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δ^2-QUANTIZATION OF GAUGE FIELDS

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On the basis of the path-integral formalism in the phase space, a new scheme of quantization of gauge fields is proposed. The path integral in the configuration space is shown to contain two functional δ-functions that reflect the gauge condition and the Gauss law. A new propagator is obtained for the vector field which, for instance, for gauges $n^\mu A_\mu = 0$ distinguishes choices between time- and space-like vectors n_μ and does not lead to contradictions in the computation of the Wilson loop.

The problems of quantization of gauge fields started as early as in 1967[1] has recently attracted much interest (see Refs. 2 and 3 and references therein). A particular reason is the discovered discrepancy between the perturbative calculations of the Wilson loop, an obvious invariant object, in different gauges.[4] The discrepancy can be explained, according to Refs. 2–5, as follows: a singularity of the propagator in the Hamiltonian gauge $A_0 = 0$ with respect to the momentum K_0 should not be understood as the principal value. However, the modified propagator proposed in these papers is translationary non-invariant. Contradictions due to this non-invariance are treated in Ref. 3.

In this note we analyze another possibility to remove the above difficulty: we start with the ordinary functional integral in the phase space,[6,7] and show that transition to the configurational space can be made so that the functional integral will contain two functional δ-functions due to which the gauge condition and the Gauss law will be fulfilled. In this way quantization is accomplished only for physical degrees of freedom of vector fields; this method will be called the δ^2-quantization.

To start with, *we consider the interaction of photons with an external field* and take the conventional path integral in the phase space in the Coulomb gauge[6,7] $\partial^i A_i = 0$

$$Z[j] = \int \mathcal{D}A\mathcal{D}\pi\,\delta(\pi_0)\delta(\partial^i\pi_i + j_0) \cdot \delta(\partial^i\partial_i A_0 + j_0)\exp\left\{i\int d^4x\left[-\frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\pi_i\pi^i + \pi^i(\partial_0 A_i - \partial_i A_0) + j^\mu A_\mu\right]\right\} \quad (1)$$

as a starting point which is fixed uniquely since upon imposing the gauge $\partial^i A_i \approx 0$ in the phase space, the secondary gauge condition $\partial^i \partial_i A_0 + j_0 \approx 0$ necessarily follows from the primary of the Hamilton equations of motion,[7] like the primary $\pi_0 \approx 0$, and secondary constraints, $\partial^i \pi_i + j_0 \approx 0$. All the notation in (1) is standard. Note that the functional (1) already contains two δ-functions that make the vector field A_μ subject to the primary gauge condition, and the Gauss law, simultaneously.

Integrating (1) over momenta we obtain

$$Z[j] = \int \mathcal{D}A\,\delta(\partial^i A_i)\delta(\partial^i \partial_i A_0 + j_0) \cdot \exp\left\{ i \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu \right] \right\} . \tag{2}$$

A standard transformation of (1) into the configuration space leaves a single δ-function of the gauge condition (a δ^1-quantization), whereas the δ-function that makes A_0 subject to the Gauss law is absent. Transformation of the δ^2-functional (2) into the conventional δ^1-functional is based on the simple possibility of writing $\exp[-\frac{i}{2}(b, K^{-1}b)]$ in two ways, either

$$\exp\left[-\frac{i}{2}(b, K^{-1}b) \right] = \int \mathcal{D}x \exp\left\{ i \left[\frac{1}{2}(x, Kx) + (bx) \right] \right\} , \tag{3}$$

or

$$\exp\left[-\frac{i}{2}(b, K^{-1}b) \right] = \int \mathcal{D}x\,\delta(x - x_0) \exp\left\{ i \left[\frac{1}{2}(x, Kx) + (bx) \right] \right\} , \tag{4}$$

where x_0 is a solution to the classical equation of motion

$$Kx_0 + b = 0 . \tag{5}$$

So, to pass from (2) to the δ^1-functional which only contains $\delta(\partial^i A_i)$, it is necessary to solve the Gauss equation $\partial^2 A_0 = j_0$ in the form $A_0 = \frac{1}{\partial^2} j_0$, (which presuppose the field A_0 to be decreasing at infinity), to integrate over A_0 by $\delta(A_0 - \frac{1}{\partial^2} j_0)$, and then to write the obtained expression as a path integral of the type (3), terming a new auxiliary integration variable, as before, A_0. It is clear that in so doing we extend the integration region in the configuration space of fields A_μ and go beyond the surface dictated by quantization in the phase space and given by two conditions: the gauge and the Gauss law.

