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STANDARD
and
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KS-TRANSFORMATION AND COULOMB-OSCILLATOR INTERBASIS EXPANSIONS

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1. ABSTRACT

The problem is exactly solved of expansions of the parabolic and spherical bases of a hydrogen atom over the bases of a four-dimensional isotropic oscillator. The results are expressed in terms of the tabulated Clebsch-Gordan coefficients of SU(2) group and the Wigner function.

2. INTRODUCTION

In the paper by Kibler et al. [1] a problem important for the physics of hydrogen atoms in external fields has been formulated as how expand parabolic and spherical bases over the bases of a four-dimensional isotropic oscillator. There an expansion was established for the parabolic basis of a hydrogen atom over the double polar basis of the four-dimensional isotropic oscillator, some selection rules and bilinear relations were derived, and programs of numerical and analytic computations were constructed.

In this note, we have found an exact solution to the above-mentioned problem. Our results are expressed in terms of the well studied and tabulated "objects", the Clebsch-Gordan coefficients and the Wigner function.

3. THE KS-TRANSFORMATION

The KS (Kuctaanheimo-Steifel) transformation [2] is a square transformation

$$\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} u_4 & u_3 & u_2 & u_1 \\ -u_3 & u_4 & -u_1 & u_2 \\ -u_1 & -u_2 & u_3 & u_4 \\ -u_2 & u_1 & u_4 & -u_3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \qquad (1)$$

which makes each point (u_1, u_2, u_3, u_4) of a four-dimensional space R^4 correspond to a point (x, y, z) of a three-dimensional space R^3; and there holds the so-called Euler identity:

$$r^2 \equiv (x^2 + y^2 + z^2) = (u_1^2 + u_2^2 + u_3^2 + u_4^2)^2 \equiv u^4 \qquad (2)$$

The KS-transformation matrix is not determined uniquely.For instance, a matrix obtained from the matrix (1) by transposing the rows also leads to the identity (2). Which our choice, it follows from (1) that

$$\begin{aligned} x &= 2(u_1u_4 + u_2u_3) \\ y &= 2(u_2u_4 - u_1u_3) \\ z &= u_3^2 + u_4^2 - u_1^2 - u_2^2 \end{aligned} \qquad (3)$$

This transformation is useful in the physics of hydrogen-like systems because in terms of the variables u_i the problem of a free hydrogen atom gets identical with the problem of a four-dimensional isotropic oscillator with some auxiliary conditions. Switching on the external electric and magnetic fields breaks that simple correspondence, and in the u-space there appear anharmonic terms. The anharmonic oscillator was studied in some particular problems of QFT, and along that way powerful approximate methods were developed [3]. It may happen that the above-mentioned methods will be more efficient than the approximate methods so far applied in the physical space x,y,z. In all cases the KS-transformation connects problems of the nuclear with atomic physics, and this may provide new computational possibilities in both the two branches of physics.

Prior to formulate the connection between the spaces R^4 and R^3 in terms of the Schrödinger equation, we write one useful formula expressing derivatives in the x-space through the ones in the u-space with the matrix (1):

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \\ \frac{1}{2\ r}\hat{X} \end{pmatrix} = \frac{1}{2\ r} \begin{pmatrix} u_4 & u_3 & u_2 & u_1 \\ -u_3 & u_4 & -u_1 & u_2 \\ -u_1 & -u_2 & u_3 & u_4 \\ -u_2 & u_1 & u_4 & -u_3 \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial u_1 \\ \partial/\partial u_2 \\ \partial/\partial u_3 \\ \partial/\partial u_4 \end{pmatrix}$$

where

$$\hat{X} = u_1\partial/\partial u_2 - u_2\partial/\partial u_1 + u_4\partial/\partial u_3 - u_3\partial/\partial u_4 \qquad (4)$$

To solve the Schrödinger equation

$$\Delta_x\Psi(\vec{r}) + \frac{2\mu}{{}^2}\left[E - V(\vec{r})\right]\Psi(\vec{r}) = 0 \qquad (5)$$

is the same as finding solutions of the equation

$$\Delta_u\Phi(\vec{u}) + \frac{8\mu}{{}^2}\left[Eu^2 - u^2V(\vec{u})\right]\Phi(\vec{u}) = 0 \qquad (6)$$

that satisfy the two auxiliary conditions

a) $\Phi(-\vec{u}) = \Phi(\vec{u})$

b) $\hat{X}\Phi(\vec{u}) = 0$

The condition a) implies that the transformation (3) is quadratic, i.e. to points $\vec{u}$ and $-\vec{u}$ there corresponds one point of the space R^3. The condition b) selects, among the solution to eq. (6), only those which depend merely on three variables x, y and z. For the Coulomb field $V= -Ze^2/r$, equation (6) transforms into the Schrödinger equation for a four-dimensional isotropic oscillator

$$\Delta_u\Phi + \frac{2\mu}{\hbar^2}\left[\mathcal{E} - \mu\omega^2u^2/2\right]\Phi = 0 \qquad (7)$$

with $-4E = \mu\omega^2/2$ and $\mathcal{E} = 4Ze^2$.

