. From Eq. (7.11) we find that

$$Z^{(1)} = \Gamma(1)J_1(\theta_0^2) + \eta\theta_0^2\Gamma(3)J_3(\theta_0^2), \tag{7.12}$$

where we define

$$J_\nu(\theta^2) = \frac{1}{\Gamma(\nu)}\int_0^\infty d\alpha\,\alpha^{\nu-1}\exp(-\alpha-\alpha^2\theta^2). \tag{7.13}$$

The function $J_\nu(\theta^2)$ has the normalized condition $J_\nu(0) = 1$. The connected part of the four-point Green function in the second order of VPT has the form

$$-G_4^{(2)}(\mu^2) = \eta g_0 J_4(\theta_0^2) + \eta^2\left[g_0\frac{\theta_0^2}{1!}\frac{\Gamma(6)}{\Gamma(4)}J_6(\theta_0^2) - \frac{3}{2}g_0^2 J_6(\theta_0^2)\ln\frac{\Lambda^2}{\mu^2}\right]. \tag{7.14}$$

In this expression we have written out only the divergence part, which we need in the following. We use the renormalization scheme with symmetric normalization point μ^2. For the bare coupling constant g_0 we write down the VPT expansion $g_0 = g(1+\eta\alpha+\cdots)$. The VPT expansions for θ_0^2 and $J_\nu(\theta_0^2)$ are introduced in a similar manner. The divergence coefficient α is defined by the expressions (7.12), (7.14) and the demand of the function finite $-Z^2G_4(\mu^2)$. If we change the normalization point $\mu \to \mu'$ and use that the bare coupling constant independent of μ, we find the connection between g and g':

$$g' = g + \eta\beta(g)\ln\frac{\mu'^2}{\mu^2}, \tag{7.15}$$

where the β function is expressed as

$$\beta(g) = \frac{3}{2}g^2\frac{J_6(\theta^2)/J_4(\theta^2)}{1-\theta^2\{[\Gamma(6)J_6(\theta^2)/\Gamma(4)J_4(\theta^2)] - 2[\Gamma(3)J_3(\theta^2)/\Gamma(1)J_1(\theta^2)]\}}. \tag{7.16}$$

Here the parameter θ^2 is connected with the renormalized coupling constant g by Eq. (7.9) with the optimal value $t = 1$.

The expansion of the β function (7.16) in the perturbation series contains all powers of the coupling constant g. It is interesting to compare the first coefficients of the VPT β function (7.16) with the well-known values of perturbation theory. From (7.16) we get

$$\beta(g) = 1.5g^2 - 2.25g^3 + 14.63g^4 - 134.44g^5 + \cdots. \tag{7.17}$$

In the considered massless case, we use counterterms which containing only divergent parts. In the framework of the dimensional regularization this conforms only to the pole part for counterterms.[31] The corresponding β function in four-loop approximation looks as follows (Ref. 32):

$$\beta_{\text{perturb}}(g) = 1.5g^2 - 2.83g^3 + 16.27g^4 - 135.80g^5 + \cdots . \tag{7.18}$$

Note that in construction of the β function (7.16) we used only the lowest order of VPT. For this approximation the expressions (7.17) and (7.18) are in agreement.

As follows from the expression (7.16), the β function is monotonously increasing and has no ultraviolet stable point (see Fig. 6). For a large coupling constant, the β function has the asymptotic behavior

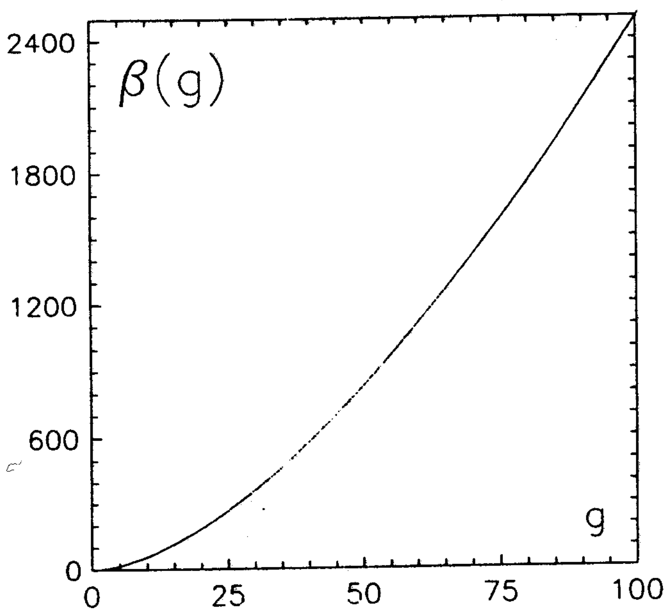

Fig. 6. The behavior of the nonperturbative β function.

$$\beta(g) \simeq 2.99g^{3/2} . \tag{7.19}$$

The degree of g in Eq. (7.19) is larger than the linear increase of the β function that was obtained in Ref. 33, and is smaller than the square increase that was found in Ref. 34.

8. Pure Yang–Mills Theory

Let us consider pure Yang–Mills theory from the viewpoint of the possibility of convergent VPT series construction. The corresponding action reads

$$S_{\text{YM}}[A] = \frac{1}{8}\,\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} , \tag{8.1}$$

where we use the conventional matrix notation:

$$\mathbf{A}_\mu = A^a_\mu T^a \,, \qquad \mathbf{F}_{\mu\nu} = F^a_{\mu\nu} T^a \,, \tag{8.2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,, \tag{8.3}$$

T_a being the anti-Hermitian generators of the SU(N) group in adjoint representation. For definiteness we shall work here with SU(2), where

$$f^{abc} = \epsilon^{abc} \tag{8.4}$$

and ϵ^{abc} are absolutely antisymmetric tensors with $\epsilon^{123} = 1$, but results hold for any group SU(N).

We split the action (8.1) into free and interaction parts:[a]

$$S_{\mathrm{YM}}[A] = S_0[A] + S_I[A] \,, \tag{8.5}$$

$$S_0[A] = \int d^d x \left[-\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\alpha} (A^a_0)^2 \right] , \tag{8.6}$$

$$\begin{aligned} S_I[A] &= S_I^{(3)}[A] + S_I^{(4)}[A] \\ &\equiv \int d^d x \left(- g \epsilon^{abc} A^b_\mu A^c_\nu \partial^\mu A^\nu_a - \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{adf} A^b_\mu A^c_\nu A^\mu_d A^\nu_f \right) . \end{aligned} \tag{8.7}$$

Let us choose a VPT functional in the form

$$B[A] = \frac{\theta}{2} \int d^d x (\partial_\mu A^a_\nu)^2 \tag{8.8}$$

and rewrite the action S_{YM} as follows:

$$S_{\mathrm{YM}}[A] = S'_0[A] + S'_I[A] \,, \tag{8.9}$$

$$S'_0[A] = S_0[A] + B^2[A] \,, \tag{8.10}$$

$$S'_I[A] = S_I[A] - B^2[A] \,, \tag{8.11}$$

which corresponds to the *anharmonic* variational procedure. As before, to study the convergence properties we interested in here, it is enough to restrict ourselves to consideration of the vacuum functional

$$Z[0] = \int DA \exp(-S_{\mathrm{YM}}[A]) \,, \tag{8.12}$$

where the functional measure is usually defined as

$$DA \equiv \prod_{\mu,a} \prod_x dA^a_\mu \,.$$

[a]The term $(1/2\alpha)(A^a_0)^2$ in (8.6) fixes the Hamiltonian (temporal) gauge. The advantages of this gauge choice will be clear later on.

The corresponding VPT series reads

$$Z = \sum_{n=0}^{\infty} Z_n , \tag{8.13}$$

$$Z_n = \frac{(-1)^n}{n!} \int DA[S_I(A) - B^2(A)]^n \exp\{-[S_0(A) + B^2(A)]\} . \tag{8.14}$$

To study the asymptotic behavior of remote terms of the VPT series, we shall now apply the $n-$ saddle point method investigated in the previous sections by considering the scalar case. After the change $A^a_\mu \to n^{1/4} A^a_\mu$ the term of the VPT series Z_n is written as

$$Z_n = \frac{(-n)^n}{n!} \int DA \exp\left(- nS_{\text{eff}}[A] - n^{-3/4} S_I^{(3)}[A] - n^{-1/2} S_0[A]\right) , \tag{8.15}$$

where

$$S_{\text{eff}}(A) = B^2(A) - \ln[S_I^{(4)}(A) - B^2(A)] \tag{8.16}$$

is an effective action that governs the leading asymptotic behavior of the VPT series terms.

It is of fundamental importance that only the part of S_I of the fourth power in the gauge fields $S_I^{(4)}[A]$, which is given by (8.7), gets into the effective action. This circumstance allows us, as we shall see later, to reduce our consideration to the scalar case investigated in the previous sections.