It is also obvious that both Eqs. (3) and (4) are admissible at the given step of consideration, i.e., δ^1- and δ^2-functionals at the given step of consideration are in fact equivalent.[a] However, as will be seen below, in other gauges, when the

[a] We leave, for the moment, the problem of boundary conditions which is beyond the scope of our analysis.

variable A_0 undergoes gauge transformations, the δ^1- and δ^2-ways of quantization lead to different results. The reason is that the loss of the Gauss law, when we expend the configuration space of integration, makes it necessary to impose a further condition on admissible state vectors (for gauges that do not fix gauge arbitrariness completely). In particular, in the δ^1-quantization, in gauge $A_0 = 0$ it is necessary to artificially impose the Gauss law on the state vectors.[2–7]

If we shift the integration variable: $A_0 - A_0 + \frac{1}{\partial^2} j_0$, Eq. (2) is rewritten in the form

$$Z[j] = \int \mathcal{D}A\delta(\partial^i A_i)\delta(\partial^2 A_0)\exp\{iS_{\text{eff}}(A,j)\} , \tag{6}$$

where

$$S_{\text{eff}} = S_0 + S_I^{\text{eff}} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2} j^\mu \delta^c_{\mu\nu} j^\nu + j^\mu A_\mu \right] ;$$
$$\delta^c_{\mu\nu} = \eta_\mu \eta_\nu / \partial^2 ; \quad \eta_\mu = (1,0,0,0) .$$

Thus, as distinct from the conventional approach, the effective action in (6) contains the instantaneous Coulomb interaction of charges, $\frac{1}{2}(j_0, \frac{1}{\partial^2} j_0)$, and the functional integration is made only over the physical degrees of freedom of the vector field, transverse photons.[b] The generating functional of the S-matrix corresponding to (6) (that defines the S-matrix of the mass shell) is of the form

$$S[A] = \exp(iS_0(A)) \int \mathcal{D}B\delta(\partial^i(A_i - B_i))\delta(\partial^2(A_0 - B_0))$$
$$\cdot \exp\left\{ -i\int d^4x A^\mu K^{\text{tr}}_{\mu\nu} B^\nu + iS_{\text{eff}} \right\} , \tag{7}$$

where

$$K^{\text{tr}}_{\mu\nu} = g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu .$$

A change-over to the gauge $\varphi A = 0$ in (7) is made with the aid of a nondegenerate transformation (see, e.g. Ref. 9)[c]

$$A_\mu \to A_\mu - \partial_\mu \frac{1}{\partial^2}[\varphi A - \partial^i A_i] . \tag{8}$$

As a result, $S|A|$ acquires the form

$$S[A] = \exp(iS_0(A)) \int \mathcal{D}B\delta[\varphi(A-B)]\delta[L(A-B)]$$
$$\cdot \exp\left\{ -i\int d^4x A^\mu K^{\text{tr}}_{\mu\nu} B^\nu + iS_{\text{eff}} \right\} , \tag{9}$$

[b] In this connection, see Ref. 8 where the operator method of quantization has been employed.
[c] φ_μ may be both a vector, for instance, $\varphi_\mu = n_\mu$; $\varphi_\mu = x_\mu$ (the Fock-Schwinger gauge condition $x^\mu A_\mu = 0$) and an operator for example, $\varphi_\mu = \partial_\mu$, and also any linear dimensionless combination of them.

