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φ^{2k} OSCILLATOR IN THE STRONG COUPLING LIMIT

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A nonperturbative method for calculating functional integrals is proposed. Only Gaussian quadratures and only those types of diagrams which occur in the standard perturbation theory are used for the formulation of this method. The proposed approach is used for the consideration of the φ^{2k} anharmonic oscillator in the strong coupling limit.

Many tasks in quantum field theory demand application of nonperturbative methods. In this direction there are a lot of papers characterized by a variety of approaches. For example, the summation method for the series of perturbation theory and the asymptotic expressions for distant terms of the series are widely used.[1] The summation task for the asymptotic series of perturbation theory has functional arbitrariness that can be removed only by using additional information about the sum of the series.[2] The characteristics of the series sum, which are necessary for unambiguous summation, are unknown for the majority of field-theoretical models.

The approaches that are not directly connected with the series of perturbation theory are being developed.[3–9] To find a "leading contribution" many of the nonperturbative approaches use one of the variational procedures. Among them, for example, is the method of the Gaussian effective potential, which has often been used in recent years.[10–12]. It should be noted, however, that variational approaches faced certain difficulties connected as a rule with the estimate of accuracy of the obtained results. The reason is that the formalism realizing the variational principle does not lead to a natural scheme for calculating corrections to the basic contribution. The existence of a procedure of corrections computation is not enough to judge the properties of convergence of the series. In this connection, there remains open the question about stability and reliability of the results obtained within the variational method.[a]

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[a]See Ref. 11 about the Gaussian approximation in this connection.

In the present paper, we formulate a nonperturbative method for the φ^{2k} anharmonic oscillator (as a one-dimensional model of field theory with interaction of $\lambda\varphi^{2k}$). The starting point of the proposed approach is the functional integral; to calculate it we use only Gaussian functional quadratures, as in standard perturbation theory. We introduce for consideration some free parameters, which are then fixed from the optimization principle. The latter plays the role of some variational procedure. However, the proposed approach is not reduced to the usual variational methods. Our approach differs essentially from the latter in that its formalism contains from the very beginning the corrections computation procedure of any order, as we are dealing with a series and can always calculate a subsequent expansion term. We deal not with the estimation of the given quantity according to the variational concept, but with the representation of an initial quantity as a series called the series of variational perturbation theory (VPT), whose convergence may be governed by variational parameters, and which, in principle, makes it possible to calculate a correction of any order and thereby to study the stability of the obtained results.

The important technical peculiarity of the proposed approach is the fact that for the Lth order of VPT only those Feynman diagrams are used which determine the same order of the standard perturbation theory. Contrary to the above-mentioned asymptotic series of standard perturbation theory, the VPT series has a finite region of convergence, and the availability of free parameters makes it possible to obtain the optimal approximation for the quantity of interest. This approach allows one to obtain the Leibniz series which provides upper and lower series estimates for a given quantity, and by varying variational parameters we can get the most exact estimates of the latter.[b] Note that owing to the functional integral formalism which allows, at least formally, the consideration of an arbitrary number of dimensions, the proposed method can be applied to quantum field theory.

We explain the idea of the VPT method by the example of the simple numerical integral which can be considered as a zero-dimensional analog of the two-component scalar model in the field theory

$$Z[g] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \exp[-(S_0 + gS_1)], \tag{1}$$

where $S_0 = x_1^2 + x_2^2 = \mathbf{x}^2$ is an analog of the free action and $S_1 = x_1^{2k} + x_2^{2k}$ is the action of interaction. If we rewrite the total action in the form $S = S_0 + \theta S_0^k + gS_1 - \theta S_0^k$, we construct a new expansion of $Z[g]$:

$$Z[g] = \sum_{n=0}^{\infty} Z_n[g,\theta], \tag{2}$$

$$Z_n[g,\theta] = \frac{1}{n!} \int d\mathbf{x} (\theta S_0^k - gS_1)^n \exp[-(S_0 + \theta S_0^k)], \tag{3}$$

[b]This problem will be discussed in detail in subsequent publications.