Taking advantage of the formula

$$\epsilon_{abc}\epsilon^{adf} = \delta_{bd}\delta_{cf} - \delta_{bf}\delta_{cd} , \tag{8.17}$$

we can represent $S_I^{(4)}[A]$ as follows:

$$S_I^{(4)}[A] = g^2 \int d^d x [(A^a_\mu A^a_\mu)^2 - (A^a_\mu A^a_\nu)^2] . \tag{8.18}$$

Making use of (8.8), (8.16), (8.18), we write the set of saddle point equations

$$\frac{\delta S_{\text{eff}}}{\delta \hat{A}^a_\mu} = 0$$

as

$$- \partial^2 \hat{A}^a_\mu - Q[\hat{A}][(\hat{A}^a_\nu \hat{A}^a_\nu)\hat{A}^a_\mu - (\hat{A}^a_\mu \hat{A}^a_\nu)\hat{A}^a_\nu] = 0 . \tag{8.19}$$

Here we have introduced the notation $\hat{A}^a_\mu$ for the saddle point function and

$$Q[\hat{A}] = \frac{2g^2}{B[\hat{A}]\theta(1 + D[\hat{A}])} , \tag{8.20}$$

where

$$D \equiv S_I^{(4)} - B^2 . \tag{8.21}$$

The saddle point equation (8.19) gives rise to the relation

$$B[\hat{A}] = \frac{\theta}{2g^2} Q S_I^{(4)}[\hat{A}] .$$

By using this relation we get

$$\begin{aligned} B^2[\hat{A}] &= 1 , \\ S_I^{(4)}[\hat{A}] &= 1 + D[\hat{A}] . \end{aligned} \tag{8.22}$$

Let us stress that the expression (8.22) has been obtained without using an implicit form of the solution to Eq. (8.19).

Let us conjecture[b] the following form of searched solution:

$$\begin{aligned} \hat{A}_0^a &= 0 , \\ \hat{A}_i^a &= \epsilon_{iab} n^b \varphi , \end{aligned} \tag{8.23}$$

where the first equation is in accordance with our gauge choice, n^a being an arbitrary vector in the color space. Using (8.17) we obtain

$$\begin{aligned} (\hat{A}_j^b \hat{A}_j^b)\hat{A}_i^a &= 2\mathbf{n}^2 \hat{A}_i^a = 2\mathbf{n}^2 \varphi^3 \epsilon_{iab} n^b , \\ (\hat{A}_i^b \hat{A}_j^b)\hat{A}_j^a &= \varphi^2(\mathbf{n}^2 \hat{A}_i^a - n_i n_j \hat{A}_j^a) = \mathbf{n}^2 \varphi^3 \epsilon_{iab} n^b . \end{aligned} \tag{8.24}$$

Substituting the ansatz (8.22) into (8.19), and making use of (8.23), (8.24), we get the equation

$$\partial^2 \varphi + \mathbf{n}^2 Q[\hat{A}(\varphi)]\varphi^3 = 0 . \tag{8.25}$$

We shall now evaluate the quantities $Q[\hat{A}(\varphi)]$, $S_I^{(4)}[\hat{A}(\varphi)]$, $B^2[\hat{A}(\varphi)]$ and $D[\hat{A}(\varphi)]$ in terms of the scalar function φ by using the relations (8.17) and (8.23). A short calculation gives

$$S_I^{(4)}[\hat{A}(\varphi)] = 2g^2(\mathbf{n}^2)^2 \int d^d x\, \varphi^4(x) \equiv 4(\mathbf{n}^2)^2 S_I[\varphi] , \tag{8.26}$$

$$B^2[\hat{A}(\varphi)] = 4(\mathbf{n}^2)^2 \left[\int d^d x\, \varphi(x)(-\partial^2 \varphi) \right] \equiv 4(\mathbf{n}^2)^2 A(\varphi) , \tag{8.27}$$

$$D[\hat{A}(\varphi)] = 4(\mathbf{n}^2)^2 (S_I[\varphi] - A^2[\varphi]) , \tag{8.28}$$

$$Q[\hat{A}(\varphi)] = \frac{2g^2}{\theta}(2\,(\mathbf{n}^2)^2 A[\varphi]\{1 + 4(\mathbf{n}^2)^2 D[\varphi]\})^{-1} . \tag{8.29}$$

[b]It reminds us of the well-known 't Hooft–Poliakov monopole solution, where the space and isotopic degrees of freedom are mixed.

It is easy to see that, except for dependence of the arbitrary color vector module $\mathbf{n}^2$, we have the same equation as a saddle point equation determining the large order behavior of the VPT series in the scalar case with the interaction action

$$S_I[\varphi] = \frac{g^2}{2} \int d^d x \, \varphi^4(x) \tag{8.30}$$

[compare (8.26)–(8.29) with the scalar case in Sec. 5]. However, this dependence of $\mathbf{n}^2$ is pure imaginary since it can be removed by means of the substitution

$$\varphi \to (2\mathbf{n}^2)^{-1/2} \varphi \, .$$

Then we have eventually the following set of equations instead of (8.25)–(8.29):

$$\partial^2 \varphi + Q[\hat{A}(\varphi)]\varphi^3 = 0 \, ,$$

$$Q[\hat{A}(\varphi)] = \frac{2g^2}{\theta}(A(\varphi)\{1 + D[\hat{A}(\varphi)]\})^{-1} \, ,$$

$$D[\hat{A}(\varphi)] = S_I[\varphi] - A^2[\varphi] \, ,$$

$$S_{\text{eff}}[\hat{A}(\varphi)] = A^2[\varphi] - \ln D[\hat{A}(\varphi)] \, ,$$

and $S_I[\varphi]$ is determined by (8.30). The subsequent consideration performed in complete analogy with the scalar case has been studied in detail in Sec. 5.

Let us note in conclusion that we have represented here one of the solutions of (8.19) providing the minimum of S_{eff}. This means that there exists a solution giving rise to an absolute minimum of S_{eff} which, in particular, may coincide with the found one. However, even if it is not so, it would not change our main conclusion about convergence of the VPT series (8.13) and could lead only to another value of the VPT parameter.

9. Coupling to Spinor Fields; Yukawa Model

In the previous section we have considered pure Yang–Mills theory and constructed convergent VPT series by using the anharmonic VPT procedure. Having in mind application to QCD, it is of great interest to introduce couplings into spinor fields and to investigate how the fermionic fields influence the convergence properties of VPT series. To this end we first consider the simplest case of coupling the self-interacting bozonic fields to the charged fermions–Yukawa model.

We consider a Yukawa theory with a bozonic self-interaction in Euclidean space–time dimension $d < 4$. The corresponding VPT series for the vacuum functional[c] looks as follows:

[c]As was earlier argued, the term with a source did not influence the convergence properties because of its lower power in fields.

$$Z[0] = \sum_{n=0}^{\infty} Z_n \,,$$
$$Z_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{n+1}} \int D\varphi D[\psi, \bar{\psi}] \times \exp\left(- \{(S_0[\varphi] + S_0[\psi, \bar{\psi}] + A^2[\varphi]) + \lambda(S_I[\psi, \bar{\psi}; \varphi] + S_I[\varphi] - A^2[\varphi])\}\right) , \tag{9.1}$$

where

$$S_0[\varphi] = \frac{1}{2} \int dx (\partial\varphi)^2 + \frac{m^2}{2} \int dx (\varphi)^2 \,,$$
$$S_I[\varphi] = h \int dx \, \varphi^4 \,.$$

The VPT functional is chosen to be in anharmonic form:

$$A^2[\varphi] = \left[\theta \int dx (\partial\varphi)^2 + \frac{\chi^2}{2} \int dx (\varphi)^2\right]^2 ,$$

$$S_0[\psi, \bar{\psi}] = - \int d^d x \, \bar{\psi}(\hat{\partial} + \mu)\psi \,, \tag{9.2}$$

$$S_I[\psi, \bar{\psi}; \varphi] = -\sqrt{g} \int d^d x \, \bar{\psi}\psi\varphi \,, \tag{9.3}$$

and the integration contour lying in the complex $\lambda-$ plane surrounds the point (value of VPT expansion parameter) $\lambda = 1$.

Integrating in (9.1) over spinor fields we obtain

$$Z_n = \oint \frac{d\lambda}{\lambda^{n+1}} \int D\varphi \exp\left(-\{(S_0[\varphi] + A^2[\varphi]) + \lambda(S_I[\varphi] - A^2[\varphi]) - \ln D[\varphi; \tilde{\lambda}]\}\right) , \tag{9.4}$$

where the normalized fermionic determinant

$$D[\varphi; \tilde{\lambda}] = \frac{\det(i\hat{\partial} + \mu + \tilde{\lambda}\varphi)}{\det(i\hat{\partial} + \mu)} \,, \tag{9.5}$$

$$\tilde{\lambda} = \sqrt{g}\lambda \,, \tag{9.6}$$

is a nonlocal functional of the field φ. It is clear that due to the positive power of the determinant only the large values of φ are "dangerous" for asymptotic behavior of Z_n, i.e. it is only for large φ that the contribution of $D[\varphi; \tilde{\lambda}]$ in asymptotic can become essential as compared with the bosonic one and may lead to nonconvergence of the series at large n. Indeed, as long as $\det(i\hat{\partial} + \mu + \tilde{\lambda}\varphi)$ is an entire function of

$\tilde{\lambda}$ in the whole complex plane[3,38] (for regular φ fields), nonanalytic terms in $\tilde{\lambda}$ may arise only from the nonconvergence of the functional integral at a large φ field.[d]

On the other hand, when

$$\tilde{\lambda}\varphi \geq \frac{\partial_\mu \varphi}{\varphi}$$

the *quasilocal* (Tomas–Fermi) approximation becomes available:[38]

$$G(x, y; \tilde{\lambda}\varphi(x)) = G_c(x, y; \tilde{\lambda}\varphi)\,, \qquad |\partial_\mu \ln \varphi(x)||x - y| \ll 1\,,$$

where $G_c(x, y; \tilde{\lambda}\varphi)$ is the fermionic propagator in a constant field $\tilde{\lambda}\varphi$.

