SUPERSYMMETRY OF A ONE-DIMENSIONAL HYDROGEN ATOM

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It is shown that a one-dimensional hydrogen atom possesses supersymmetry that manifests itself in the momentum representation. The explicit form of charge operators and analogs of boson and fermion sectors are found.

A one-dimensional atom (1DH) is a system described by the Schrödinger equation ($\hbar=e=\mu=1$)

$$\hat{\mathscr{H}}\psi(x)=(-\tfrac{1}{2}\partial_x^2-|x|^{-1})\psi(x)=E\psi(x)\,, \tag{1}$$

which also describes the realistic hydrogen atom in a strong magnetic field. For instance, for a pulsar $B\approx10^{12}$ G and when $\boldsymbol{B}=(B,0,0)$ we have $\sqrt{y^2+z^2}=\rho_c=(c\hbar/eB)^{1/2}\approx3\times10^{-11}$, i.e. the transversal dimensions of the atom are almost three orders smaller than its longitudinal dimensions (the Bohr radius), which means that in pulsar magnetic field the hydrogen atom approaches the one-dimensional system (1).

A first systematic study of the 1DH was performed by Loudon in 1959 [1]. He has established three remarkable properties:

(a) the excited levels of the 1DH discrete spectrum are described by the formula $E_n=-1/2n^2$, $n=0, 1, 2, ...$ (atomic units);

(b) the excited levels are double-degenerate, which contradicts the known assertion that the one-dimensional discrete spectrum should be nondegenerate;

(c) there exists a normal level $E_0=-\infty$ described by the wave function

$$\psi_0(x)=\lim_{\alpha\to0}\psi_\alpha(x)$$
$$=\lim_{\alpha\to0}\frac{e^{-|x|/\alpha}}{\sqrt{\alpha}}=0\,,\quad x\neq0\,,$$
$$=\infty\,,\quad x=0\,, \tag{2}$$

the integral of which over the region $(-\infty,\infty)$ equals zero, unlike the δ-function. Also zero is the scalar product of the function $\psi_0(x)$ with any square integrable function $f(x)$:

$$\int_{-\infty}^{\infty}\psi_0(x)f(x)\,dx=\lim_{\alpha\to0}\int_{-\infty}^{\infty}\psi_\alpha(x)f(x)\,dx\,. \tag{3}$$

The latter property was first indicated by Andrews [2] and Nunez Yepez and Salas Brito [3]. They threw doubt on the physical reliability of state (2). However, this problem cannot be considered as completely solved. Indeed,

$$|\psi_0(x)|^2=\lim_{\alpha\to0}|\psi_\alpha(x)|^2=\delta(x) \tag{4}$$

and therefore it cannot be stated that $\psi_0(x)\equiv0$. Maybe, in mathematics, certain objects (functions) can be introduced the square of which may be interpreted as a generalized function and the function $\psi_0(x)$ belongs to this class.

The property (b) and the reason for breaking of the assertion that the discrete spectrum is nondegenerate in one-dimensional quantum mechanics was explained in ref. [1]. From more general considerations this problem was investigated by Andrews [4] who found that the potential $U(x)=-|x|^{-1}$ is singular at the point $x=0$, as a result of which the regions $x>0$ and $x<0$ are physically not connected. A particle with a given energy may be either in the right or left region, i.e. there are two states with different wave functions. The scenario of formation of double-degenerate levels was traced for the potential $U=kx^2+\Omega\delta(x)$ in ref. [5]. The spectroscopy of this system is such that there exist levels with even and odd wave functions and these levels are alternating. The odd levels do not depend on the parameter Ω, whereas the even ones do. With growing Ω (we are so far speaking about positive Ω) every even level approaches the above odd level and at $\Omega=+\infty$ they merge thus forming a double-degenerate energy level. In this example there is a fall onto the centre realized in the limit $\Omega\to-\infty$, the normal level $E_0=-\infty$, like for 1DH, being described by the wave function (2).