where θ is the arbitrary parameter so far. Since $Z[g]$ is independent of the parameter θ, we can choose its value so that a finite number of series terms in (2) would provide the best approximation value (1). One can propose different versions of the optimization procedure. First, one can determine the variational parameter θ from the minimality requirement for the absolute value of the sum of the last series terms in VPT being minimal:

$$\min\left|\sum_{i=k}^{N} Z_i[g,\theta]\right|, \quad 1 \le k < N .$$

Second, since the exact value of $Z[g]$ is independent of the parameter θ, the optimization procedure can be $\frac{\partial Z^{(N)}[g,\theta]}{\partial\theta} = 0$. Third, one can require the contribution of the distant terms in the VPT series to be minimal (the so-called asymptotic optimization). The asymptotic behavior of the coefficients $Z_n[g,\theta]$ with large n is

$$Z_n[g,\theta] \underset{n\to\infty}{\sim} \frac{\sqrt{2\pi}}{n\theta^{1/k}}\sqrt{\frac{t-1}{k^3(k-1)}}\left(\frac{t-1}{t}\right)^n \exp\left(-\frac{\sqrt{n}}{\theta^{1/k}}\right), \tag{4}$$

where

$$t = \frac{2^{k-1}\theta}{g}. \tag{5}$$

Hence, we can see that the VPT series has the finite region of convergence for $g \le 2^{k-1}\theta$. In the case of $t = 1$, which corresponds to the asymptotic optimization, the VPT series becomes an alternating sign convergent series of Leibniz, and there is a possibility of carrying out upper and lower bilateral estimates of the sum of series proceeding from the first terms.

In order to reduce the integral in (3) to the Gaussian form, we carry out a number of transformations, specifically the forward and inverse Fourier transformation:

$$\exp(-A^k) = \int_{-\infty}^{\infty} du\, F(u)\exp(-iuA), \tag{6}$$

where $F(u)$ is the Fourier image of the function $\exp(-A^k)$ and

$$\frac{1}{a^\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty d\alpha\,\alpha^{\nu-1}\exp(-\alpha a). \tag{7}$$

We obtain the next expression:

$$Z_n[g,\theta] = \int_0^\infty d\alpha\,\alpha^{kn}\exp(-\alpha-\theta\alpha^k)\sum_{j=0}^{n}\frac{\theta^{n-j}}{(n-j)!\,\Gamma(kj+1)}Z_j[g], \tag{8}$$

$$Z_j[g] = \frac{1}{j!}\int d\mathbf{x}[-g(x_1^{2k}+x_2^{2k})]^j\exp(-\mathbf{x}^2), \tag{9}$$

where $Z_j[g]$ is the ordinary coefficient of perturbation theory. Then, in the first nontrivial order we find that

$$Z_0[g,\theta] = \pi \int_0^\infty d\alpha \, \exp(-\alpha - \alpha^k \theta)\,, \tag{10}$$

$$Z_1[g,\theta] = \pi \int_0^\infty d\alpha \, \alpha^k \theta \, \exp(-\alpha - \alpha^k \theta) \left[1 - \frac{g}{k!\theta} \frac{(2k-1)!!}{2^{k-1}}\right] . \tag{11}$$

Using the optimization procedure in conformity with versions 1 and 2, we obtain

$$t = t_1 = t_2 = \frac{(2k-1)!!}{k!} \,. \tag{12}$$

The results for different k are shown in Table 1. Note that the interval $(Z^{(1)}[g,\theta]$, $Z^{(0)}[g,\theta])$ θ_3 is the variational parameter for asymptotic optimization) determines the upper and lower estimates for $Z[g]$, which corresponds to the Leibniz series.