The intuitive basis for this approximation is the following: It is well known that the large order behavior of the fermionic determinant is governed by the order ρ of the determinant as an entire function of a charge.[38] On the other hand the value ρ is determined by the number of zero eigenvalues of $\hat{p} + \mu - \tilde{\lambda}\varphi$:

$$(\hat{p} + \mu - \tilde{\lambda}_n \varphi)\psi_n = 0\,.$$

The number of zero eigenvalues of $\hat{p} + \mu - \tilde{\lambda}\varphi$ is the same as that for $(\hat{p} + \mu)/\tilde{\lambda} - \varphi$. Their asymptotical distribution as $\tilde{\lambda} \to \infty$ for a given "external" field φ (which is equivalent to $\varphi \to \infty$ for a given fixed $\tilde{\lambda}$) determines ρ. Evidently, large $\tilde{\lambda}$ is the same as small $\hbar$ and μ, and one expects the semiclassical approximation to be good as $\tilde{\lambda} \to \infty$ (or, equivalently, $\varphi \to \infty$). Therefore we can conjecture that to estimate large order behavior of $D[\varphi; \tilde{\lambda}]$ it is enough to treat φ in $D[\tilde{\lambda}\, \varphi]$ as if it were quasiconstant by setting

$$[\hat{p}, \varphi] = O(\hbar) = 0\,.$$

Thus, we conclude that if we are interested in the leading contribution in n, then we can restrict ourselves to the large, smooth enough values of φ.

Let us return to the quantity (9.5), which we wish to calculate. It can be rewritten using the reflection symmetry:

$$D^2[\varphi; \tilde{\lambda}] = \frac{\det[(i\hat{\partial} + \mu + \tilde{\lambda}\varphi)(-i\hat{\partial} + \mu + \tilde{\lambda}\varphi)]}{\det[(i\hat{\partial} + \mu)(-i\hat{\partial} + \mu)]}\,. \tag{9.7}$$

Calculating the corresponding products and making use of the well-known relation

$$\ln \det ||A|| = \mathrm{sp} \ln ||A||\,,$$

we get

$$\ln D[\varphi; \tilde{\lambda}] = \frac{1}{2} \mathrm{Sp} \ln\{[-\partial^2 + (\mu + \tilde{\lambda}\varphi)^2 + \tilde{\lambda}\hat{\partial}\varphi](-\partial^2 + \mu^2)^{-1}\}\,, \tag{9.8}$$

[d]This has to be contrasted with what would happen if ψ and $\bar{\psi}$ were boson fields. Then the integration over spinor fields would give rise to $\det^{-1}$, so that the zeros of the determinant would give singularities which would contribute to the large order behavior.

where the symbol Sp means the trace operation with respect to Lorentz indices and spatial variables.

In accordance with our approximation we neglect the derivative of $\varphi(x)$. Then the trace over $\gamma-$ matrices yields a factor $N = \mathrm{sp}\,\mathbf{1}$ and the remaining part of the calculation is identical to the scalar case of the Fredholm determinant $D[V;\tilde{\lambda}]$:

$$\ln D[V;\tilde{\lambda}] \equiv \ln\det[AB^{-1}] = \mathrm{tr}\ln\{[-\partial^2+\mu^2+V(x)](-\partial^2+\mu^2)^{-1}\} \tag{9.9}$$

with the potential

$$V(x) = \tilde{\lambda}^2\varphi^2 + 2\tilde{\lambda}m\varphi\,. \tag{9.10}$$

Taking advantage of the formula

$$\ln(AB^{-1}) = \int_0^\infty \frac{dt}{t}(e^{-tB} - e^{-tA}) \tag{9.11}$$

and noticing that for $|V(x)|$ large the integral over t is dominated by the small t region, and therefore we can use the approximation

$$e^{-tA} \sim e^{-tB}e^{-t(B-A)}\,,$$

neglecting the term

$$\frac{t^2}{2}[A,B] = \frac{t^2}{2}[-\partial^2\,V(x) + 2\partial_\mu V(x)\partial_\mu]\,,$$

we obtain

$$\mathrm{tr}\ln AB^{-1} = \int_0^\infty \frac{dt}{t}\int\frac{d^dp}{(2\pi)^d}\,e^{-t(p^2+\mu^2)}\int d^dx(1-e^{-tV(x)})\,, \tag{9.12}$$

where the symbol tr means the trace operation with respect to spatial variables only. Integrating over p and then over t we get

$$\mathrm{tr}\ln AB^{-1} = -\frac{1}{(4\pi)^{d/2}}\Gamma\left(-\frac{d}{2}\right)\int d^dx\{[\mu^2+V(x)]^{d/2} - (\mu^2)^{d/2}\}\,. \tag{9.13}$$

Neglecting the mass μ for $V(x)$ large we get eventually

$$D[\varphi;\tilde{\lambda}] \sim \exp\left[-\frac{N}{2}\frac{\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int d^dx\,V^{d/2}(x)\right]\,. \tag{9.14}$$

Let us now return to (9.9) and derive (9.14) in another way, which will be useful in the following and will help us to clarify the sense of quasilocal approximation we use here.

We again use the formula (9.11) to integrate over t in (9.12) and obtain

$$\mathrm{tr}\ln[AB^{-1}] = \int d^dx\int\frac{d^dp}{(2\pi)^d}\ln\left[1+\frac{V(x)}{p^2+\mu^2}\right]\,. \tag{9.15}$$

We can see that the latter equation coincides with the equation obtained as if we regarded $V(x)$ as a constant, $V(x) = V$, and "recalled" that $V(x)$ is a function of x only at the final stage of calculations. Indeed, let us regard V as a constant from the very beginning. Then the operator AB^{-1} satisfies the translation invariance property and we can use the formula

$$\operatorname{tr}\ln M(x-y) = \Omega \int \frac{d^dp}{(2\pi)^d} \ln M(p)\,,$$

where $\Omega = \int d^dx$ is a volume of d-dimensional space. We obtain the equation

$$\operatorname{tr}\ln[AB^{-1}] = \int d^dx \int \frac{d^dp}{(2\pi)^d} \ln\left(1 + \frac{V}{p^2+\mu^2}\right),$$

which, after restoration of the x dependence in V, coincides with (9.15).

By introducing the notation $M^2(t) \equiv \mu^2 + tV(x)$, making use of the identity

$$\ln(1+x) = \int_0^1 dt\, \frac{1}{t+1/x}\,,$$

we can rewrite (9.15) as

$$\operatorname{tr}\ln AB^{-1} = \int d^dx\, V(x) \int_0^1 dt \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+M^2(t)}\,.$$

Performing in this equation the integration over p,

$$\int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+M^2} = \frac{1}{(4\pi)^d/2} (M^2)^{d/2-1} \Gamma\left(1-\frac{d}{2}\right),$$

and then over t, we again obtain just Eq. (9.13).

Thus we conclude that in the quasiconstant field approximation the expression for $\ln D[\varphi;\tilde{\lambda}]$, though computed for constant φ, is used at the final stage of calculations as the effective addition to the action, where the coordinate dependence is restored.