The Andrews model is of a rather general character. There the particular form of the singular potential around the point $x=0$ ($U\sim-|x|^{-1}$, $U\sim-|x|^{-2}$, and so on) is not important. The only important thing is that the regions $x>0$ and $x<0$ are not connected physically.

Another aspect of the problem of degenerateness of the discrete spectrum was developed by us in ref. [6]; we found that among the systems with a potential singular at the point $x=0$ the 1DH occupies an important place. Just for the 1DH an alternative mechanism of degenerateness, hidden symmetry O(2), works. Like in three dimensions this symmetry reveals itself only in the momentum representation. In a sense, the hidden symmetry O(2) may be considered as more hidden than the Fock symmetry O(4) of a realistic hydrogen atom: in one dimension the hidden symmetry is completely "stifled" by the competing Andrews mechanism absent in three dimensions.

In this note, we suggest one more approach to the problem of degenerateness in the 1DH: we will prove that the 1DH is a supersymmetric system, which is far from being trivial. Supersymmetry of the 1DH was considered earlier in refs. [7] and [8], but the consideration was based on the fact of double degenerateness of 1DH excited states. It is, of course, a necessary but not sufficient condition of supersymmetry. We recall that the ground state energy for supersymmetric systems $E_0=0$, whereas for 1DH $E_0=-\infty$. Therefore this problem should be further analysed, and we apply the momentum representation.

When $x\neq0$, eq. (1) can be multiplied by $2|x|$ and reduced to the form

$$|x|(\hat{p}^2-2E)\psi=2\psi\,, \tag{5}$$

where $\hat{p}=-\mathrm{i}\partial_x$. We apply to eq. (5) the operator

$$\hat{K}=(\hat{p}^2-2E)|x|(\hat{p}^2-2E)$$

from the left and pass from the function $\psi(x)$ to the function

$$\chi(x)=(\hat{p}^2-2E)\psi(x)\,.$$

(A similar procedure was used in the theory of the hydrogen atom, in the Hylleraas three dimensions [9]. Details can be found in ref. [10].)

The above transformations lead to the equation

$$(\hat{p}^2-2E)|x|(\hat{p}^2-2E)|x|\chi(x)=4\chi(x)\,. \tag{6}$$

It may easily be verified that

$$|x|\hat{p}^2-\hat{p}^2|x|=2\mathrm{i}\,\mathrm{sgn}\,x\,\hat{p}+2\delta(x)\,,$$

from which we get the equality

$$|x|(\hat{p}^2-2E)|x|=(\hat{p}^2-2E)x^2+2\mathrm{i}\hat{p}x\,. \tag{7}$$

Taking (7) into account we obtain, instead of eq. (6), the equation

$$[(\hat{p}^2-2E)^2x^2+2\mathrm{i}(\hat{p}^2-2E)\hat{p}x]\chi(x)=4\chi(x)\,,$$

which does not contain the absolute value $|x|$ and we may pass to the momentum representation using the ansatz

$$\hat{p}=-\mathrm{i}\partial_x\to p\,,\quad x\to-\partial_p\,.$$

As a result, we arrive at the equation

$$(p^2-2E)^2\frac{\mathrm{d}^2\phi}{\mathrm{d}p^2}+2(p^2-2E)p\frac{\mathrm{d}\phi}{\mathrm{d}p}+4\phi=0\,, \tag{8}$$

which replaces the initial equation (5); $\phi(p)$ is the Fourier transform of the function $\chi(x)$.

We started with the second-order equation (5) and arrived again at the second-order equation (8); thus losing nothing and acquiring nothing. In ref. [11], instead of eq. (1) in the momentum representation two first order equations have been obtained. This approach is based on the ansatz

$$|x| \to \mathrm{i}\,\mathrm{sgn}\,x \frac{\mathrm{d}}{\mathrm{d}p},$$

which is valid only for the class of functions which are zero either in the region $x>0$ or in the region $x<0$ (in connection with this see ref. [12]).