Table 1. The results of calculation of $Z[g,\theta]$ for different k in the first order of VPT, where $Z^{(1)}[g,\theta] = Z_0[g,\theta] + Z_1[g,\theta]$, $Z^{(0)}[g,\theta] = Z_0[g,\theta]$ and θ_1, θ_2, θ_3 are the variational parameters for different optimization procedures.

k	g	Z^{exact}	$Z^{(1)}[g,\theta]$ $\theta=\theta_1=\theta_2$	$Z^{(1)}[g,\theta]$ $\theta=\theta_3$	$Z^{(0)}[g,\theta]$ $\theta=\theta_3$
2	0.1	2.8025	2.7994	2.7902	2.8929
	1.0	1.8726	1.8585	1.8153	2.0599
	10.0	0.8500	0.8369	0.7920	0.9841
	100.0	0.3076	0.3016	0.2801	0.3642
	10000.0	0.0326	0.0320	0.0294	0.0391
3	0.1	2.7046	2.6881	2.6222	2.8813
	1.0	1.9919	1.9556	1.7716	2.2763
	10.0	1.2138	1.1769	0.9642	1.4680
	100.0	0.6496	0.6247	0.4711	0.8126
	10000.0	0.1552	0.1482	0.1033	0.1992
4	0.1	2.6220	2.5873	2.3596	2.8620
	1.0	2.0535	1.9974	1.5435	2.3914
	10.0	1.4391	1.3806	0.8439	1.7692
	100.0	0.9276	0.8806	0.4178	1.1837
	10000.0	0.3334	0.3131	0.0986	0.4417

In the strong coupling limit for $g \to \infty$ we find from (8) the expression for $Z[g]$ in the Nth order of VPT:

$$Z^{(N)}[g] = x^{1/k} \, \frac{\Gamma\left(N + 1 + \frac{1}{k}\right)}{g^{1/k}} \sum_{j=0}^{N} \frac{(-x)^j}{(kj+1)!\,(N-j)!} \, a_j \,, \tag{13}$$

where

$$x = \frac{2^{k-1}}{t}\,, \quad a_j = \sum_{m=0}^{j} \frac{\Gamma\left(km + \frac{1}{2}\right)\,\Gamma\left[k(j-m) + \frac{1}{2}\right]}{m!\,(j-m)!}\,. \tag{14}$$

The same results for $Z[g]$ in the first order of VPT ($g \to \infty$) for different k are shown in Table 2.

Table 2. The value $Z^{(1)}[g]$ in the first order of VPT ($g \to \infty$).

k	$Z^{\text{exact}}[g]$	$Z^{(1)}[g]$	Error (%)
2	$3.28626\, g^{-1/2}$	$3.21488\, g^{-1/2}$	2.172
3	$3.44265\, g^{-1/3}$	$3.28119\, g^{-1/3}$	4.689
4	$3.54752\, g^{-1/4}$	$3.31130\, g^{-1/4}$	6.658

Further, let us consider the task of calculating the ground state energy for the anharmonic oscillator with the Euclidean action

$$S = S_0 + \frac{1}{2}\, m^2 \tilde{S} + g S_1\,, \tag{15}$$

where

$$S_0 = \frac{1}{2}\int_{-\infty}^{\infty} dt\;\dot{\varphi}^2\,, \quad \tilde{S} = \int_{-\infty}^{\infty} dt\;\varphi^2\,, \quad S_1 = \int_{-\infty}^{\infty} dt\;\varphi^{2k} \tag{16}$$

in the strong coupling limit as $g/m^{k+1} \to \infty$. We proceed to dimensionless variables

$$\varphi \to g^{-\frac{1}{2(k+1)}}\varphi\,, \quad t \to g^{-\frac{1}{k+1}}t\,,$$

and consider the $\frac{\partial E_0}{\partial g}$ quantity, which is connected with the $2k$-point Euclidean Green function by the expression

$$\frac{\partial E_0}{\partial g} = g^{-\frac{k}{k+1}}\, G_{2k}(0)\,, \tag{17}$$

where

$$G_{2k}(0) = N^{-1}\int D\varphi\;\varphi^{2k}(0)\,\exp\left[-\left(S_0 + \frac{\omega^2}{2}\tilde{S} + S_1\right)\right], \tag{18}$$

$$N = \int D\varphi\,\exp\left[-\left(S_0 + \frac{\omega^2}{2}\tilde{S} + S_1\right)\right], \tag{19}$$