One notices that if $d/2$ is an integer, the expression (9.14) has singularities which correspond to ultraviolet divergences of Feynman diagrams and are removed by the corresponding (one-loop) counterterms. For instance, for $d = 2$ we get instead of (9.14) the equation

$$D[\varphi;\tilde{\lambda}]\big|_{d\to 2} \sim \exp\left\{-\frac{1}{(4\pi)^{d/2}} \int d^dx \left[-\Gamma\left(-\frac{d}{2}\right) V^{d/2}(x) - \Gamma\left(1-\frac{d}{2}\right) V(x)\right]\right\}.$$

Let us clarify this procedure. We expand $\ln D[V]$ in powers of V treating $V(x)$, in accordance with our approximation, as a constant. Thus we take as a basis for expansion the relation (9.12) and can write

$$\operatorname{tr}\ln AB^{-1} = \sum_{n=1}^{\infty} D_n[\varphi;\tilde{\lambda}]\,, \tag{9.16}$$

$$D_n[\varphi;\tilde{\lambda}] = \frac{(-1)^n}{n!} \int d^dx\, V^n(x) \int_0^\infty dt\, t^{n-1} \int \frac{d^dp}{(2\pi)^d} e^{-t(p^2+\mu^2)}\,. \tag{9.17}$$

In dimensions $d < 4$, which we are interested in here, all of the terms of the expansion (9.16) are convergent except for the first,

$$D_1[\varphi;\tilde{\lambda}] = \frac{1}{(4\pi)^{d/2}}(\mu^2)^{d/2-1}\Gamma\left(1-\frac{d}{2}\right)\int d^d x\, V(x)\,,$$

which is singular in dimension $d = 2$. In this case we must subtract this addendum which is divergent as $\epsilon \to 0$ ($d = 2 - 2\epsilon$) from the divergent "exact" value (9.13) of the determinant to obtain the finite[e] (renormalized) quantity,

$$\ln D_{\text{ren}}[\varphi;\tilde{\lambda}] = \frac{N}{(8\pi)}\int d^2x\left(V - \mu^2\ln\frac{\mu^2+V}{\mu^2} - V\ln\frac{\mu^2+V}{\mu^2}\right), \tag{9.18}$$

or, as $V/\mu^2 \gg 1$,

$$D_{\text{ren}}[\varphi;\tilde{\lambda}] = \exp\left[\frac{N}{8\pi}\int d^2x\left(V - V\ln\frac{V}{\mu^2}\right)\right]. \tag{9.19}$$

In odd and fractional dimensions (9.14) is a finite quantity. In particular, as $d = 3$ we get instead of (9.14) the relation

$$\ln D[\varphi;\tilde{\lambda}]\big|_{d\to 3} \sim -\frac{N}{12\pi}\int d^3x\, V^{3/2}(x)\,. \tag{9.20}$$

Thus, neglecting the low powers of φ in (9.14) and (9.19), we now can write the following large VPT order estimate instead of (9.4):

$$\begin{aligned} Z_n \sim \oint \frac{d\lambda}{\lambda^{n+1}}\int D\varphi\, \exp[-([S_0(\varphi) + A^2(\varphi)] \\ + \lambda\{S_I(\varphi) - A^2(\varphi)\} - \Delta S_F[\varphi;\lambda])]\,. \end{aligned} \tag{9.21}$$

$\Delta S_F[\varphi;\lambda]$ is an effective addition to the action that arises due to integration over the spinor fields, which are treated in our approximation as coupling with the large quasiconstant scalar fields. It has the form

$$\Delta S_F[\varphi;\lambda] = -\frac{N}{2}\frac{\Gamma(\frac{d}{2})}{(4\pi)^{d/2}}g^{d/2}\lambda^d\int d^dx\,\varphi^d(x)\,, \tag{9.22}$$

as $d < 4$ ($d \not\equiv 2$), and

$$\Delta S_F[\varphi;\lambda] = \frac{N}{2}\frac{1}{4\pi}g\lambda^2\int d^2x\,\varphi^2(x)\ln\frac{\varphi^2}{\mu^2} \tag{9.23}$$

in two dimensions.

[e]As usual, this subtraction may be justified by adding an appropriate mass counterterm to the action. The latter, however, being quadratic in fields, does not influence VPT series convergence since it is suppressed by the anharmonic VPT functional $A^2[\varphi]$ and does not get into the effective action.

Let us now apply the functional steepest descent method to Z_n for further estimates of its large n behavior. For large n the integrals in (9.21) are dominated by a simultaneous saddle point in the variables φ and λ. Making the change of variables[f] $\varphi \to n^{1/4}\varphi$, we obtain

$$Z_n \sim \int D\varphi \oint d\lambda \exp\Big(-nS_{\text{eff}}[\varphi;\lambda] - \sqrt{n}S_0[\varphi] - \ln\lambda$$
$$+ n^d \Delta S_F[\varphi;\lambda] + \frac{1}{2}\delta_{d,2}\, n^2 \ln n\Delta S_F[\varphi;\lambda]\Big), \tag{9.24}$$

where the quantity $S_{\text{eff}}[\varphi;\lambda]$, which determines the leading-in-n behavior of VPT series terms, has the form

$$S_{\text{eff}}[\varphi;\lambda] = A^2[\varphi] + \lambda\big[S_I(\varphi) - A^2(\varphi)\big] + \ln\lambda\,. \tag{9.25}$$

The saddle point equations look thus:

$$\frac{d}{d\lambda_0}S_{\text{eff}}[\varphi_0;\lambda_0] = 0\,,$$

$$\frac{\delta}{\delta\varphi_0(x)}S_{\text{eff}}[\varphi_0;\lambda_0] = 0\,,$$

and thus:

$$\lambda_0 = \big[A^2(\varphi_0) - S_I(\varphi_0)\big]^{-1} \equiv D^{-1}(\varphi_0)\,, \tag{9.26}$$

$$0 = 2A(\varphi_0)\frac{\delta A(\varphi_0)}{\delta\varphi_0(x)} - \frac{1}{D(\varphi_0)}\left[\frac{\delta S_I(\varphi_0)}{\delta\varphi_0(x)} - 2A(\varphi_0)\frac{\delta A(\varphi_0)}{\delta\varphi_0(x)}\right]. \tag{9.27}$$

We notice that Eq. (9.27) is just the same as the saddle point equation determining the large VPT order behavior in the pure scalar φ^4 model and gives rise to (5.22). Indeed, in this case we would have

$$Z_n = \frac{1}{2\pi i}\oint\frac{d\lambda}{\lambda^{n+1}}\int D\varphi$$
$$\times \exp\big(-\big\{(S_0[\varphi] + A^2[\varphi]) + \lambda(S_I[\varphi] - A^2[\varphi])\big\}\big)\,,$$

and it does not depend on whether we apply the steepest descent procedure to the integral over λ first and then to the integral over φ, or we implement the steepest descent procedure looking for simultaneous saddle points in the variables φ and λ; the result would be the same. Namely, we would get for the saddle point in λ Eqs. (9.26), and the asymptotic for Z_n in both cases would be the same as that obtained by using the direct n-steepest descent method (see Sec. 5) and would be given by (5.34).

[f]It is important that λ remains unchanged at that.

Thus we can conclude that the coupling with spinor fields does not essentially influence the large VPT order behavior in the Yukawa model, which as well as in the pure scalar case is determined by the main cutting factor $\exp[-nS_{\rm eff}(\varphi_0)]$, where φ_0 submits to (5.22).

10. Yang–Mills Theory with Fermions

Up to now we have discussed the example of a pure gauge theory without the matter fields. Let us now consider the gauge theory involving the fermions. The corresponding action reads

$$S[\psi,\bar{\psi};A] = S_{\rm YM}[A] + S_F[\psi,\bar{\psi};A]\,, \tag{10.1}$$

where the Yang–Mills action $S_{\rm YM}$ (together with the temporal gauge-fixing term) is given by (8.5)–(8.7), and $S_F[\psi,\bar{\psi};A]$ is the action of spinor fields minimally coupling with the gauge fields:

$$S_F[\psi,\bar{\psi};A] = \int d^dx[\bar{\psi}(i\gamma^\mu D_\mu - m)\psi]\,, \tag{10.2}$$

where

$$D_\mu = \partial_\mu + gA^a_\mu t_a \tag{10.3}$$

is a covariant derivative, and T_a and t_a are the generators of the color group SU(N) in the adjoint and fundamental representations, respectively.

We choose the VPT functional in the form (8.8), and the corresponding VPT series in the d-dimensional Euclidean space reads

$$\begin{aligned} Z[0] &= \sum_{n=0}^{\infty} Z_n\,, \\ Z_n &= \oint \frac{d\lambda}{\lambda^{n+1}} \int DA\, D[\psi\bar{\psi}] \exp\big[-(\{S_0(A) + S_0[\psi,\bar{\psi}] + B^2[A]\} \\ &\quad + \lambda[S_I^{(3)}(A) + S_I^{(4)}(A) + S_I(\psi,\bar{\psi};A) - B^2(A)])\big]\,. \end{aligned} \tag{10.4}$$

Here $S_0[A]$, $S_I^{(3)}[A]$ and $S_I^{(4)}[A]$ are given by Eqs. (8.6) and (8.7),

$$S_0[\psi,\bar{\psi}] = -\int d^dx\,\bar{\psi}(\hat{\partial} + m)\psi\,, \tag{10.5}$$

$$S_I[\psi,\bar{\psi};A] = -g\int d^dx\,\bar{\psi}\gamma^\mu\Gamma(\mathbf{A}_\mu)\psi\,, \tag{10.6}$$

where

$$\mathbf{A}_\mu = A^a_\mu T_a\,,$$

and the symbol Γ stands for transmission of the matrices for the Lie algebra of the SU(N) group from adjoint into fundamental representation:

$$\Gamma(T_a) = t_a\,.$$

Integrating in (10.4) over spinor fields we get

$$Z_n = \oint \frac{d\lambda}{\lambda^{n+1}} \int DA \exp\left[- (\{S_0(A) + B^2[A]\} + \lambda\{S_I^{(3)}(A) + S_I^{(4)}(A) - B^2[A]\} - \ln D[A;\tilde{\lambda}])\right], \tag{10.7}$$

where the fermionic determinant

$$D[A;\tilde{\lambda}] = \frac{\det[i\hat{\partial} + m + \tilde{\lambda}\Gamma(\hat{\mathbf{A}})]}{\det[i\hat{\partial} + m]}, \tag{10.8}$$

$$\tilde{\lambda} = g\lambda, \tag{10.9}$$

contains the whole information concerning the spinor sector of the theory. Thus, our main goal now is to explore the quantity $\ln D[A;\tilde{\lambda}]$ as a function of the "external" field A_μ^a. To this end we again use quasilocal approximation for the gauge field, whose eligibility have been argued in the previous section.