where $L_\mu = K^{\rm tr}_{0\mu}$.
The corresponding generating functional of the Green functions looks as follows:

$$Z[j] = \int \mathcal{D}A\delta(\varphi A)\delta(LA)\exp\left\{i\left[S(A) + \int d^4x\left(\frac{1}{2}j^\mu\delta^c_{\mu\nu}j^\nu + j^\mu A_\mu\right)\right]\right\}$$
$$= \exp\left\{\frac{i}{2}\int d^4x(j^\mu\delta^c_{\mu\nu}j^\nu)\right\}Z^{\rm tr}[j]\,. \tag{10}$$

The functional $Z^{\rm tr}[j]$ determines the propagator of transverse photons $\mathcal{D}_{\mu\nu}$ in the gauge $\varphi A = 0$:

$$Z^{\rm tr}[j] = \exp\left\{\frac{i}{2}\int d^4xd^4yj^\mu(x)\mathcal{D}_{\mu\nu}(x,y)j^\nu(y)\right\}\,. \tag{11}$$

Then, according to (10), the propagator $\Delta_{\mu\nu}$ that appears in the diagram technique is given by the expression

$$\Delta_{\mu\nu} = \mathcal{D}_{\mu\nu} + \delta^c_{\mu\nu}\,. \tag{12}$$

The propagator $\mathcal{D}_{\mu\nu}$ can easily be calculated for all usable gauges. Its connection with the propagator in the δ^1-scheme can be found without difficulty. If $\tilde{\Delta}_{\mu\nu}$ is a propagator arising in the standard δ^1-approach in a certain gauge, the propagator in the same gauge is connected with $\tilde{\Delta}_{\mu\nu}$ by the formula

$$\mathcal{D}_{\mu\nu} = \tilde{\Delta}_{\mu\nu} - \frac{[\tilde{\Delta}_{\mu\alpha}L^\alpha][L^\beta\tilde{\Delta}_{\beta\nu}]}{[L^\mu\tilde{\Delta}_{\mu\nu}L^\nu]}\,. \tag{13}$$

In the gauge $nA = 0$ we get

$$\mathcal{D}_{\mu\nu}(K) = -\frac{1}{K^2+i0}\left[g_{\mu\nu} - \frac{K_\mu n_\nu + K_\nu n_\mu}{(nK)} + n^2\frac{K_\mu K_\nu}{(nK)^2} + \frac{K^2(n\eta)}{(nK)[\eta^2K^2-(\eta K)^2]}(K_\mu\eta_\nu + K_\nu\eta_\mu) - \frac{(n\eta)^2K^2K_\mu K_\nu}{(nK)^2[\eta^2K^2-(\eta K)^2]} - \frac{K^2\eta_\mu\eta_\nu}{\eta^2K^2-(\eta K)^2}\right]\,. \tag{14}$$

The propagator (14) obeys two conditions, $n^\mu\mathcal{D}_{\mu\nu} = 0$ and $L^\mu\mathcal{D}_{\mu\nu} = 0$, simultaneously. The latter term in (14) is nothing else but $-\delta^c_{\mu\nu}$. Thus the propagator $\Delta_{\mu\nu}$ defined by (12) and appearing in the Feynman diagram technique does not contain the Coulomb term. This circumstance is, of course, necessary for gauge invariance. At the same time, we see that the propagator $\Delta_{\mu\nu}$ (Eq. (14) without the last summand) arising in the δ^2-approach essentially differs from the conventional propagator $\tilde{\Delta}_{\mu\nu}$ in the gauge $nA = 0$ (the first three terms in (14)). The equality $\Delta_{\mu\nu} = \tilde{\Delta}_{\mu\nu}$ occurs when $(n\eta) = 0$, for instance, for the gauge $A_3 = 0$; but precisely for space-like n_μ the standard propagator $\tilde{\Delta}_{\mu\nu}$ does not lead to contradictions in computations of the Wilson loop. Also, attention is to be paid to the fact that the unitarity of the S-matrix in the conventional approach can explicitly be verified only for the space-like n_μ.[10]

