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Convergent series in variational perturbation theory

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The problem of convergence of series of the variational perturbation theory is analyzed for the $\lambda\varphi^4_{(d)}$-model. It is shown that there exist such methods of choosing variational additions that lead to convergent series for any values of the coupling constants.

For the approaches to nonperturbative problems that are not solvable exactly it is important the following: the method should be capable of computing corrections to the so-called leading contribution found, for instance, by a variational procedure. However, the algorithm of computation of corrections is not sufficient to judge the reliability of the results obtained by a nonperturbative method or to speak about the range of validity of derived relations. Of fundamental importance here are properties of convergence of the series that approximates an initial quantity. Indeed, if in perturbation theory even a divergent series may be considered to be asymptotic and to contain a certain information on small values of the coupling constant, then in the absence of a small parameter, the approximating series are to be subjected to more rigid requirements. Actually, to obtain reasonable information, we should require this series to be convergent. It would be even more desirable to deal not only with a convergent series but with the convergent Leibniz series (an alternating series with terms decreasing in absolute value). In this case it would be possible to make upper and lower estimations on the basis of the first terms of the series, and with extra free parameters these estimates could be made maximally close to each other.

To this end we will consider a method of variational perturbation theory (VPT) [1–3]. Though the word "perturbation" is present in this approach, the VPT method is not perturbative and does not, generally, use any small parameter. The corrections can be computed in the VPT method because only calculable gaussian functional quadratures are used, like in the standard perturbation theory. And which is more, the VPT series can be written so that its terms could be computed with the use of Feynman diagrams (the series is, of course, different in structure from the perturbation series and the propagator is also modified in form).

We will start with a simple numerical example that in the functional-integral formalism can be considered as a zero-dimensional analog of the φ^4-model. Consider the integral

$$Z[g] = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \, \exp[-S_2(x,y) - gS_4(x,y)], \qquad (1)$$

where

$$S_2(x,y) = x^2 + y^2, \quad S_4(x,y) = x^4 + y^4.$$

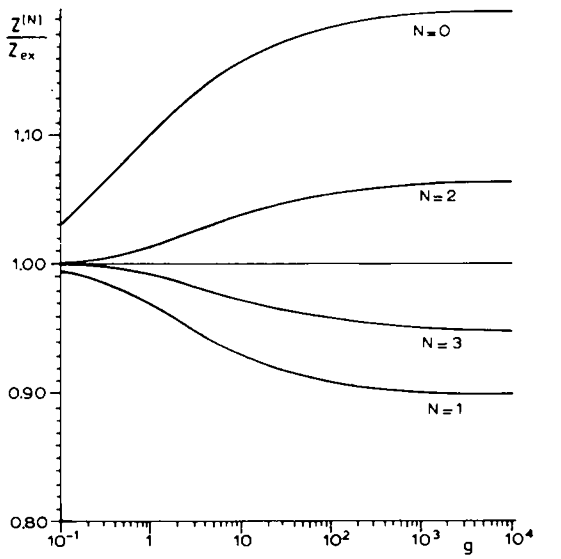

Fig. 1. A corridor of estimates of upper and lower bounds defined by the functions $Z^{(N)}/Z_{ex}$ at $N = 1,2,3$ and $t = 1$.

We take [#1] the variational term in the action in the form $g\,\frac{1}{2}tS^2$, then the VPT series is written as follows

$$Z[g] = \sum_{n=0}^{\infty} Z_n(g,t), \quad Z_n(g,t) = \frac{(-g)^n}{n!} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, (S_4 - \tfrac{1}{2}tS_2^2)^n \exp(-S_2 - g\,\tfrac{1}{2}tS_2^2)\,. \tag{2}$$

The sum of the VPT series does not depend on the parameter t therefore it can be chosen on the basis of a certain criterion of the VPT expansion being optimal.

It is easy to find out that the series (2) converges when $t > \frac{1}{2}$, which is valid for all positive g. An analog of the Sobolev inequality here is the relation

$$S_4/S_2^2 \leqslant 1\,, \tag{3}$$

from which it follows that for $t > 1$ the VPT series (2) is of positive sign. At $t = 1$ the regime changes and for $\frac{1}{2} < t < 1$ the series is the Leibniz series. Note that the value of the variational parameter, $t = 1$, passing which the alternating series becomes a series of fixed sign, is determined from the criterion of asymptotic optimization of the VPT series which minimizes the contribution of higher-order expansion terms [1].

For the Leibniz series the exact value Z_{ex} has the following bilateral estimate:

$$Z^{(2N+1)} < Z_{ex} < Z^{(2N)}, \tag{4}$$

where $Z^{(2N+1)}$ and $Z^{(2N)}$ are, respectively, odd and even partial sums of the VPT series. In fig. 1 we plot a corridor of estimates of upper and lower bounds defined by the functions $Z^{(N)}/Z_{ex}$ at $N = 1,2,3$ and $t = 1$. It is seen that even the first partial sums provide a reasonable accuracy for the whole range of coupling constant.