The KS-transformation is thoroughly described in ref. [5], and application of it to the connection between a hydrogen atom and a four-dimensional isotropic oscillator can be found, e.q., in refs. [1] and [6].

4. COORDINATES AND BASES

We shall make use of the following coordinates and bases:

1. Four-dimensional coordinates and bases of an oscillator

a) Cartesian coordinates

$$-\infty < u_1 < \infty\ ,\ -\infty < u_2 < \infty\ ,\ -\infty < u_3 < \infty\ ,\ -\infty < u_4 < \infty$$

$$|n_1n_2n_3n_4\rangle = \lambda^2\bar{\varkappa}_{n_1}(\lambda u_1)\bar{\varkappa}_{n_2}(\lambda u_2)\bar{\varkappa}_{n_3}(\lambda u_3)\bar{\varkappa}_{n_4}(\lambda u_4) \qquad (8)$$

where $\lambda = \sqrt{\mu\omega/\hbar}$, $n_0 = n_1+ n_2+ n_3+ n_4$ is the principal quantum number, $\mathcal{E} = \hbar\omega(n_0+2)$, is the energy spectrum, and

$$\bar{\varkappa}_n(x) = \frac{e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}}\,\varkappa_n(x) \; ;$$

b) Canonical hyperspherical coordinates

$$\begin{aligned} &u_1 = u\cdot\sin\psi\cdot\sin\vartheta\cdot\sin\varphi \;, && u_3 = u\cdot\sin\varphi\cdot\cos\vartheta \\ &u_2 = u\cdot\sin\psi\cdot\sin\vartheta\cdot\cos\varphi \;, && u_4 = u\cdot\cos\psi \qquad (9) \\ &0 \le \psi \le \pi \;, \quad 0 \le \vartheta \le \pi \;, && 0 \le \varphi \le 2\pi \end{aligned}$$

$$|n_o j\; m_1 m_2\rangle = R_{n_o j}(u)\cdot Y_{jm_1m_2}(\psi,\vartheta,\varphi)$$

where $j = 0,2,\dots,n_o$ and $j = 1,3,\dots,n_o$, respectivety, for even and odd n_o, $0 \le m_1 \le j$, $|m_2| \le m_1$ *)

c) Noncanonical hyperspherical coordinates

$$\begin{aligned} &u_1 = u\cdot\sin\alpha\cdot\sin\beta \;, && u_3 = u\cdot\cos\alpha\cdot\sin\gamma \\ &u_2 = u\cdot\sin\alpha\cdot\cos\beta \;, && u_4 = u\cdot\cos\alpha\cdot\cos\gamma \qquad (10) \\ &0 \le \alpha \le \pi/2 \;, \quad 0 \le \beta \le 2\pi \;, && 0 \le \gamma \le 2\pi \end{aligned}$$

$$|n_o j\; k_1 k_2\rangle = R_{n_o j}(u)\cdot \tilde{Y}_{jk_1k_2}(\alpha,\beta,\gamma)$$

where $|k_1| \le j$, $|k_2| = 0,2,\dots,j-|k_1|$, or $|k_2| = 1,3,\dots,j-|k_1|$;

d) Double polar coordinates

$$u_1 = \rho_1\sin\beta \;, \qquad u_3 = \rho_2\sin\gamma$$

*) The bases of a four-dimensional isotropic oscillator and a hydrogen atom are explicitly given in Appendix A.

