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THREE VIEWS OF THE PROBLEM OF DEGENERATION IN THE ONE-DIMENSIONAL QUANTUM MECHANICS

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A review is given of the modern status of the degeneration problem in the one-dimensional quantum mechanics. Special attention is paid to the problem in a one-dimensional hadrogen atom. We expound three points of view on the degeneration phenomenon presently known in quantum one dimension.

Thinking about the theme of a paper devoted to the eightieth anniversary of N.N.Bogolubov we decided to consider the problem of degeneration in one-dimensional quantum mechanics. We were inspired by the idea that it were just the systems with degeneration that led N.N.Bogolubov to the discovery of two pearls of modern theoretical physics, the concept of quasiaverages and spontaneous symmetry breaking.

30 years ago it was proved that the spectrum of a one-dimensional hydrogen atom (in what follows denoted by HI) contains a ground state $E_0=-\infty$ and an infinite number of excited doubly degenerated states [1]. The latter property (degeneration) is not only strange, but it also contradicts one of basic principles of the one-dimensional quantum mechanics. Therefore it is not surprising that this problem has evoked a lot of questions, some of which have been solved quite recently.

HI is a system with a potential $U=-|x|^{-1}$; this object is perhaps usual in astrophysics. Let a hydrogen atom in a field of a pulsar ($B\sim10^{12}$) and direct the axis x along the field. It can easily be estimated that $\sqrt{y^2+z^2} = (c\hbar/eB)^{1/2} \approx 3\cdot 10^{-11}$sm, i.e. the atomic sizes perpendicular to the field are three orders as small as its longitudinal sizes. This atom diminishes in its transverse sizes, acquires a needle shape, and practically becomes an HI, i.e. there occurs a reduction of the hydrogen atom into one dimension.

The SO model. The dynamics of formation of doubly degenerated discrete levels in one dimension may be conveniently analysed for a system with a potential $\mathcal{U}(x) = \mu\omega^2 x^2/2 + \Omega\,\delta(x)$, which we have called a singular linear oscillator (SO) . The SO spectroscopy was investigated by us two years ago [2] .

To start with, we introduce, for convenience, dimensionless quantities, $\lambda = E/\hbar\omega - \frac{1}{2}$ and $\gamma = (\Omega/\hbar)(2\mu/\hbar\omega)^{1/2}$. The energy spectrum of a SO consists of two series; the first of them is usual odd levels of a linear oscillator; the second is determined by the transcendental equation with gamma-functions

$$-\gamma = \Gamma\left(\frac{1-\lambda}{2}\right)/\Gamma\left(-\frac{\lambda}{2}\right) \tag{1}$$

and it is associated with even wave functions.

From equation (1) it follows that at $\gamma = 0$ the index λ runs only over nonnegative even integers. Dependence of the first two even levels on parameter γ is given by eq. (1) and is shown in Fig. 1. Other excited levels have the same qualitative behaviour as the level $\lambda(0) = 2$.

Analysing this graph we arrive at the following conclusions: Once the delta-potential is switched on, the levels $\lambda_0 = 2,4,6...$ acquire corrections for the sign of parameter γ . With growing $|\gamma|$, these levels approach odd levels $\lambda = 3,5,7...$ and $\lambda = 1,3,5...$, resp. for $\gamma > 0$ and $\gamma < 0$. The ground level $\lambda_0 = 0$ with growing γ approache the first odd level, and for negative γ is steadily lowers and goes at negative infinity.

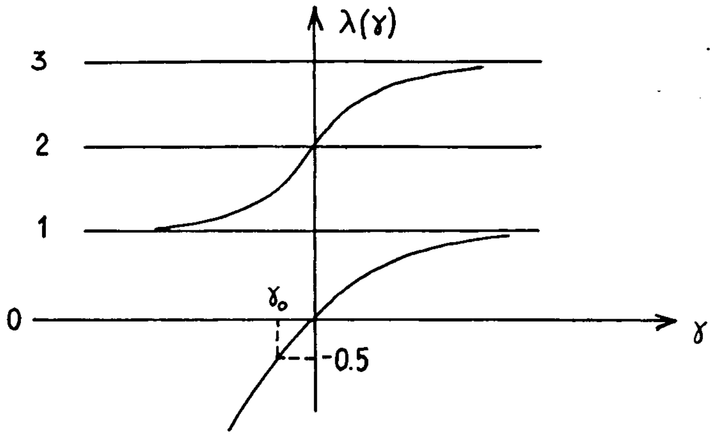

Fig. 1. Dependence of the index λ on the parameter γ (γ_0=-0.6)

Thus, in the limits $\gamma=+\infty$ and $\gamma=-\infty$ two series are formed: In the first of them ($\gamma=+\infty$) all the levels are doubly degenerated, including the ground levels, whereas in the second degenerated are only excited levels, for the ground state $\lambda=-\infty$.

Note is to be made that the second series of a SO resembles the spectrum of an HI.

A singular mechanism. Now we are ready to expound the three points of view announced in the title. The first of them deals with the so-called singular mechanism; its consistent accound may be found in ref. [3].

Let us demonstrate that mechanism for the SO. In the limits $|\gamma|=\infty$ the potential is strongly singular at point $x = 0$. The probability of a particle to penetrate from the region $x>0$ in the region $x<0$ and back is zero; thus, these regions are completely isolated from each other. A particle with a given energy may be either in the left or in the right region. These alternatives lead to the double degeneration of the spectrum, since the potential is even. Out of the wave functions of the states we may compose linear combinations of a given parity which have been discussed in the previous section.