As is known,[39] in non-Abelian theories there are only two types of gauge fields which produce a constant field strength tensor:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c = \text{const}. \tag{10.10}$$

The first is an "Abelian" gauge field,

$$A_\mu^a(x) = -\frac{1}{2}\,\eta^a x^\nu F_{\mu\nu}, \tag{10.11}$$

$$F_{\mu\nu}^a = \eta^a F_{\mu\nu} = \text{const}, \tag{10.12}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{10.13}$$

is a constant Abelian field strength tensor and η_a is a constant unit vector in color space. The second is a constant gauge field,

$$A_\mu^a = \text{const}, \tag{10.14}$$

which is purely non-Abelian and produces a constant field strength tensor:

$$F_{\mu\nu}^a = g f^{abc} A_\mu^b A_\nu^c = \text{const}. \tag{10.15}$$

We note, however, that there exist such gauges that are inconsistent with the choice of gauge potentials in the form (10.14). One of these gauges is the Fock–Shwinger one, $x^\mu A_\mu = 0$, which is obviously consistent only with the zero value of the constant potential A_μ^a. Another is a temporal (Hamiltonian) gauge we just used here, whose combination this the condition (10.14) gives rise to the zero value of the chromo-electric field $\mathbf{E}_a$. Thus, we can dispose of necessity to consider the case $A_\mu^a = \text{const}$

through the appropriate choice of gauge in which we build the VPT series from the very beginning, and the task is to evaluate the fermionic determinant (10.8) for the quasiconstant background field (10.11).

By analogy with four-dimensional QED we call the part of action arising due to integration over spinor fields by the Eulier–Heisenberg (EH) effective action:

$$S_{\mathrm{EH}}[A] = -i \ln D[A; \tilde{\lambda}] . \tag{10.16}$$

We first calculate (10.16) considering the gauge field as a pure Abelian background field with a constant field strength tensor, and then we will easily generalize obtained results to the case (10.11) under consideration.

To calculate (10.16) in arbitrary dimensions d we can use the proper time method that was developed by Schwinger[40] to perform similar calculations in four-dimensional QED. For convenience we calculate the EH action in Minkowski space. Using the fact that the trace of an odd number of γ matrices is equal to zero, we easily prove the identity

$$\mathrm{Sp} \ln(b + B^{\mu}\gamma_{\mu}) = \mathrm{Sp} \ln(b - B^{\mu}\gamma_{\mu}) ,$$

where the symbol Sp means the trace operation with respect to Lorentz indices as well as spatial variables, and B and b are arbitrary operators acting in coordinate space. Making use of the last equation we get

$$\begin{aligned} S_{\mathrm{EH}}[A] &= -i \ \mathrm{Sp} \ln \left[(\hat{D} - m + i\epsilon)(\hat{\partial} - m + i\epsilon)^{-1} \right] \\ &= -i \ \mathrm{Sp} \ln \left[(\hat{D} + m - i\epsilon)(\hat{\partial} + m - i\epsilon)^{-1} \right] , \end{aligned} \tag{10.17}$$

where

$$D_{\mu} \equiv i\partial_{\mu} - \tilde{\lambda} A_{\mu} , \tag{10.18}$$

and thus

$$S_{\mathrm{EH}}[A] = -\frac{i}{2} \mathrm{Sp} \ln \left[(\hat{D}^2 - m^2 + i\epsilon)(\partial^2 - m^2 + i\epsilon)^{-1} \right] . \tag{10.19}$$

By virtue of the relation

$$\hat{D}^2 = (i\partial_{\mu} - \tilde{\lambda} A_{\mu})^2 - \frac{\tilde{\lambda}}{2} \sigma_{\mu\nu} F^{\mu\nu} , \tag{10.20}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] ,$$

and by using the identity

$$\ln \frac{a}{b} = \int_0^{\infty} \frac{ds}{s} \left(e^{is(b+i\epsilon)} - e^{-is(a+i\epsilon)} \right) ,$$

we can rewrite (10.19) in the form[g]

$$S_{\rm EH}[A] = \int d^dx\, L_{\rm EH}(x)\,,$$
$$L_{\rm EH}(x) = -\frac{i}{2}\int_{-\infty}^{0} \frac{d\tau}{\tau}\,{\rm sp}[U_A(x,x';\tau) - U_0(x,x';\tau)]_{x\to x'}\,. \tag{10.21}$$

Here the operator

$$\hat{U}_A(\tau) = e^{-i\hat{H}\tau} \tag{10.22}$$

is one which describes the evolution of a quantum-mechanical system (a particle interacting with an external electromagnetic field in d-dimensional space) in the proper time τ, governed by the "Hamiltonian"

$$\hat{H} = (p_\mu - \tilde{\lambda}A_\mu)^2 - \frac{\tilde{\lambda}}{2}\sigma_{\mu\nu}F^{\mu\nu} - m^2 + i\epsilon\,, \tag{10.23}$$

taken in x representation:

$$p_\mu = i\partial_\mu\,, \qquad [x_\mu, p_\nu] = -ig_{\mu\nu}\,, \tag{10.24}$$

$$\langle x|x'\rangle = \delta^{(d)}(x - x')\,. \tag{10.25}$$

The corresponding "Feynman transition amplitude" reads

$$U_A(x,x';\tau) = \langle x|\hat{U}_A(\tau)|x'\rangle = \langle x(\tau)|x'(0)\rangle\,, \tag{10.26}$$

where $|x(\tau)\rangle = \hat{U}_A^+(\tau)|x\rangle$ is the eigenfunction of the coordinate operator in Heisenberg representation.

It is easy to show that the quantity $U_A(x,x';\tau)$ obeys the equation

$$i\partial_\tau U_A(x,x';\tau) = H_A(x,p)U_A(x,x';\tau) \tag{10.27}$$

and the boundary conditions

$$\lim_{\tau\to 0} U_A(x,x';\tau) = \delta^{(d)}(x - x')\,, \tag{10.28}$$

$$\lim_{\tau\to -\infty} U_A(x,x';\tau) = 0\,. \tag{10.29}$$

The former boundary condition is obvious and the latter is fulfilled due to the presence of an imaginary addition[h] $i\epsilon$ to m^2 in (10.23).

Equation (10.27) may be rewritten in the form

$$i\partial_\tau\langle x(\tau)|x'(0)\rangle = \langle x(\tau)|H_A(x(\tau),p(\tau))|x'(0)\rangle\,, \tag{10.30}$$

[g]The remaining symbol, sp, means the trace operation only with respect to Lorentz indices.

[h]As is easy to see, this addition provides the convergence of the integral over τ in (10.21) as $\tau \to -\infty$.

where $x(\tau) = U^+(\tau)xU(\tau)$ and $p(\tau) = U^+(\tau)pU(\tau)$ are the coordinate and impulse operators in Heisenberg representation. To find the solution of (10.39) with boundary conditions (10.28), (10.29) we first solve the proper time dynamical equations[i]

$$\frac{dx_\mu}{d\tau} = i[H_A, x_\mu(\tau)] = -2D_\mu(\tau)\,, \tag{10.31}$$

$$\frac{dD_\mu(\tau)}{d\tau} = i[H_A, D_\mu(\tau)] = -2\tilde{\lambda}F_{\mu\rho}D^\rho(\tau)\,. \tag{10.32}$$

Besides, the equations

$$[i\partial_\mu^x - \tilde{\lambda}A_\mu(x)]\langle x(\tau)|x'(0)\rangle = \langle x(\tau)|D_\mu(\tau)|x'(0)\rangle\,, \tag{10.33}$$

$$[i\partial_\mu^{x'} - \tilde{\lambda}A_\mu(x')]\langle x(\tau)|x'(0)\rangle = \langle x(\tau)|D_\mu(0)|x'(0)\rangle\,, \tag{10.34}$$

reflecting the right relations between coordinate and impulse, must be taken into account.