$$u_2 = \rho_1 \cos\beta \;, \qquad u_4 = \rho_2 \cos\gamma \qquad (11)$$

$$0 \le \rho_1 \le \infty \;, \quad 0 \le \rho_2 \le \infty \;, \quad 0 \le \beta \le 2\pi \;, \quad 0 \le \gamma \le 2\pi$$

$$|t_1 t_2 q_1 q_2\rangle = \lambda^2/2 \cdot f_{t_1 q_1}(\lambda^2 \rho_1^2) \cdot f_{t_2 q_2}(\lambda^2 \rho_2^2) \cdot e^{iq_1\beta + iq_2\gamma}$$

with $n_0 = 2t_1 + 2t_2 + |q_1| + |q_2|$

2. Three-dimensional coordinates and bases of a hydrogen atom

a) Spherical coordinates

$$x = r \cdot \sin\vartheta \cdot \sin\varphi \;, \qquad y = r \cdot \sin\vartheta \cdot \cos\varphi \;, \qquad z = r \cdot \cos\vartheta$$

This choice corresponds to the so-called canonical tree [7] and is convenient for further computations. The KS-transformation transforms the three-dimentional spherical coordinates into the noncanonical coordinates :

$$\vartheta = 2\alpha \;, \qquad \varphi = \beta + \gamma \qquad (12)$$

The spherical basis of a hydrogen atom is of the form :

$$|n\ l\ m\rangle = R_{nl}(r) \cdot Y_{lm}(\vartheta, \varphi) \qquad (13)$$

where n is the principal quantum number, $l = 0,1,\ldots,n$ is the orbital moment, and $|m| \le l$ is the azimuthal number.

b) Parabolic coordinates

$$x = \sqrt{\nu\mu}\,\sin\phi \;, \qquad y = \sqrt{\nu\mu}\,\cos\phi \;, \qquad z = (\nu - \mu)/2$$

The KS-transformation connects the parabolic and double polar coordinates as follows

$$\nu = 2 \cdot \rho_1^2 \;, \qquad \mu = 2 \cdot \rho_2^2 \;, \qquad \phi = \beta + \gamma \qquad (14)$$

The parabolic basis is

$$|p_1p_2m\rangle = \sqrt{2\alpha^3/n}\ \ f_{p_1m}(\alpha\nu)\cdot f_{p_2m}(\alpha\mu)\frac{e^{im\phi}}{\sqrt{2\pi}} \qquad (15)$$

where $n = p_1 + p_2 + |m| + 1$, $\alpha = \dfrac{\mu Z e^2}{\hbar^2}$.

5. THE KS-TRANSFORMATIONS OF STATE VECTORS (A)

Now let us write all the eight expansions that connect the bases of a hydrogen atom with those of a four-dimensional isotropic oscillator :

$$|p_1p_2m\rangle = \Sigma\ \langle t_1t_2q_1q_2|p_1 p_2m\rangle|t_1t_2q_1q_2\rangle \qquad (16)$$

$$|n\ l\ m\rangle = \Sigma\ \langle n_oj\ k_1k_2|n\ l\ m\rangle|n_oj\ k_1k_2\rangle \qquad (17)$$

$$|p_1p_2m\rangle = \Sigma\ \langle n_oj\ k_1k_2|p_1p_2m\rangle|n_oj\ k_1k_2\rangle \qquad (18)$$

$$|n\ l\ m\rangle = \Sigma\ \langle t_1t_2q_1q_2|n\ l\ m\rangle|t_1t_2q_1q_2\rangle \qquad (19)$$

$$|n\ l\ m\rangle = \Sigma\ \langle n_oj\ m_1m_2|n\ l\ m\rangle|n_oj\ m_1m_2\rangle \qquad (20)$$

$$|n\ l\ m\rangle = \Sigma\ \langle n_1n_2n_3n_4|n\ l\ m\rangle|n_1n_2n_3n_4\rangle \qquad (21)$$

$$|p_1p_2m\rangle = \Sigma\ \langle n_oj\ m_1m_2|p_1p_2m\rangle|n_oj\ m_1m_2\rangle \qquad (22)$$

$$|p_1p_2m\rangle = \Sigma\ \langle n_1n_2n_3n_4|p_1p_2m\rangle|n_1n_2n_3n_4\rangle \qquad (23)$$

As mentioned above, the KS-transformation transforms three-dimensional spherical and parabolic coordinates into noncanonical hyperspherical and double polar coordinates, respectively. Therefore, it may be said that the expansion coefficients or transition matrices $\langle t_1t_2q_1q_2|p_1p_2m\rangle$ and $\langle n_oj\ \ k_1k_2|n\ \ l\ \ m\rangle$ are diagonal. That the matrix $\langle t_1t_2q_1q_2|p_1p_2m\rangle$ is diagonal has been proved by Kibler et al. [1] :

$$\langle t_1t_2q_1q_2|p_1p_2m\rangle = \frac{1}{2n}\left(\frac{\pi Z}{a}\right)^{1/2}\delta_{p_1t_1}\delta_{p_2t_2}\delta_{mq_1}\delta_{mq_2} \qquad (24)$$

Here a is the Bohr radius, and n = 1,2,..... determines the energy spectrum of a hydrogen atom.