Now consider a massless $\varphi^4_{(4)}$ theory in the four-dimensional euclidean space with the action

$$S[\varphi] = S_0[\varphi] + \lambda S_4[\varphi], \quad S_0[\varphi] = \tfrac{1}{2}\int dx\,(\partial\varphi)^2, \quad S_p[\varphi] = \int dx\,\varphi^p. \tag{5}$$

For the generating functional of the Green functions $W[J]$ with the variational addition taken in the form $gtS_0^2[\varphi]$ we obtain the following VPT series:

[#1] We keep in mind that our computations should be based on gaussian quadratures.

$$W[J] = \int D\varphi \exp(-S[\varphi] + \langle J\varphi\rangle) = \sum_{n=0}^{\infty} W_n[J,t],$$

$$W_n[J,t] = \frac{(-g)^n}{n!} \int D\varphi\, (S_4/4C_s - t S_0^2)^n \exp(-S_0 - g t S_0^2 + \langle J\varphi\rangle) . \quad (6)$$

Here we made use of the constant $C_s = 4!/(16\pi^2)^2$ from the Sobolev inequality (see, for instance, refs. [4,5]):

$$S_4[\varphi]/S_0^2[\varphi] \leqslant 4 C_s \quad (7)$$

and set $\lambda = g/4C_s$.

The method of computation of functional integrals of the form (6) can be found in refs. [1–3]. In the given case we are interested in the problem of convergence of the series. Asymptotic estimate of remote expansion terms can be made by the functional saddle-point method [6–9]. To this end, we represent $W_n[g,t]$ in the form

$$W_n[J,t] = (-g)^n \frac{n^n}{n!} \int D\varphi \exp(-n S_{\rm eff} - n^{1/2} S_0 + n^{1/4}\langle J\varphi\rangle) , \quad (8)$$

where

$$S_{\rm eff}[\varphi] = g t S_0^2[\varphi] - \ln D[\varphi] , \quad D[\varphi] = \frac{S_4[\varphi]}{4 C_s} - t S_0^2[\varphi] . \quad (9)$$

The main contribution to the integral (8) in the leading order in the large saddle-point parameter n comes from the functions φ_0 obeying the equation $\delta S_{\rm eff}/\delta\varphi = 0$, i.e.,

$$\partial^2 \varphi_0 + \frac{a}{3!}\varphi_0^3 = 0, \quad (10)$$

and leaving the action functional to be finite. Their explicit form is as follows:

$$\varphi_0(x) = \sqrt{\frac{48}{a}} \frac{\mu}{(x - x_0)^2 + \mu^2} , \quad (11)$$

$$a = \frac{32\pi^2}{t S_0[\varphi_0]\{1 + g D[\varphi_0]\}} . \quad (12)$$

Arbitrary parameters x_0 and μ in (11) reflect the translational and scale invariance of the theory under consideration. From (11) and (12) it follows that $a^2 = g t\,(32\pi^2)^2$; as a result we obtain

$$W_n[J,t] \sim (-1)^n \frac{n^n}{n!} \left(\frac{1-t}{t}\right)^n \exp\left(-n - \sqrt{\frac{n}{g t}} + n^{1/4}\langle J\varphi_0\rangle\right) . \quad (13)$$

From this expression it is clear that irrespective of the values of the coupling constant g, the VPT series (6) absolutely converges when $t > \frac{1}{2}$ and when $t > 1$, as follows from the Sobolev inequality (7), that series is of positive sign. In the interval $\frac{1}{2} < t < 1$ at large n the series (6) is the Leibniz series. Here again the value $t = 1$ corresponds both to the change of the regime of the VPT series and to its asymptotic optimization. Note is to be made that the expression (13) determines only the leading contribution to the functional dependence of W_n on the large parameter n. In particular, in (13) we do not reproduce a certain multiplier that appears in the next to leading order in n. However, the properties of convergence of the series can be quite well analyzed in the leading order in n.

Now consider a more general case, a two-parameter VPT for the φ^4-model in the d-dimensional euclidean space. It is evident from the above analysis that the term with a source in the action and the mass term are both for nothing in studying the properties of convergence. Therefore, for simplicity we will only consider the vacuum functional $W[0]$. We perform variation of the action with the use of a two-parameter addition,

$$\tilde{S}[\varphi] = \lambda A^2[\varphi], \quad A[\varphi] = \theta S_0[\varphi] + \frac{\chi}{2} S_2[\varphi], \tag{14}$$

where θ and χ are free parameters of the variational type. The VPT series is constructed by the expansion in powers of the new interaction action $S_I = \lambda S_4 - \tilde{S}$. The asymptotic behavior of terms of the VPT series in the leading order in n is as follows:

$$W_n[0,\theta,\chi] \sim n^{-1/2} \lambda^n D^n[\varphi_0] \exp\{-n(\lambda A^2[\varphi_0] - 1)\}, \tag{15}$$

where

$$D[\varphi_0] = A^2[\varphi_0] - S_4[\varphi_0]. \tag{16}$$

The range of parameters θ and χ in which the VPT series converges is given by the inequality

$$|\lambda D[\varphi_0]| < \exp(\lambda A^2[\varphi_0] - 1).$$

The condition of asymptotic optimization $D[\varphi_0] = 0$ results in the following connection between the parameters θ and χ

$$\chi = \left(\frac{16}{\theta^n C_d^2}\right)^{1/(4-d)}, \tag{17}$$

where C_d is a known constant dependent on the space dimension d [8]. Specifically, in the one-dimensional case $C_1 = \frac{16}{3}$ and the condition (17) turns into the condition of asymptotic optimization for the anharmonic oscillator [1,2]. The two-parameter VPT is convenient because the parameter θ that remains free upon the asymptotic optimization can be fixed on the basis of optimization of the first terms of the VPT series employed in computations.

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