Solving the set of equations (10.28)–(10.34) we get

$$\begin{aligned} U_A(x,x';\tau) &= a_d \exp\left[i\tilde{\lambda}\int_x^{x'} d\xi^\mu A_\mu(\xi)\right]\frac{1}{\tau^{d/2}} \\ &\times \exp\left\{\frac{i}{4}(x-x')^\mu[\tilde{\lambda}F\coth(\tilde{\lambda}F\tau)]_{\mu\nu}(x-x')^\nu\right\} \\ &\times \exp[-M(\tau)]\exp\left(\frac{i}{2}\tilde{\lambda}\tau\sigma^{\mu\nu}F_{\mu\nu}\right)\exp[i\tau(m^2-i\epsilon)]\,, \end{aligned} \tag{10.35}$$

where integration over ξ is performed along the straight line joining points x and x',

$$a_d \equiv (2\sqrt{\pi})^{-d}\exp\left(i\frac{\pi d}{4}\right), \tag{10.36}$$

$$M(\tau) \equiv \frac{1}{2}\,\mathrm{sp}\,\ln\left[(\tilde{\lambda}F\tau)^{-1}\sinh(\tilde{\lambda}F\tau)\right]. \tag{10.37}$$

Substituting (10.35) into (10.21) we obtain

$$\begin{aligned} L_{\mathrm{EH}}(A) = &-\frac{i}{2}a_d\int_{-\infty}^0 \frac{d\tau}{\tau^{1+\frac{d}{2}}}\,e^{i(m^2-i\epsilon)} \\ &\times \mathrm{sp}\left[e^{-M(\tau)}\exp\left(\frac{i}{2}\tilde{\lambda}\tau\sigma^{\mu\nu}F_{\mu\nu}\right)-1\right]. \end{aligned} \tag{10.38}$$

[i]In the second equation two terms containing derivatives of $F_{\mu\nu}$ are omitted, in accordance with our condition $F_{\mu\nu} = \mathrm{const}$.

We shall now calculate the quantities $M(\tau)$ and $\operatorname{sp}\exp\left(\frac{i}{2}\tilde{\lambda}\tau\sigma^{\mu\nu}F_{\mu\nu}\right)$ in $d = 2$, $d = 3$ and $d = 4$.

Let us first recall that in two dimensions, Dirac fields are two-component objects[j] and Dirac matrices may be chosen to be Pauli matrices:

$$\gamma^0 = \sigma^1\,, \quad \gamma^1 = i\sigma^2\,, \quad \gamma_5 = -\sigma^3\,, \tag{10.39}$$

with the following obvious connection among them:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5 \quad (\mu,\nu = 0,1)\,, \tag{10.40}$$

where $\epsilon^{\mu\nu}$ is an absolutely antisymmetric tensor with

$$\epsilon^{01} = -\epsilon_{01} = 1\,.$$

In three-dimensional space–time the Dirac algebra is also realized with 2×2 Pauli matrices, and γ matrices may be chosen as follows:[k]

$$\gamma^0 = \sigma^3\,, \quad \gamma^1 = i\sigma^2\,, \quad \gamma_3 = i\sigma^1\,. \tag{10.41}$$

They satisfy the relation

$$\gamma^\mu\gamma^\nu = i\epsilon^{\mu\nu\alpha}\gamma_\alpha + g^{\mu\nu} \quad (\mu,\nu = 0,1,2)\,, \tag{10.42}$$

where $\epsilon^{\mu\nu\alpha}$ is an absolutely antisymmetric tensor with

$$\epsilon^{012} = \epsilon_{012} = 1\,.$$

In four dimensions we shall use a conventional representation of (4×4) γ matrices.

To evaluate the quantity $\exp[-M(\tau)]$, which in virtue of (10.37) is determined by a determinant:

$$\exp[-M(\tau)] = \det{}^{-\frac{1}{2}}\|L(\tau)\|\,, \tag{10.43}$$

$$\|L(\tau)\| \equiv L(\|F\|) = (\tilde{\lambda}\|F\|\tau)^{-1}\sinh(\tilde{\lambda}\|F\|\tau)\,, \tag{10.44}$$

it is first necessary to find the eigenvalues of the matrix $\|F\|$ in different dimensions. To this end we can iterate the eigenvalue equation

$$F^{\mu\nu}\Phi_\nu = q\Phi^\mu \tag{10.45}$$

by introducing the auxiliary matrix $\|K\|$:

$$K_{\mu\nu} = F_{\mu\alpha}F^{\alpha\beta}F_{\beta\nu} \quad (d = 2,3)\,,$$

$$K_{\mu\nu} = F_\mu^\alpha F_\alpha^\beta F_\beta^\gamma F_{\gamma\nu} \quad (d = 4)\,.$$

[j] Aside from any further degrees of freedom associated with internal symmetry.

[k] There is no γ_5 matrix in three dimensions since no matrix anticommutes with all three Pauli matrices.

One then easily gets

$$\|K\| = -\frac{1}{2}F^2\|F\| = \tilde{F}^2\|F\| \quad (d=2)\,, \tag{10.46}$$

$$\|K\| = -\frac{1}{2}F^2\|F\| = -\tilde{F}^2\|F\| \quad (d=3)\,, \tag{10.47}$$

$$\|K\| = -\frac{1}{2}F^2\|F\|^2 + \frac{1}{16}(\tilde{F}F)^2\|\mathbf{1}\| \quad (d=4)\,, \tag{10.48}$$

where the corresponding dual strength tensors are defined as follows:

$$F_{\mu\nu} = -\epsilon_{\mu\nu}\tilde{F}\,, \qquad \tilde{F} = \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu} \quad (d=2)\,, \tag{10.49}$$

$$F_{\mu\nu} = \epsilon_{\mu\nu\alpha}\tilde{F}^{\alpha}\,, \qquad \tilde{F}^{\mu} = \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \quad (d=3)\,, \tag{10.50}$$

$$F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tilde{F}^{\alpha\beta}\,, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,, \quad (d=4)\,, \tag{10.51}$$

and

$$\begin{aligned} &F^2 \equiv F^{\mu\nu}F_{\mu\nu}\,, && \tilde{F}^2 = -\frac{1}{2}F^2 \quad (d=2)\,,\\ &\tilde{F}^2 \equiv \tilde{F}^{\mu}\tilde{F}_{\mu} = \frac{1}{2}F^2 \quad (d=3)\,, && (\tilde{F}F) \equiv \tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \quad (d=4) \end{aligned} \tag{10.52}$$

are scalar functions.

Combining then Eqs. (10.45)–(10.48) we find eigenvalues q:

$$q_{1,2} = \pm\tilde{F} \quad (d=2)\,, \tag{10.53}$$

$$q_0 = 0\,, \quad q_{1,2} = \pm i|\tilde{F}| \equiv \sqrt{\tilde{F}^{\mu}\tilde{F}_{\mu}} \quad (d=3)\,, \tag{10.54}$$

$$(q_{1,2})^2 = -\frac{1}{4}\left[F^2 \pm \sqrt{(F^2)^2 + (F\tilde{F})^2}\right] \quad (d=4)\,. \tag{10.55}$$

Making use of (10.43), (10.53)–(10.55) we calculate $e^{-M(\tau)}$. So,

$$\begin{aligned} e^{-M(\tau)} &= [L(q_1)L(q_2)]^{-\frac{1}{2}}\\ &= \left[\frac{\tilde{\lambda}\tau q_1}{\sinh(\tilde{\lambda}\tau q_1)}\,\frac{\tilde{\lambda}\tau q_2}{\sinh(\tilde{\lambda}\tau q_2)}\right]^{\frac{1}{2}} = \frac{\tilde{\lambda}\tau\tilde{F}}{\sinh(\tilde{\lambda}\tau\tilde{F})} \end{aligned} \tag{10.56}$$

in two-dimensional space,

$$\begin{aligned} e^{-M(\tau)} &= [L(q_0)L(q_1)L(q_2)]^{-\frac{1}{2}}\\ &= \left[\frac{\tilde{\lambda}\tau q_0}{\sinh(\tilde{\lambda}\tau q_0)}\,\frac{\tilde{\lambda}\tau q_1}{\sinh(\tilde{\lambda}\tau q_1)}\,\frac{\tilde{\lambda}\tau q_2}{\sinh(\tilde{\lambda}\tau q_2)}\right]^{\frac{1}{2}} = \frac{\tilde{\lambda}\tau|\tilde{F}|}{\sin(\tilde{\lambda}\tau|\tilde{F}|)} \end{aligned} \tag{10.57}$$

in three-dimensional space, and

$$e^{-M(\tau)} = \prod_{i=1}^{4} L^{\frac{1}{2}}(q_i) = \tilde{\lambda}^2 \tau \frac{Q_1 Q_2}{\sin(\tilde{\lambda}\tau Q_1)\sin(\tilde{\lambda}\tau Q_2)}, \tag{10.58}$$

where

$$Q_{1,2} = \frac{1}{2}\left[F^2 \pm \sqrt{(F^2)^2 + (\tilde{F}F)^2}\right]^{\frac{1}{2}}, \tag{10.59}$$

in four dimensions. The last equation may be reduced to the more convenient (conventional) form making the change of variables:

$$Q_1 = b, \qquad Q_2 = -ia.$$

Then we obtain

$$e^{-M(\tau)} = \frac{\tilde{\lambda}\tau a}{\sinh(\tilde{\lambda}\tau a)} \frac{\tilde{\lambda}\tau b}{\sin(\tilde{\lambda}\tau b)}, \tag{10.60}$$

where[1]

$$a^2 - b^2 = \mathbf{E}^2 - \mathbf{H}^2, \qquad ab = \mathbf{EH}. \tag{10.61}$$

The only remaining thing to be done now in (10.38) to accomplish calculation of L_{EH} is taking the trace over the Lorentz indices of the matrix

$$\Sigma = \exp\left(\frac{i}{2}\tilde{\lambda}\tau\sigma^{\mu\nu}F_{\mu\nu}\right). \tag{10.62}$$