We will now show that the matrix $\langle n_o j\ k_1 k_2 | n\ l\ m \rangle$ is diagonal as well. Start with the identity (17). In the left-hand side of (17) with the KS-transformation (3) we transform the spherical coordinates into noncanonical ones ($\vartheta = 2\alpha$, $\phi = \beta + \gamma$), take into account that the functions $\exp(i k_1 \beta)$ and $\exp(i k_2 \gamma)$ are orthogonal and that the Jacobi polynomials are connected with the Legendre polynomials [8] :

$$(L + |b|)! \cdot (\sin 2\alpha)^{|b|} P_{L-|b|}^{(|b|,|b|)}(\cos 2\alpha) = 2^{|b|} P_L^{|b|}(\cos 2\alpha)$$

The result is

$$\langle n_o j\ k_1 k_2 | n\ l\ m \rangle = \frac{1}{2n} \left(\frac{\pi Z}{a}\right)^{1/2} \xi_m \delta_{l,j/2} \delta_{mk_1} \delta_{mk} \quad (25)$$

where $\xi_m = \exp\{i\pi(m + |m|)/2\}$

Thus, the matrices (24) and (25) are both diagonal. The bases of a hydrogen atom and a four-dimensional isotropic oscillator are expressed through each other only by the KS-transformation.

Now let us calculate the coefficients $\langle n_o j\ k_1 k_2 | p_1 p_2 m \rangle$ and $\langle t_1 t_2 q_1 q_2 | n\ l\ m \rangle$. Expansion of the parabolic basis of a hydrogen atom over the spherical basis is given by the Tarter matrix [9]. In accordance with ref. [10],

$$\langle n\ l\ m | p_1 p_2 m \rangle = (-1)^{l+(n-1)/2} \cdot \xi_m \cdot T_{(n-1)/2,\, p_1}^{l,m} \quad (26)$$

where the Tarter matrix is expressed in terms of the Clebsch-Gordan coefficients :

$$T^{l,m}_{a,b} = C^{l,|m|}_{a,a-b;a,b-a+|m|} \tag{27}$$

Inserting into the identity

$$\langle n_o\ j\ k_1 k_2 | p_1 p_2 m \rangle = \sum \langle n_o j\ k_1 k_2 | n\ l\ m \rangle \langle n\ l\ m\ | p_1 p_2 m \rangle \tag{28}$$

the relations (25) and (26) we get

$$\langle n_o j\ k_1 k_2 | p_1 p_2 m \rangle = \\ = \frac{1}{2n} \left(\frac{\pi Z}{a} \right)^{1/2} (-1)^{p_1 + |m| + j/2} \delta_{mk_1} \delta_{mk_2} T^{j/2,m}_{(n-1)/2,p_1} \tag{29}$$

In a similar manner we can establish the following formula

$$\langle t_1 t_2 q_1 q_2 | n\ l\ m \rangle = \frac{1}{2n} \left(\frac{\pi Z}{a} \right)^{1/2} (-1)^{t_1 + j/2} \xi_{-m} T^{l,m}_{(n-1)/2,t_1} \tag{30}$$

Thus, to complete solution of the problem, we are left to compute the coefficients $\langle n_o j m_1 m_2 | nlm \rangle$, $\langle n_1 n_2 n_3 n_4 | nlm \rangle$, $\langle n_o j\ m_1 m_2 | p_1 p_2 m \rangle$ and $\langle n_1 n_2 n_3 n_4 | p_1 p_2 m \rangle$. To this end we shall make use of the identities :

$$\langle n_o j m_1 m_2 | nlm \rangle = \sum \langle n_o j\ m_1 m_2 | n_o j\ k_1 k_2 \rangle \langle n_o j k_1 k_2 | nlm \rangle \tag{31a}$$

$$\langle n_1 n_2 n_3 n_4 | nlm \rangle = \sum \langle n_1 n_2 n_3 n_4 | n_o j\ k_1 k_2 \rangle \langle n_o j\ k_1 k_2 | nlm \rangle \tag{31b}$$

$$\langle n_o j m_1 m_2 | p_1 p_2 m \rangle = \sum \langle n_o j m_1 m_2 | t_1 t_2 q_1 q_2 \rangle \langle t_1 t_2 q_1 q_2 | p_1 p_2 m \rangle \tag{31c}$$

$$\langle n_1 n_2 n_3 n_4 | p_1 p_2 m \rangle = \sum \langle n_1 n_2 n_3 n_4 | t_1 t_2 q_1 q_2 \rangle \langle t_1 t_2 q_1 q_2 | p_1 p_2 m \rangle \tag{31d}$$