Taking advantage of the replacement

$$p=\sqrt{-2E}\,\mathrm{tg}(\varphi/2) \tag{9}$$

$(-\pi \leqslant \varphi \leqslant \pi)$ we can greatly simplify eq. (8). This replacement is a one-dimensional analog of the stereographic projection used by Fock [13,14]. Introducing the notation $\phi(p) \equiv G(\varphi)$ and using (9) we obtain, instead of (8), the following equation,

$$\frac{\mathrm{d}^2 G}{\mathrm{d}\varphi^2} - \frac{1}{2E} G = 0 .$$

Thus, we have arrived at a remarkable conclusion: between the 1DH and a two-dimensional rotator there exists a one-to-one correspondence, namely, to the 1DH corresponds a two-dimensional rotator with the Hamiltonian $\hbar = (-\mathrm{i}\partial_\varphi)^2$,

$$\hbar G \equiv (-\mathrm{i}\partial_\varphi)^2 G = \epsilon G . \tag{10}$$

Here ϵ is connected with E as

$$\epsilon = -\frac{1}{2E} = n^2, \quad n=0, 1, 2, \ldots . \tag{11}$$

Note that to the ground level of the rotator $\epsilon_0 = 0$ there corresponds the 1DH ground level $E_0 = -\infty$.

Now it is easy to prove the 1DH supersymmetry. For $n=0$, 1, 2, ... to every level of the rotator (11) correspond two wave functions, even and odd:

$$G_n^{(+)} = c \cos n\varphi, \quad G_n^{(-)} = c \sin n\varphi .$$

To the ground state corresponds the wave function $G_0^{(+)} = \mathrm{const}$. Thus, we have a supersymmetric spectrum.

Let us introduce the operators

$$Q = \tfrac{1}{2}(1-\mathscr{P})(-\mathrm{i}\partial_\varphi), \quad Q^{(+)} = \tfrac{1}{2}(1+\mathscr{P})(-\mathrm{i}\partial_\varphi),$$

where $\mathscr{P}$ is the parity operator,

$$\mathscr{P} f(\varphi) = f(-\varphi) .$$

Obviously, the operators $\mathscr{P}$ and $-\mathrm{i}\partial_\varphi$ anticommute with each other, therefore it may be shown that

$$\hbar = QQ^+ + Q^+Q, \quad Q^2 = 0, \quad (Q^+)^2 = 0 ,$$

$$[Q, \hbar] = [Q^+, \hbar] = 0 .$$

From these relations it follows that $\hbar$ is a super-Hamiltonian and Q and Q^+ are the change operators. The super-Hamiltonian $\hbar$ can be represented as a sum of superpartners,

$$\hbar_{\mathrm{B}} = Q^+Q = \tfrac{1}{2}(1+\mathscr{P})(-\mathrm{i}\partial_\varphi)^2 ,$$

$$\hbar_{\mathrm{F}} = QQ^+ = \tfrac{1}{2}(1-\mathscr{P})(-\mathrm{i}\partial_\varphi)^2 .$$

The pairs $(\hbar_{\mathrm{B}}, G_n^{(+)})$ and $(\hbar_{\mathrm{F}}, G_n^{(-)})$ realise, respectively, boson and fermion sectors; the operators Q^+ and Q can be shown to connect these sectors:

$$QG_n^{(+)} = \mathrm{i}nG_n^{(-)}, \quad QG_n^{(-)} = 0 ,$$

$$Q^+G_n^{(+)} = 0, \quad Q^{(+)}G_n^{(-)} = -\mathrm{i}nG_n^{(+)} .$$

In conclusion, in the coordinate representation the 1DH falls outside the scope of the systems described by the Witten scheme [15]; whereas in the momentum representation and with stereographic projection (9) the Witten scheme becomes valid. Therefore we conclude that the 1DH supersymmetry is hidden and reveals itself only upon passing to a plane rotator equivalent to it.

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