In two dimensions, by using (10.39) and (10.40) we obtain

$$\sigma^{\mu\nu}F_{\mu\nu} = 2i\gamma_5\tilde{F} \equiv 2i\sigma_3\tilde{F}, \tag{10.63}$$

and hence

$$\Sigma = \cosh(\tilde{\lambda}\tau\tilde{F}) + \sigma_3\sinh(\tilde{\lambda}\tau\tilde{F}), \tag{10.64}$$

$$\mathrm{sp}\,\Sigma = 2\cosh(\tilde{\lambda}\tau\tilde{F}). \tag{10.65}$$

In three-dimensional space, by using (10.41) and (10.42) we have

$$\sigma^{\mu\nu}F_{\mu\nu} = -2\tilde{F}^{\alpha}\gamma_{\alpha}, \tag{10.66}$$

and, by expanding Σ in powers of $\tilde{F}^{\alpha}\gamma_{\alpha}$, using (10.42), and noting that

$$\epsilon^{\alpha\beta\gamma}\tilde{F}_{\alpha}\tilde{F}_{\beta} = 0,$$

[1]The validity of (10.61) can be easily verified by using the identities:

$$\frac{1}{4}(\tilde{F}F) = -\mathbf{EH}, \quad \frac{1}{2}F^2 = -\frac{1}{2}\tilde{F}^2 = \mathbf{H}^2 - \mathbf{E}^2.$$

we find that

$$\Sigma = \cos(\tilde{\lambda}\tau|\tilde{F}|) + i\gamma^{\alpha}\frac{\tilde{F}_{\alpha}}{|\tilde{F}_{\alpha}|}\sin(\tilde{\lambda}\tau|\tilde{F}|), \tag{10.67}$$

$$\mathrm{sp}\,\Sigma = 2\cos(\tilde{\lambda}\tau|\tilde{F}|). \tag{10.68}$$

In four dimensions one can perform calculation of sp Σ by using one of the well-known representations (for instance, "standard") of γ matrices. A direct calculation yields

$$\mathrm{sp}\,\Sigma = 4\cosh(\tilde{\lambda}\tau a)\sin(\tilde{\lambda}\tau b). \tag{10.69}$$

By substituting Eqs. (10.56), (10.57), (10.60) and (10.65), (10.68), (10.69) into (10.38), we get eventually

$$L_{\mathrm{EH}}(A) = -\frac{1}{4\pi}\int_{-\infty}^{0}\frac{d\tau}{\tau^2}\,e^{i\tau(m^2-i\epsilon)}\left[\tilde{\lambda}\tau\tilde{F}\coth(\tilde{\lambda}\tau\tilde{F}) - 1\right], \tag{10.70}$$

in two dimensions,

$$L_{\mathrm{EH}}(A) = -\frac{i}{8\pi^{3/2}}\,e^{\frac{3}{4}\pi i}\int_{-\infty}^{0}\frac{d\tau}{\tau^{5/2}}\,e^{i\tau(m^2-i\epsilon)}\left[\tilde{\lambda}\tau|\tilde{F}|\cot(\tilde{\lambda}\tau|\tilde{F}|) - 1\right] \tag{10.71}$$

in three dimensions, and

$$L_{\mathrm{EH}}(A) = -\frac{1}{8\pi^2}\int_{-\infty}^{0}\frac{d\tau}{\tau}\,e^{i\tau(m^2-i\epsilon)}\left[\tilde{\lambda}^2 ab\,\frac{\cosh(\tilde{\lambda}a\tau)\cos(\tilde{\lambda}b\tau)}{\sinh(\tilde{\lambda}a\tau)\sin(\tilde{\lambda}b\tau)} - \frac{1}{\tau^2}\right], \tag{10.72}$$

$$ab = \mathbf{EH}, \qquad a^2 - b^2 = \mathbf{E}^2 - \mathbf{H}^2, \tag{10.73}$$

in four-dimensional space.

Let us now analyze obtained expressions for S_{EH}. It is easy to show that $S_{\mathrm{EH}}[A]$ in the case $d < 4$, when expressions for this quantity are free of divergences, behaves at large $|F| \equiv \sqrt{F^{\mu\nu}F_{\mu\nu}}$ as a polynomial of the power $d/2$ in gauge fields and reduces to the form

$$S_{\mathrm{EH}}[A] \sim \tilde{\lambda}^{d/2}\int d^dx|F|^{d/2}, \tag{10.74}$$

where the symbol $\sim$ denotes the quality up to a numerical constant. We shall now illustrate this statement by considering, for instance, the case $d = 3$. Deforming the path of integration, $\tau \longrightarrow i\tau$, we get instead of (10.71) the equation

$$L_{\mathrm{EH}}(A) = -\frac{i}{8\pi^{3/2}}\int_{0}^{\infty}\frac{d\tau}{\tau^{5/2}}\,e^{-m^2\tau}\left[\tilde{\lambda}\tau|\tilde{F}|\coth(\tilde{\lambda}\tau|\tilde{F}|) - 1\right]. \tag{10.75}$$

Passing then in (10.75) to dimensionless variables,

$$x = \tilde{\lambda}\tau|\tilde{F}|, \qquad \epsilon = \frac{m^2}{\tilde{\lambda}|\tilde{F}|},$$

we obtain

$$L_{\mathrm{EH}}(A) = -\frac{i}{8\pi^{3/2}}\,(\tilde{\lambda}|\tilde{F}|)^{3/2}\int_0^\infty \frac{dx}{x^{5/2}}\,e^{-\epsilon x}(x\coth x - 1)\,. \tag{10.76}$$

We notice then that our approximation of large (constant) $|\tilde{F}|$ corresponds to $\epsilon \ll 1$ and, therefore, the leading asymptotic behavior of S_{EH} is determined by the equation

$$S_{\mathrm{EH}}(A) \simeq C\tilde{\lambda}^{3/2}\int d^3x|\tilde{F}|^{3/2}\,, \tag{10.77}$$

where the constant C is

$$C = \int_0^\infty \frac{dx}{x^{5/2}}\,(x\coth x - 1)\,.$$

On the other hand, when $d = 4$ the expression for L_{EH} (10.72) suffers ultraviolet divergence as $s \to 0$ and the additional subtraction of the second term in the Laurent expansion in s of the integrand[m] is necessary. We first consider, for simplicity, the case $\mathbf{E} = 0$. Then, deforming the path of integration, $\tau \to i\tau/m^2$, and passing to dimensionless field variables

$$a^* = \frac{\tilde{\lambda}}{m^2}\,a\,, \qquad b^* = \frac{\tilde{\lambda}}{m^2}\,b\,, \tag{10.78}$$

which corresponds to

$$\mathbf{E}^* = \frac{\tilde{\lambda}}{m^2}\,\mathbf{E}\,, \qquad \mathbf{H}^* = \frac{\tilde{\lambda}}{m^2}\,\mathbf{H}\,,$$

we obtain instead of (10.72) the equation

$$L_{\mathrm{EH}}(A) = -\frac{m^4}{8\pi^2}\int_0^\infty \frac{d\tau}{\tau^3}\,e^{-\tau}\left[\tau H^*\coth(\tau H^*) - 1 - \frac{\tau^2 H^{*2}}{3}\right], \tag{10.79}$$

where $H^* \equiv \sqrt{\mathbf{H}^{*2}}$ and the third term of (10.79) just corresponds to the additional subtraction and ensures convergence of the integral at the low limit. By making the change of variables $\tau = x/H^*$, the expression for L_{EH} can be rewritten in the form

$$L_{\mathrm{EH}}(A) = \int_0^\infty dx\,\frac{e^{-\epsilon x}}{x}f(x)\,, \tag{10.80}$$

where the parameter $\epsilon = 1/H^*$ satisfies the condition

$$\epsilon = \frac{1}{H^*} \ll 1\,,$$

in accordance with our approximation of a large (constant) strength tensor, and the function

[m] Which, as usual, is justified by introducing the counterterm $\sim F^2$ into the action, which being quadratic in gauge fields does not influence the convergence properties.

$$f(x) = -\frac{m^4}{8\pi^2}\frac{H^{*2}}{x^2}\left(x\coth x - 1 - \frac{1}{3}x^2\right) \tag{10.81}$$

goes to zero as $x \to 0$ and behaves for $x \gg 1$ as follows:

$$f(x) = f(\infty) + O\left(\frac{1}{x}\right),$$

where

$$f(\infty) = \frac{\tilde{\lambda}^2}{24\pi^2} H^{*2}.$$

It is easy to show that asymptotically, as $\epsilon \to 0$, for functions with such behavior the following estimate holds:

$$\int_0^\infty \frac{e^{-\epsilon x}}{x} f(x) \simeq f(\infty)\ln\frac{1}{\epsilon}, \quad \epsilon \ll 1,$$

and hence we obtain the following asymptotic expression for $L_{\rm EH}$ for $H^* \gg 1$, $E^* = 0$:

$$L_{\rm EH} \simeq \frac{\tilde{\lambda}^2}{24\pi^2} H^2 \ln H^*, \quad H^* \gg 1. \tag{10.82}$$