As the matrices $\langle n_o j k_1 k_2 | nlm \rangle$ and $\langle t_1 t_2 q_1 q_2 | p_1 p_2 m \rangle$ are diagonal, the above identities provide summation over "intermediate states", and only the coefficients connecting the corresponding basis of a four-dimensional isotropic oscillator are to be calculated.

5. INTERBASIS EXPANSIONS IN A FOUR-DIMENSIONAL ISOTROPIC OSCILLATOR

In total, there are 12 expansions between bases in a four-dimensional isotropic oscillator.Owing to the unitarity condition, the coefficients of "direct" and "inverse" expansions follow from each other by complex conjugation. So, it suffices to analyse the following six expansions :

$$|n_1 n_2 n_3 n_4\rangle = \sum \langle t_1 t_2 q_1 q_2 | n_1 n_2 n_3 n_4\rangle | t_1 t_2 q_1 q_2\rangle \tag{32a}$$

$$|n_1 n_2 n_3 n_4\rangle = \sum \langle n_0 j\ k_1 k_2 | n_1 n_2 n_3 n_4\rangle | n_0 j\ k_1 k_2\rangle \tag{32b}$$

$$|n_1 n_2 n_3 n_4\rangle = \sum \langle n_0 j\ m_1 m_2 | n_1 n_2 n_3 n_4\rangle | n_0 j\ m_1 m_2\rangle \tag{32c}$$

$$|t_1 t_2 q_1 q_2\rangle = \sum \langle n_0 j\ k_1 k_2 | t_1 t_2 q_1 q_2\rangle | n_0 j\ k_1 k_2\rangle \tag{32d}$$

$$|n_0 j\ m_1 m_2\rangle = \sum \langle t_1 t_2 q_1 q_2 | n_0 j\ m_1 m_2\rangle | t_1 t_2 q_1 q_2\rangle \tag{32e}$$

$$|n_0 j\ m_1 m_2\rangle = \sum \langle n_0 j\ k_1 k_2 | n_0 j\ m_1 m_2\rangle | n_0 j\ k_1 k_2\rangle \tag{32f}$$

The six coefficients in (32a-f) are explicitly expressed in terms of three structure elements : a "reduced" Wigner function $d^{\tau}_{\lambda,\nu}(\pi/2)$, Clebsch-Gordan coefficients $C^{c\gamma}_{a\alpha;b\beta}$ and objects $M^{c\gamma}_{a\alpha;b\beta}$, analytic continuations of the usual Clebsch-Gordan coefficients into one-fourth-integer values of indices. In accordance with ref. [11],

$$d^{\tau}_{\lambda,\nu}(\pi/2) = \frac{(-1)^{\lambda-\nu}}{2^{\tau}} \left\{ \frac{(\tau+\lambda)!(\tau-\lambda)!}{(\tau+\nu)!(\tau-\nu)!} \right\}^{1/2} \sum_{\sigma} (-1)^{\sigma} \binom{\tau+\nu}{\sigma} \binom{\tau-\nu}{\sigma+\lambda-\nu}$$

$$M^{c\gamma}_{a\alpha;b\beta} = \frac{\delta_{\gamma,\alpha+\beta} \cdot \Delta(a,b,c) \cdot \sqrt{2c+1}}{\Gamma(a+b-c+1)\Gamma(c-b+\alpha+1)\Gamma(c-a-\beta+1)} \cdot$$

$$\left\{ \frac{\Gamma(a+\alpha+1)\Gamma(b-\beta+1)\Gamma(c+\gamma+1)\Gamma(c-\gamma+1)}{\Gamma(a-\alpha+1)\Gamma(b+\beta+1)} \right\}^{1/2} \cdot {}_3F_2 \left\{ \begin{matrix} c-a-b, \alpha-a, -b-\beta \\ c-a-\beta+1, c-b+\alpha+1 \end{matrix} \middle| 1 \right\}$$

In the latter formula,

$$\Delta(a,b,c) = \left\{\frac{\mho(a+b-c+1)\mho(a-b+c+1)\mho(b-a+c+1)}{\mho(a+b+c+2)}\right\}^{1/2}$$

Note that $M^{c\gamma}_{a\alpha;b\beta}$ appeared earlier in the problem of expansions between bases in a three-dimensional isotropic oscillator [14] and in some problems of the quantum theory of the angular momentum [15].Tables of those $M^{c\gamma}_{a\alpha;b\beta}$ can be found in ref. [14].