We see that the singular mechanism is equally successful in explaining the spectrum degeneration of any one-dimensional system with a singular symmetric potential. During the last ten years it was the only standpoint on the degeneration in one dimension.

Hidden symmetry. From a quantum-theoretical point of view, the singular mechanism possesses one significant drawback: degeneration is not related to symmetry (parity of the potential does not matter as it does not lead to degeneration without singularity). May we believe that the hydrogen atom upon reduction into one dimension still preserves some information on the hidden symmetry inherent of it before reduction?

We have answered this question in ref. [4]. We have proved that an HI in the momentum space is described by the integral equation

$$\frac{P_0}{2\pi}\int_{-\infty}^{\infty}\frac{q'^2+1}{(q'-q)^2}\,\alpha(q')\,dq'=-\alpha(q) \tag{2}$$

where $P_0=-\sqrt{-2E}$, $\alpha(q)$ is the wave function of HI in the momentum representation; the proncipal value of the integral in taken. Equation (2) in one dimension represents the equation famous in the hydrogen-atom theory [5]. Using the substitution $q'=tg(\varphi'/2)$ and $q=tg(\varphi/2)$ we can rewrite eq. (2) as follows

$$\frac{P_0}{2\pi}\int_{0}^{2\pi}\frac{\Psi(\varphi')}{1-\cos(\varphi'-\varphi)}\,d\varphi'=-\Psi(\varphi) \tag{3}$$

Here $\Psi(\varphi)=(q^2+1)\alpha(q)$ is a periodic function of φ with period 2π . In view of this property it may be proved that eq. (4) is invariant under engular translations $\varphi\to\varphi+\varphi_0$, with arbitrary φ_0 .

Thus, upon the reduction into one dimension the hydrogen atom does preserve symmetry O(2) within which the double degeneration of the HI spectrum can be naturally explained from a quantum-theoretical point of view.

Supersymmetry (SUSY). That the spectrum of HI is doubly degenerated does not mean that HI may not be a supersymmetric system [6]. Does this suggest the existence of one more, third mechanism of degeneration in one dimension?

From the analysis of SO we already know that double degeneration is a result of two "mishaps": either the ground level is doubly degenerated ($\gamma=+\infty$), or it has an infinite negative energy.

Neither the first, nor second is acceptable for SUSY. Does then it follows that HI is not a supersymmetric system?

We have answered this question in ref. [7]. We made use of a method [8] forgotten for a long time to derive the differential equation [9]:

$$(q^2+1)^2\frac{d^2W}{dq^2}+2q(q^2+1)\frac{dW}{dq}+\frac{4}{P_0^2}W=0 \qquad (4)$$

where $W=(q^2+1)\alpha(q)$, and other quantities are defined in eq. (2). As in eq. (2) we also perform the substitution $q=tg(\varphi/2)$ that transforms eq. (4) into

$$\hat{h}\,G(\varphi)=\epsilon\,G(\varphi) \qquad (5)$$

where $\hat{h}=(-i\partial\varphi)$, $\epsilon=P_0^{-2}$

From eq. (5) it follows that there exists a one-to--one correspondence between HI and a two-dimensional rotator, i.e. we have shown that in the momentum space the HI is a two-dimensional rotator.

The spectrum of the rotator is $\epsilon=n^2$, where $n=$ $=0,1,2,\ldots$ all excited levels of the energy are doubly degenerated, and the ground level is described by a zero energy.

The analogy between HI and a two-dimensional rotator has been established in ref. [10], independently of us;

however, the authors do not relate that analogy with SUSY.

SUSY for the rotator is constructed as follows. We introduce the operators

$$Q=\frac{1}{2}(1-\mathcal{P})(-i\partial\varphi) \qquad Q^{+}=\frac{1}{2}(1+\mathcal{P})(-i\partial\varphi)$$

where $\mathcal{P}$ is the parity operator in variable φ. The operators Q, $\mathcal{P}$ anticommute, and therefore, Q^{+} is Hermitian conjugate to Q. It is easy to verify that

$$\hat{h}=QQ^{+}+Q^{+}Q \quad , \quad Q^{2}=0 \quad , \quad (Q^{+})^{2}=0 \qquad (6)$$

and that the operator $\hat{h}$ commutes with the operators Q and Q^{+}. In terms of SUSY relationship (6) implies that $\hat{h}$ is a super-Hamiltonian, whereas Q and Q^{+} are charge operators. According to (6), the super-Hamiltonian $\hat{h}$ splits into a sum of Hamiltonians, superpartners, $\hat{h}_{B}=Q^{+}Q$ and $\hat{h}_{F}=QQ^{+}$. The pairs ($\hat{h}_{B}$, $\cos n\varphi$) and ($\hat{h}_{F}$, $\sin n\varphi$) correspond to the so-called boson and fermion sectors of supersymmetric theories. It may be verified that the operators Q and Q^{+} connect the boson and fermion sectors with each other, as it should be made by charge operators in SUSY.

In this way one more mechanism of degeneration in one dimension was put into operation.

Conclusion. We have shown that there exist three mechanisms of double degeneration of the discrete spectrum in one dimension. In HI, all the three mechanisms are equally valid. Which of them is primary for HI? We think this question is incorrect: Indeed, one must remember that HI is only an abstraction which we obtain by reduction from three dimensions. Switching-on of a magnetic field breaks the hidden O(4) symmetry of the hydrogen atom. At the "threshold" of one dimension we have the field $U(x)=-(x^{2}+\alpha^{2})^{-1/2}$ where α is an infinitesimal number. When $\alpha\neq 0$, there is no degeneration, no singularity, no O(2) symmetry, and no SUSY. All of them appear at $\alpha=0$ instantly.

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