$$\omega^2 = m^2 g^{-\frac{2}{k+1}}\,. \tag{20}$$

We introduce an auxiliary functional in the form

$$A = \theta S_0 + \frac{\nu}{2}\tilde{S}\,,$$

with arbitrary parameters θ and ν for constructing a new expansion, and rewrite the action (15) as follows:[c]

$$S = S_0' + S_1' ,$$

where

$$S_0' = S_0 + \frac{\omega^2}{2}\tilde{S} + A^k , \quad S_1' = S_1 - A^k .$$

Carry out the expansion in powers of the new action of the interaction S_1'. Then, the VPT series is written as

$$G_{2k}(0) = N^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int D\varphi\, \varphi^{2k}(0)(-S_1')^n \exp(-S_0') . \tag{21}$$

Again, due to the fact that the exact value of $G_{2k}(0)$ is independent of θ and ν, we can use any optimization condition. Let us use the asymptotic optimization. For that we must find the asymptotic form of the functional integral

$$\int D\varphi (A^k - S_1)^n \exp[-(S_0 + A^k)] \tag{22}$$

for large n. We use substitution $\varphi \to n^{\frac{1}{2k}}\varphi$ and the functional method of steepest descent to determine the saddle point function φ_0, which gives the basic contribution to the functional integral (22):[13]

$$\varphi_0(t) = \pm\left(\sqrt{\frac{ka}{b}}\{\cosh[(k-1)\sqrt{a}(t-t_0)]\}^{-1}\right)^{\frac{1}{k-1}} ,$$

$$a = \frac{\nu}{\theta} , \quad b = \frac{2}{\theta(1 - D[\varphi_0])A^{k-1}[\varphi_0]} , \quad D[\varphi_0] = A^k[\varphi_0] - S_1[\varphi_0] ,$$

where the parameter t_0 reflects the translational invariance of the theory. The contribution of the distant terms of the series (21) will be minimal when $D[\varphi_0] = 0$. This requirement leads to the relation between the parameters θ and ν:

$$\nu_{\mathrm{opt}}[\theta] = \left(\frac{2^{k-2}}{k}\left\{\frac{(k^2-1)\,\Gamma\left[\frac{2}{k-1}\right]}{k\sqrt{\theta}\,\Gamma\left[\frac{1}{k-1}\right]}\right\}^{k-1}\right)^{\frac{2}{k+1}} . \tag{23}$$

There is a limit: $\lim_{k\to\infty} \nu(\theta) = 1/\theta$. The remaining variational parameter θ will be fixed proceeding from the finite number of expansion terms of VPT. Using again the formulas (6) and (7) and having in mind the intermediate dimensional regularization, as well as introducing differentiation with respect to the parameter α in order to achieve any power of A, we find that in the strong coupling limit

$$G_{2k}(0) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{1}{(n-j)!}\left(\frac{d}{d\alpha}\right)^{n-j} \int_{-\infty}^{\infty} du\, F(u) \frac{g_j(z^2)}{\left[1 + iu\theta(1-\alpha)^{1/k}\right]^{kj+k}} \Bigg|_{\alpha=0} , \tag{24}$$

[c]A similar method was applied in Refs. 4, 8 and 9.

where

$$z^2 = \frac{\omega^2 + iu\nu(1-\alpha)^{1/k}}{1 + iu\theta(1-\alpha)^{1/k}},$$

$$g_j(z^2) = \frac{(-1)^j}{j!} \int D\varphi\, \varphi^{2k}(0) S_1^j \exp\left[-\left(S_0 + \frac{z^2}{2}\tilde{S}\right)\right]. \tag{25}$$

$g_j(z^2)$ are ordinary coefficients for perturbation theory series. One can establish their connection with the A_n coefficients for the expansion $E_0(g)$ in the perturbation theory series[3]

$$E_0(g) = \frac{m}{2} + m \sum_{n=1}^{\infty} A_n \left(\frac{g}{m^{k+1}}\right)^n.$$