Now we shall write explicit form of the studied expansion coefficients :

$$\langle t_1 t_2 q_1 q_2 | n_1 n_2 n_3 n_4 \rangle = e^{i\pi\phi}\, d^{\tau_1}_{\lambda_1\nu_1}(\pi/2)\, d^{\tau_2}_{\lambda_2\nu_2}(\pi/2) \qquad (33a)$$

$$\Phi = (2t_1 + 2t_2 - n_1 - n_3)/2\ ,\quad \lambda_1 = q_1/2\ ,\quad \lambda_2 = q_2/2$$

$$\tau_1 = (n_1 + n_2)/2\ ,\ \tau_2 = (n_3 + n_4)/2\ ,$$

$$\nu_1 = (n_2 - n_1)/2\ ,\ \nu_2 = (n_4 - n_3)/2$$

$$\langle n_o j\ k_1 k_2 | n_1 n_2 n_3 n_4 \rangle =$$

$$= e^{i\pi\phi}\, d^{\tau_1}_{s_1\nu_1}(\pi/2)\, d^{\tau_2}_{s_2\nu_2}(\pi/2)\, C^{c_1\gamma_1}_{a_1\alpha_1;b_1\beta_1} \qquad (33b)$$

$$\Phi=(2n_1+n_2+n_3+n_o-j-|k_1|)/2,\ s_1=k_1/2,\ s_2=k_2/2,$$

$$c_1=j/2,\ \gamma_1=(|k_1|+|k_2|)/2,\ a_1=(n_o-|k_1|+|k_2|)/4,$$

$$b_1=(n_o+|k_1|-|k_2|)/4,\ \alpha_1=(n_3+n_4-n_1-n_2+|k_1|+|k_2|)/4,$$

$$\beta_1=(n_1+n_2-n_3-n_4+|k_1|+|k_2|)/4$$

$$\langle n_o j\ m_1 m_2 | n_1 n_2 n_3 n_4 \rangle=$$

$$= e^{i\pi\phi} d^{\tau_1}_{r_1\nu_1}(\pi/2)\, M^{c_2\gamma_2}_{a_2\alpha_2;b_2\beta_2} M^{c_3\gamma_3}_{a_3\alpha_3;b_3\beta_3} \qquad (33c)$$

$$\Phi = (|m_2|-n_2)/2\ ,\quad r_1 = m_2/2\ ,\quad c_2 = j/2\ ,\quad \gamma_2 = m_1/2$$

$$a_2 = (n_0+m_1+1)/4\ ,\qquad \alpha_2 = (n_0-2n_4+m_1+1)/4$$

$$b_2 = (n_0-m_1-1)/4\ ,\qquad \beta_2 = (2n_4-n_0+m_1+1)/4$$

$$c_3 = (2m_1-1)/4\ ,\qquad \gamma_3 = (2|m_2|-1)/4$$

$$a_3 = (n_1+n_2+n_3+|m_2|)/4\ ,\qquad \alpha_3 = (n_1+n_2-n_3+|m_2|)/4$$

$$b_3 = (n_1+n_2+n_3-|m_2|-1)/4\ ,\quad \beta_3 = (n_3-n_1-n_2+|m_2|-1)/4$$

$$\langle n_0 j\ k_1 k_2 | t_1 t_2 q_1 q_2 \rangle = e^{i\pi\phi}\, \delta_{q_1 k_1}\, \delta_{q_2 k_2}\, C^{c_4\gamma_4}_{a_4\alpha_4;b_4\beta_4} \tag{33d}$$