In the gauge $A_0 = 0$ ($n_\mu = \eta_\mu$) where the δ^1-approach meets with difficulties, the propagator $\Delta_{\mu\nu}$ does not coincide with the standard one $\tilde{\Delta}_{\mu\nu}$ and reduces to the usual Coulomb propagator $\Delta_{\mu\nu}|_{n=\eta} = -\frac{1}{K^2+i0}\left\{g_{\mu\nu} + \frac{K_\mu K_\nu - K_0(\eta_\mu K_\nu + K_\mu \eta_\nu)}{\mathbf{K}^2}\right\}$, that secures the consistency of the Wilson loop calculations in various gauges.

Thus, if the δ^1-approach in the gauge $nA = 0$ leads to the same propagator for all directions of n_μ (up to going around poles $(n \cdot k)$), the δ^2-quantization essentially discriminates between the time- and space-like vectors n_μ.

The generating δ^2-functional in QED is of the form

$$Z[j;\eta,\bar{\eta}] = \int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}A\delta(\varphi A)\delta(\partial^i F_{i0} + J_0) \cdot \exp\left\{i\left[S(A;\bar{\psi},\psi) + \int d^4x(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta)\right]\right\}, \tag{15}$$

where $S(A;\bar{\psi},\psi)$ is the total action in QED, and $J_\mu = g\bar{\psi}\gamma_\mu\psi + j_\mu$. The second δ-function in (15) provides the Gauss law to hold in the functional integrand. Equation (15) can be rewritten as

$$Z[j;\bar{\eta},\eta] = \int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}A\delta(\varphi A)\delta(\partial^i F_{i0})\exp\left\{i\left[S(A,\bar{\psi},\psi) + \int d^4x(j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta) + \frac{1}{2}(J_0\boldsymbol{\partial}^{-2}J_0)\right]\right\}. \tag{16}$$

The second δ-function in (16) gives the free Gauss law, and the action contains the additional Coulomb term.

It is interesting to apply the δ^2-approach to the well known *exactly solvable gauge models*. Consider the *Bloch-Nordsiek* model differing from QED in that it contains a vector u_μ, $u^2 = 1$, instead of the Dirac matrices γ_μ and compute the gauge-invariant spinor propagator

$$G(x,y|C) = iP\exp\left\{ig\int dZ^\mu \frac{\delta}{i\delta j^\mu(Z)}\right\} \cdot \left.\frac{\delta^2 Z[j,\bar{\eta},\eta]}{\delta\bar{\eta}(x)\delta\eta(y)}\right|_{j=0;\eta=\bar{\eta}=0}. \tag{17}$$

Integration in the exponent that makes Eq. (7) gauge-invariant is performed along an arbitrary contour joining points x and y. Constructions of objects of type (17) within the Lagrangian formalism and their study can be found in Refs. 11 and 12.

Making use of the generating functional (16), writing the Coulomb term in the action as a functional integral over the variable Λ, and integrating over spinor fields we obtain

$$G(x,y|C) = \int \mathcal{D}\Lambda \exp\left[-\frac{i}{2}(\Lambda, \partial^2\Lambda)\right] \int \mathcal{D}A\delta(\varphi A)\delta(LA) \cdot \exp\left\{i\left[S_0(A) + ig\int dZ^\mu(A_\mu + \eta_\mu\Lambda)\right]\right\} \cdot G(x,y|A_\mu + \eta_\mu\Lambda)\ , \tag{18}$$

where $G(x,y|B_\mu)$ is the Green function in the external field B_μ. In deriving (18) we have taken into account that the determinant arising from integration over spinor fields is equal to unity owing to the absence of the vacuum polarization in the model. Explicitly, the Green function is

$$G(x,y|A_\mu + \eta_\mu\Lambda) = i\int_0^\infty d\nu\delta(x - y - u\nu)\exp[-i\nu(m^2 - i0)] \cdot \exp\left[-ig\int_x^y dZ^\mu(A_\mu + \eta_\mu\Lambda)\right]. \tag{19}$$