By analogy, one can show that in the case $E^* \gg 1$, $H^* = 0$ the function $L_{\rm EH}$ admits an estimate:

$$L_{\rm EH} \simeq -\frac{\tilde{\lambda}^2}{24\pi^2} E^2 \ln E^*, \quad E^* \gg 1. \tag{10.83}$$

It is not difficult to generalize Eqs. (10.82) and (10.83) to the case of arbitrary **E** and **H**. We have an estimate:

$$S_{\rm EH} \sim \tilde{\lambda}^2 \int d^4x\, F^2 \ln|F^*|.$$

Let us now return to the non-Abelian case of the quasiconstant gauge field (10.11) under consideration. We have seen that $S_{\rm EH}(A)$ in the Abelian case behaves at large gauge fields as follows:

$$S_{\rm EH}[A] \sim \tilde{\lambda}^{d/2} \int d^dx |F|^{d/2}, \quad d < 4, \tag{10.84}$$

$$S_{\rm EH}[A] \sim \tilde{\lambda}^2 \int d^4x\, F^2 \ln|F^*|, \quad d = 4, \tag{10.85}$$

where

$$|F| = \sqrt{F^2} \equiv \sqrt{F^{\mu\nu}F_{\mu\nu}}$$

and the symbol $*$ stands for dimensionless field variables [see (10.78)]. The generalization of this expression to the case (10.11) is obvious and looks as follows:

$$S_{\rm EH}[A] \sim \tilde{\lambda}^{d/2}\,{\rm tr}\int d^dx \|F^2\|^{d/4}\,, \quad d<4\,, \tag{10.86}$$

$$S_{\rm EH}[A] \sim \tilde{\lambda}^{2}\,{\rm tr}\int d^4x \|F^2\| \ln \|F^{*2}\|^{1/2}\,, \quad d=4\,, \tag{10.87}$$

where the matrix $\|F^2\|$ is defined as

$$\|F^2\| \equiv F^a_{\mu\nu}F^{\mu\nu}_b t^a t^b$$

and the symbol tr means the trace operation with respect to color indices.

To perform subsequent calculations we consider, for definiteness, an SU(2) gauge theory coupled to fermions in the fundamental representation with generators

$$t_a = \frac{\tau_a}{2i} \quad (a = 1,2,3)\,,$$

where τ_a are the Pauli matrices, but obtained results are easily generalized to other gauge groups.

With such choice of t_a one can see that due to the identity $\tau^a\tau^b = -\tau^b\tau^a$ the nondiagonal terms in the sum $F^a_{\mu\nu}F^{\mu\nu}_b t_a t_b$ cancel each other and we have

$$\begin{aligned}\|F^2\| &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_b \tau_a\tau_b \\ &= -\frac{1}{4}(F^1_{\mu\nu}F^{\mu\nu}_1\tau_1^2 + F^2_{\mu\nu}F^{\mu\nu}_2\tau_2^2 + F^3_{\mu\nu}F^{\mu\nu}_3\tau_3^2) = F^a_{\mu\nu}F^{\mu\nu}_a \mathbf{1}\,,\end{aligned} \tag{10.88}$$

where $\mathbf{1}$ stands for the unit matrix in the color space with tr $\mathbf{1} = 2$. Thus, up to an inessential numerical factor, we get instead of (10.86) and (10.87) the equations

$$S_{\rm EH}[A] \sim \tilde{\lambda}^{d/2}\int d^dx [F^a_{\mu\nu}F^{\mu\nu}_a]^{d/4}\,, \quad d<4\,, \tag{10.89}$$

$$S_{\rm EH}[A] \sim \tilde{\lambda}^{2}\int d^4x [F^a_{\mu\nu}F^{\mu\nu}_a]^2 \ln[F^a_{\mu\nu}F^{\mu\nu}_a]^{\frac{1}{2}}\,, \quad d=4\,, \tag{10.90}$$

where $F^a_{\mu\nu}$ is given by (10.12), and we can see that the power of $S_{\rm EH}[A]$ in gauge fields is less than that for the anharmonic VPT addition $B^2[A]$. Therefore, $S_{\rm EH}[A]$ does not get into the effective action[n] that (through simultaneous saddle points in $\tilde{\lambda}$ and A variables) determines the large VPT order behavior and, therefore, does not influence the VPT series convergence properties.

11. Conclusion

We have proposed a method for nonperturbative calculation of the functional integral, which we have called variational perturbation theory. The method is based on the mere computation of the Gaussian functional quadratures. It does not

[n]The latter, as in the pure Yang–Mills theory case, has the form (8.16).

require new diagrams and uses only those that appear in standard perturbation theory in the same order of approximation.

Very important for a nonperturbative method is the problem stability. It must be noted that the possibility of calculating corrections is not still enough for one to conclude about stability. Here the properties of convergence of the series play a special role. Indeed, if the small parameter coupling constant is present in the theory, then even divergent perturbative series regarded as asymptotical can give useful information concerning the region of the small coupling constant. Quite a different picture arises when such a small parameter is absent from the very beginning and does not emerge in a certain effective way. Here we may hope to derive reliable results only when we deal with the convergent series. Thus, in the nonperturbative approaches the tasks of calculating corrections to the main contribution and analyzing the properties of series convergence have to accompany each other.

Within this method, a quantity we are interested in is represented by a series whose convergence may be governed by variational parameters. The VPT approach allows one to obtain convergent series, for instance the Leibniz series which provides upper and lower series estimates for a given quantity. The method implies the optimal choice of parameters. However, unlike many other variational approaches, the VPT method allows us to compute corrections since we are dealing with a series and can always calculate a subsequent expansion term. Therefore we avoid the problem typical of variational approaches of the determination of stability and reliability of the result obtained. The proposed VPT given a regular method of computation of corrections, and without going beyond its scope, allows us to answer the question concerning the realistic degree of dominance of the "leading contribution."

In this paper the GEP as a first nontrivial VPT order has been derived by using one or another variational addition to the action. In other words, we have shown that VPT series possessing different structures may give rise to the same result when only the leading contribution is retained.

We have shown here how the GEP emerges in the framework of VPT in the first nontrivial order. It is important that from the very beginning we deal with a series that, in principle, allows us to calculate the corrections and, thus, to explore the question about stability of the results obtained by using the "main contribution." The possibility of calculating corrections advantageously distinguishes the VPT method from other nonperturbative approaches, where the question about stability of the results obtained, for example, by using the variational method, turns into a serious problem because of the absence of a simple algorithm for calculating corrections. Moreover, the VPT method allows one to construct a series whose convergence properties can be influenced through special parameters. It is particularly important in the essentially nonperturbative tasks, where, despite the absence of a small initial parameter, reliable results can be obtained on the basis of a series whose convergence is fast enough.

In this paper we have analyzed the properties of convergence of VPT series for the $\lambda\varphi^4$ theory obtained by various methods of varying the action functional. When the variational addition is harmonic, the VPT series is asymptotic and its higher order terms behave like the terms of standard perturbation theory. Nevertheless, the harmonic variational addition produces a certain stabilization of the results for further radiative corrections. In the regions where the partial sums of conventional perturbation theory suffer oscillations specific for asymptotic series, the VPT series gives a stable result.

The VPT method gives us the possibility of combining the advantage of the variational approaches in nonperturbative effects investigations with a possibility of calculating the corrections to the main contributions. A similar approach has been developed in Ref. 15, where a procedure analogous to our harmonic VPT method has been used. As has been noticed in this paper, the harmonic procedure gives rise to the divergent series if we keep the variational parameter values equal in all the VPT series orders. However, it was observed empirically in Refs. 24 and 41 that the results seem to converge if the variational parameter is chosen, in each order, according to the optimization condition. This induced convergence phenomenon has been discussed in detail in Ref. 42, and in Refs. 43 and 44 it has been proved that δ expansion based on a harmonic type procedure does converge in dimensions 0 and 1. We have shown that the anharmonic VPT procedure gives rise to the convergent series without redefinition of the variational parameters from order to order. This can be explained as follows: for higher order terms of the VPT series the major contribution to the functional integral comes from the field configurations that are proportional to the positive degree of the large saddle point parameter. Therefore, the effective interaction $\lambda S_4[\varphi] - \tilde{S}[\varphi]$ is dominated by the conventional term $\lambda S_4[\varphi]$, which, as in perturbation theory, leads to an asymptotic series. A different picture arises when the action is varied with the help of an anharmonic functional. Here the degrees of fields in $\lambda S_4[\varphi]$ and $\tilde{S}[\varphi]$ are the same and the variational addition greatly influences the asymptotic behavior of higher order terms of the VPT series. In this paper we have shown that there exists a finite region of values of the variational parameters where the VPT series converges for all positive coupling constants.

An analogous situation arises also in the other theories considered here. So, in pure Yang–Mills theory the four-gluon part of interaction suppresses the three-gluon term in the large orders of VPT series. This occurs because only the four-gluon term gets into the effective action which governs the asymptotic properties of VPT series. It is remarkable that the coupling to fermionic fields does not influence the convergence properties, as was argued here.

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