$$\phi = (2t_2+n_2-j)/2\ ,\quad c_4 = j/2\ ,\quad \gamma_4 = (|q_1|+|q_2|)/2$$

$$a_4 = (t_1+t_2+|q_2|)/2\ ,\qquad \alpha_4 = (t_2-t_1+|q_2|)/2$$

$$b_4 = (t_1+t_2+|q_1|)/2\ ,\qquad \beta_4 = (t_1-t_2+|q_1|)/2$$

$$\langle t_1 t_2 q_1 q_2 | n_0 j\ m_1 m_2 \rangle =$$

$$= \quad e^{i\pi\phi}\, \delta_{q_1 m_2}\, C^{c_4\gamma_4}_{a_4\alpha_4;b_4\beta_4}\, C^{m_1 m_2}_{j/2,\lambda_1-\lambda_2;j/2,\lambda_1+\lambda} \tag{33e}$$

$$\phi = (n_0+2t_2+2m_1+2|q_2|+|m_2|-q_2)/2$$

$$\langle n_0 j\ k_1 k_2 | n_0 j\ m_1 m_2 \rangle = e^{i\pi\phi}\, \delta_{m_2 k_1}\, C^{m_1 m_2}_{j/2,s_1-s_2;j/2,s_1+s_2} \tag{33f}$$

$$\phi = (j+m_1+2|k_1|+|k_2|)/2$$

The coefficients (33b), (33c) and (33f) were computed by the methods proposed in refs. [12] and [13] which can be used for an isotropic oscillator of any dimensionality. Formula (33a) was derived as follows : First, we substituted thee xplicit form of bases into (32a), passed over to the double polar coordinates in the left-hand side of (32a), tended in both its sides ρ_1 and ρ_2 to infinity, used the orthogonality of the functions $\exp(iq_1\beta)$ and $\exp(iq_2\gamma)$, and finally compare the result with the integral representation for the Wigner function [11]

$$d^{J}_{MM'}(\beta) = \frac{i^{M-M'}}{2\pi} \left\{\frac{(J+M)!(J-M)!}{(J+M)!(J-M)!}\right\}^{1/2} \cdot$$

$$\int_0^{2\pi} \left[e^{i\frac{\varphi}{2}} \cos\frac{\beta}{2} + ie^{-i\frac{\varphi}{2}} \sin\frac{\beta}{2}\right]^{J-M'} \cdot$$

$$\left[e^{-i\frac{\varphi}{2}} \cos\frac{\beta}{2} + ie^{i\frac{\varphi}{2}} \sin\frac{\beta}{2}\right]^{J+M'} e^{iM\varphi} d\varphi$$

To derive formula (33d) it is necessary : to insert the explicit form of bases into (32d), to pass over to the noncanonical coordinates in the left-hand side of (32d), to tend in both its sides to infinity, to use the orthogonality of functions $\exp(ik_1\beta)$, $\exp(ik_2\gamma)$, and the Jacobi polynomials [8]. Then, we are to calculate the integral

$$\mathfrak{L} = \int_{-1}^{1} (\sin^2\alpha)^{t_1+|k_1|} (\cos^2\alpha)^{t_2+|k_2|} P_L^{(|k_1|,|k_2|)}(\cos 2\alpha) d\cos 2\alpha$$

where $L = (j-|k_1|-|k_2|)/2$.

The last step is to apply the Rodrigues formula for the Jacobi polynomials and the following integral representation for the Clebsch-Gordan coefficients [11] :

$$C^{c\gamma}_{a\alpha;b\beta} =$$

$$= \frac{(-1)^{a-c+\beta}}{2^{J+1}} \left\{\frac{(c+\gamma)!(J-2c)!(2c+1)}{(a-\alpha)!(a+\alpha)!(b-\beta)!(b+\beta)!(c-\gamma)!(J-2a)!(J-2b)!}\right\}^{\frac{1}{2}} \cdot$$

$$\int_{-1}^{1} (1-x)^{a-\alpha} (1+x)^{b-\beta} \frac{d^{c-\gamma}}{dx^{c-\gamma}} \left\{ (1-x)^{J-2a} (1+x)^{J-2b} \right\} dx$$

$$J = a + b + c.$$

Formula (33e) is derived by substituting (33f) and the complex conjugation of (33d) into the formula :

$$\langle t_1 t_2 q_1 q_2 | n_0 j\ m_1 m_2 \rangle = \Sigma \langle t_1 t_2 q_1 q_2 | n_0 j\ k_1 k_2 \rangle \langle n_0 j\ k_1 k_2 | n_0 j\ m_1 m_2 \rangle$$

6.THE KS-TRANSFORMATION OF STATE VECTORS (B)

We have proved the following assertion. Coefficicients of transitions between bases of a hydrogen atom and those of a four-dimensional isotropic oscillator are expressed through the coefficients of certain expansions relating one basis with another of the four-dimensional oscillator. We are only left to load those coefficients with information on the quantum numbers of the corresponding states of a hydrogen atom.