The corresponding expression has the form

$$g_j(z^2) = \frac{(1+j)A_{1+j}}{z^{(k+1)j+k}}.$$

Then, in the Nth order of our approximation we obtain (if $\nu = \nu_{\text{opt}}$ and $\omega^2 = 0$)

$$E_0^{(N)} = (k+1)g^{\frac{1}{k+1}} \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{(1+j)A_{1+j}}{(n-j)!} \left(\frac{1}{\nu}\right)^{\frac{k(j+1)+j}{2}}$$
$$\times \left\{\Gamma\left[\frac{k(j+1)-j}{2}\right] \Gamma\left[\frac{k(j+1)+j}{2}\right]\right\}^{-1} R_{n,j}(\theta), \tag{26}$$

where

$$R_{n,j}(\theta) = \int_0^\infty dx\, e^{-x}\, x^{\frac{k(j+1)-j}{2}-1}$$
$$\times \int_0^\infty dy\, y^{\frac{k(j+1)+j}{2}-1} (\theta x + y)^{k(n-j)} \exp\left[-(\theta x + y)^k\right]. \tag{27}$$

The calculational results for different k and for various optimization procedures are shown in Table 3. The "exact" numerical results for E_0 were taken from Ref. 14.

Table 3. The ground state energy $E_0^{(1)}[g]$ for different k ($g \to \infty$).

k	θ	$E_0^{\text{exact}}(g)$	$E_0^{(1)}(g)$
2	0.027926	$0.668\,g^{1/3}$	$0.663\,g^{1/3}$
3	0.038009	$0.680\,g^{1/4}$	$0.698\,g^{1/4}$
4	0.040149	$0.704\,g^{1/5}$	$0.709\,g^{1/5}$

Apart from the dominant contribution to E_0, the VPT method allows one to determine corrections to the basic contribution. This is reached by expanding in power of ω^2. As a result, for the φ^4 oscillator ($k = 2$) in the first order of VPT ($m^2 = 0$) we find that

$$E_0^{(1)} = g^{1/3}\left(0.663 + g^{-2/3}0.1407 - g^{-4/3}0.0085 + \cdots\right).$$

We can compare this result with the exact expression:

$$E_0^{\text{exact}} = g^{1/3}\left(0.668 + g^{-2/3}0.1437 - g^{-4/3}0.0088 + \cdots\right).$$

The ground state energy $E_0^{(5)}$ ($k = 2$) for different g ($m^2 = 1$) is shown in Table 4.

Table 4. The ground state energy $E_0^{(5)}(k = 2)$ for different g ($m^2 = 1$).

g	E_0^{exact}	$E_0^{(5)}$	θ_{opt}	Error (%)
0.1	0.559	0.56407	0.0255	0.906
0.5	0.696	0.69793	0.0246	0.277
1.0	0.804	0.80557	0.0241	0.220
2.0	0.952	0.95334	0.0218	0.141
50	2.499	2.50322	0.0215	0.141
200	3.931	3.93627	0.0215	0.134
1000	6.694	6.70317	0.0215	0.137
8000	13.367	13.38603	0.0229	0.142
20000	18.137	18.16315	0.0229	0.144

We use the ratio connecting the ground energy level with the propagator:[15]

$$E_0 = \frac{k+1}{2k}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[1 - G_0^{-1}(p)G(p)\right]. \tag{28}$$

Assuming that $\theta = 0$ we can rewrite expression (24) in the form

$$G_{2k}(p) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{1}{(n-j)!}\left(\frac{d}{d\alpha}\right)^{n-j}\int_{-\infty}^{\infty} du\, F(u)G_j(p, z^2)\bigg|_{\alpha=0}, \tag{29}$$

$$z^2 = \omega^2 + iu\nu(1-\alpha)^{1/k},$$

where

$$G(p) = \frac{1}{p^2 + z^2} - g(k)\frac{1}{z^{k-1}(p^2 + z^2)^2} + \cdots; \tag{30}$$

$g(k)$ and the results of the calculation are shown in Table 5.