Contour integration in (19) is made along the straightline joining points x and y. Substituting (19) into (18) we get

$$G(x,y|C) = G_0(x-y)W_c[\Gamma_{xy}]W_{tr}[\Gamma_{xy}]\ , \tag{20}$$

where G_0 is the free propagator, $W_c[\Gamma_{xy}]$ and $W_{tr}[\Gamma_{xy}]$ are the Wilson loops defined, respectively, by the Coulomb term and transverse photons.

$$\begin{aligned} W_c[\Gamma_{xy}] &= \int \mathcal{D}\Lambda\exp\left\{-\frac{i}{2}(\Lambda,\partial^2\Lambda) + ig\oint_{\Gamma_{xy}} dZ^0\Lambda\right\},\\ W_{tr}[\Gamma_{xy}] &= \int \mathcal{D}A\delta(\varphi A)\delta(LA)\exp\left\{iS_0(A) + ig\oint_{\Gamma_{xy}} dZ^\mu A_\mu(Z)\right\}. \end{aligned} \tag{21}$$

The integration contour C_{xy} in (21) is shown in Fig. 1.

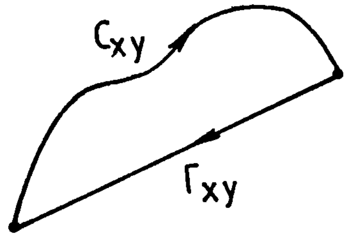

Fig. 1. The integration contour in Eq. (21).

All the possible contours, C_{xy}, can be distinguished geometrically by the straightline connecting the points x and y, and in this case the propagator $G(x,y|C)$ equals the free propagator. Therefore, the straightline contour adequately reflects the physical nature of the Bloch-Nordsiek model in which the interaction vanishes in the gauge $u^\mu A_\mu = 0$ and there occurs splitting into fermion and photon sectors non-interacting with each other.

The δ^2-quantization allows a somewhat different outlook on *two-dimensional gauge theories.* The point is that in two dimensions transverse photons are absent, their propagator $\mathcal{D}_{\mu\nu} = 0$ (which may be easily verified with the aid of Eq. (14)). Therefore, the integration over a vector field in two dimensions is trivial, and the functional (16) can be written in the form

$$Z[j;\bar{\eta},\eta] = \int \mathcal{D}[\bar{\psi}\psi] \exp\left\{ i\left[S_0(\bar{\psi},\psi) + \frac{1}{2}(J_0, \partial_1^{-2} J_0) + (\bar{\eta}\psi) + (\bar{\psi}\eta) \right] \right\} . \tag{22}$$

The functional (22) describes fermions interacting with each other by the Coulomb law and it is the same for all gauges $\varphi A = 0$. Thus, the δ^2-quantization does not admit any gauge arbitrariness in two dimensions.

In the massless case (the Schwinger model) the bosonization (22) leads to a massive scalar field with a mass $g\sqrt{\pi}$.

In the non-Abelian QCD_2 model the S-matrix is of the form (time asymptotics of the vector field obey the standard Feynman conditions of radiation):

$$S = \int_{\substack{A \longrightarrow A_{\substack{\text{out}\\ \text{in}}} \\ t\to\pm\infty}} \mathcal{D}A\,\delta(\nabla^\mu F_{\mu 0})\delta(\varphi^\mu A_\mu) \exp iS_{\text{YM}}(A) ,$$

where S_{YM} is the non-Abelian action, and ∇_μ is the conventional covariant derivative. It can be seen that in all gauges $\varphi A = 0$ including the Lorentz gauge $\partial A = 0$, there are no ghosts and self-interaction of gluon fields, and the propagator $\Delta_{\mu\nu} = -\eta_\mu\eta_\nu/\mathbf{K}^2$ is the same for all choices of the operator φ_μ fixing the gauge.

In conclusion we note that the same results can be obtained if we make use of any gauge consistent with the phase space, for instance, $A_3 = 0$, as the initial gauge for the δ^2-quantization.

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