Table 5. The results of calculation of $E_0^{(1)}[g]$ in the case $\theta = 0$.

k	$g(k)$	$E^{\text{exact}}(g)$	$E_0^{(1)}(g)$	Error (%)
2	12	$0.668\, g^{1/3}$	$0.645\, g^{1/3}$	3.41
3	22.5	$0.680\, g^{1/4}$	$0.602\, g^{1/4}$	11.49
4	105	$0.704\, g^{1/5}$	$0.602\, g^{1/5}$	14.45

Finally, we consider the construction of a nonperturbative effective potential using the proposed method. To calculate it the generating functional of connected Green functions will be necessary:

$$Z[J] = (i\Omega)^{-1} \ln W[J] , \tag{31}$$

where

$$W[J] = \int D\varphi \exp\{i[S(\varphi) + \langle J\varphi \rangle]\} .$$

The effective potential is found by the formula

$$V_{\text{eff}}[\varphi_0] = J\varphi_0 - Z[J] , \tag{32}$$

where J is the function of φ_0,

$$\varphi_0 = \frac{\partial Z[J]}{\partial J} , \tag{33}$$

since it is sufficient to use only constant source $J = \text{const}$. We introduce a variational parameter, by analogy with the previous case, as follows:

$$S[\varphi] = \left(S_0 - m^2 \tilde{S} - \frac{a^k}{\Omega^{k-1}} \tilde{S}^k \right) - \left(gS_1 - \frac{a^k}{\Omega^{k-1}} \tilde{S}^k \right) ,$$

where S_0 and S_1 are as in (16), $\tilde{S} = \frac{1}{2} \int dt\, \varphi^2$, and Ω is the "volume" of the one-dimensional space. Further, expanding the integrand exponent in powers of the new interaction action and making some transformations (in particular the forward and inverse Fourier transform), we obtain

$$W[J] = \frac{\Omega}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dC$$

$$\times \exp\left[i\Omega \left(vC - C^k\right)\right] \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} \left(\frac{d}{d\epsilon}\right)^{n-j} w_j[J, M^2] ,$$

$$w_j[J, M^2] = \frac{(-ig)^j}{j!} \left[\int dx \frac{\partial^{2k}}{\partial J^{2k}(x)} \right]^j \exp\left(-\frac{i}{2} \langle J\Delta J \rangle \right),$$

$$\Delta(p) = (p^2 - M^2 + i0)^{-1} ,$$

$$M^2 = m^2 + \epsilon^{1/k} av .$$

For example, for the case of $k = 2$ in the first order for the generating functional of coupled Green functions (31) in the limit of strong coupling we get

$$Z^{(1)}[J] = \frac{J^2}{M^2} - \frac{1}{4}(M^2)^{\frac{1}{2}} - g\left[\frac{3}{4M^2} + 3\frac{J^2}{(M^2)^{3/2}} + \frac{J^4}{(M^2)^4} \right] , \tag{34}$$

where M^2 is a new variational parameter computed by the optimization procedure. The effective potential is constructed with the use of (32)–(34), and the corresponding graphs are shown in Fig. 1.

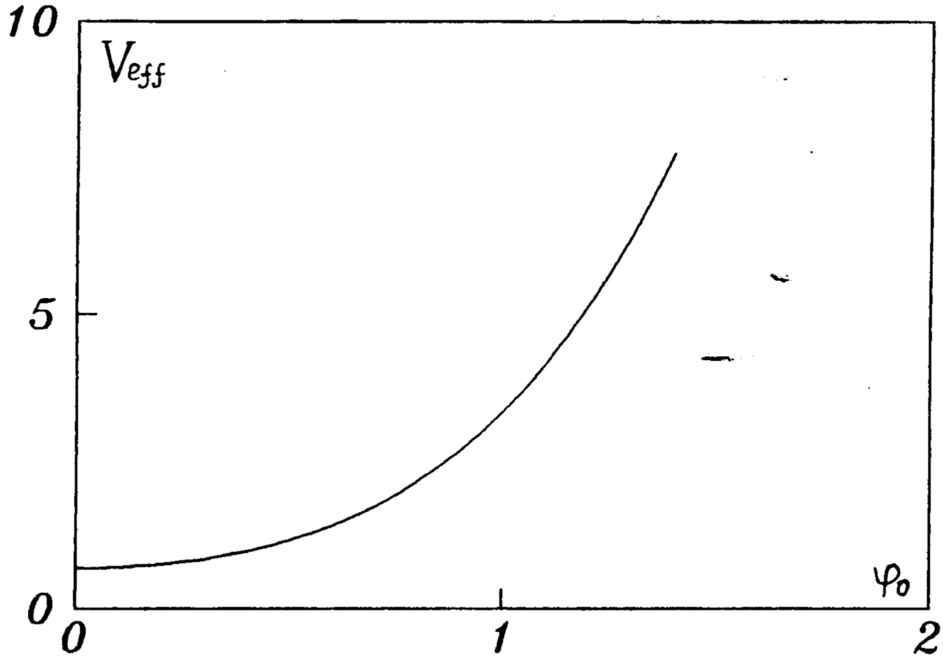

Fig. 1. The graph for the effective potential corresponding to the first order of VPT for $k = 2$ [Eqs. (32)–(34)].

We consider the anharmonic oscillator for $k = 2, 3, 4$. In the first order of VPT ($J = \text{const}$) we find that

$$W[J] = \frac{\Omega}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dC \, \exp[i\Omega S(v, C)][1 + i\Omega \, \Delta S(v, C)] \, , \tag{35}$$

where

$$S = Cv - C^k - \frac{1}{2} (M^2)^{\frac{1}{2}} \, , \tag{36}$$

$$\Delta S(v, C) = \frac{1}{4k} (M^2)^{\frac{1}{2}} - g \, (2k - 1)!! \left[-\frac{1}{2} (M^2)^{-\frac{1}{2}} \right]^k \, . \tag{37}$$

We require the optimum value of the parameter M^2 to correspond to the minimum of the absolute value of $\Delta S(v, C)$:

$$M^2 : \; \min |\Delta S(v, C)| \, .$$

In the case of k being even, the optimization condition is $\Delta S(v, C) = 0$, and since

$$V_{\text{eff}}^{(1)}(\varphi_0) = E_0^{(1)} + O(\varphi_0^2) \tag{38}$$

the ground state energy is found from the stationarity condition for the function $S(v, C)$:

$$E_0^{(1)} = -S(v_0, C_0) \, , \tag{39}$$

where

$$\left.\frac{\partial S}{\partial C}\right|_{\substack{C=C_0\\v=v_0}} = 0\,, \quad \left.\frac{\partial S}{\partial v}\right|_{\substack{C=C_0\\v=v_0}} = 0\,.$$

Then we have for even k

$$(M^2)^{1/2} = \frac{1}{2}\,[8kg(2k-1)!!]^{\frac{1}{(k+1)}}\,, \tag{40}$$

$$E_0^{(1)} = \frac{k+1}{4k}\,(M^2)^{1/2}\,,$$

$$k=2: \quad E_0^{(1)} = 0.681\; g^{1/3}\,,$$

$$k=4: \quad E_0^{(1)} = 0.792\; g^{1/5}\,.$$

For the case of $k = 3$ the optimization will consist in choosing such a real positive value of M^2 at which $|\Delta S| = \min$. Since

$$\Delta S = \frac{(M^2)^{1/2}}{12} + g\,\frac{15}{8}\,(M^2)^{-3/2}\,,$$

the parameter $(M^2)^{1/2}$ is $2.866\,\lambda^{1/4}$.

Further, two ways are possible. First, we can use the method of a stationary phase for the expression

$$\exp[i\Omega(S + \Delta S)] = \exp(i\Omega\tilde{S})\,.$$

Then, we get

$$E_0^{(1)} = -\tilde{S}(v_0, C_0) = 0.6396\,g^{1/4}\,.$$

The second method implies that owing to the smallness of ΔS we consider the expression $[1+i\Omega\,\Delta S(v,C)]$ to be a frequency factor and apply the above-mentioned method to $S(v,C)$. Then, we write

$$[1 + i\Omega\,\Delta S(v,C)] = \exp(i\Omega\,\Delta S)\,.$$

In this case, we get

$$E_0^{(1)} = -S - \Delta S = 0.6396\,g^{1/4}\,.$$

Thus, we obtain the same result as in the first case, which is a criterion of internal conformity of our approach.

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