Inserting formulae (33a), (33b), (33e), (33f), (30) and (29) into (31a)-(31d), resp., we get

$$\langle n_0 j\ m_1 m_2 | n\ l\ m \rangle =$$

$$= \frac{1}{2n}\left(\frac{\pi\ Z}{a}\right)^{\frac{1}{2}} (-1)^{l+|m|}\, i^{|m|-m_1}\, \delta_{m_2 m}\, \delta_{l,j/2}\, T^{\,m,m}_{\,l,l} \qquad (34a)$$

$$\langle n_1 n_2 n_3 n_4 | n\ l\ m \rangle =$$

$$\frac{1}{2n}\left(\frac{\pi\ Z}{a}\right)^{\frac{1}{2}} (-1)^{l-|m|}\, i^{m+n_4-n_1}\, d^{\,\tau_1}_{\,m/2,\nu}(\pi/2)\cdot$$

$$d^{\,\tau_2}_{\,m/2,\nu_2}(\pi/2)\ T^{\,l,m}_{\,(n-1)/2,(n_1+n_2-|m|)/2} \qquad (34b)$$

$$\langle n_0 j\ m_1 m_2 | p_1 p_2 m \rangle = \frac{1}{2n}\left(\frac{\pi\ Z}{a}\right)^{\frac{1}{2}} (-1)^{p_1}\, i^{|m|-m_1}\, \delta_{m_2 m}\cdot$$

$$T_{j/2,j/2} T_{(n-1)/2,p_1} \qquad (34c)$$

$$\langle n_1 n_2 n_3 n_4 | p_1 p_2 m \rangle = \frac{1}{2n}\left(\frac{\pi\ Z}{a}\right)^{\frac{1}{2}} (-1)^{p_1+p_2}\, (-i)^{n_1+n_3}\cdot$$

$$d^{\tau_1}_{m/2,\nu_1}(\pi/2)\; d^{\tau_2}_{m/2,\nu_2}(\pi/2) \qquad (34d)$$

So, the searched matrices are expressed in terms of the Kronecker symbols, Clebsch-Gordan coefficients and Wigner functions $d^J_{M,M'}(\beta)$ at $\beta=\pi/2$.

7. CONCLUSION

We have calculated the coefficients of expansions of the parabolic and spherical bases of a hydrogen atom over the double polar, canonical, noncanonical and Cartesian bases of a four-dimensional isotropic oscillator. The problem was solved in three steps : i) diagonal coefficients were found ; ii) the Tarter matrix was used ; iii) explicit form was utilized of the coefficients of expansions between bases in the oscillator. Along that way, no difficulties arise characteristic of the method based on direct investigation of the overlap integrals [1] which in this problem represent multiple integrals of the product of special functions of "very tangled" arguments. This merit allowed exact solution of the problem.

8. APPENDIX

We shall here write the explicit form of bases of a four-dimensional isotropic oscillator and a three-dimensional hydrogen atom.

1. Canonical and noncanonical hyperspherical bases

$$|n_o l\; m_1 m_2\rangle = R_{n_o j}(u)\; Y_{j\; m_1 m_2}(\psi,\vartheta,\varphi)$$

$$|n_o j\; k_1 k_2\rangle = R_{n_o j}(u)\; \tilde{Y}_{j\; k_1 k_2}(\alpha,\beta,\gamma)$$

$$R_{n_o j}(u) = \left\{\frac{2\lambda^4 (n'+j+1)!}{(n')!}\right\}^{1/2} \frac{(\lambda u)^j}{(j+1)!}\, e^{-\lambda^2 u^2/2}.$$

$$\cdot \quad F(-n';j+2;\lambda^2u^2)$$

$$n' = (n_o-j)/2$$

$$Y_{j\,m_1m_2}(\psi,\vartheta,\varphi) = \frac{(-1)^{\bar{m}_2}\,(j)!\,\{(2j+2)(j-m_1)!(j+m_1+1)!\}^{1/2}}{\pi^{1/2}\,(2j+1)!\,2^{m_1-2j+1}}$$

$$(\sin\psi)^{m_1}\,P_{j-m_1}^{(\bar{m}_1,\bar{m}_1)}(\cos\psi)\;Y_{m_1,m_2}(\vartheta,\varphi)$$

where $Y_{l,m}(\vartheta,\varphi)$ - is the conventional spherical function, $P_n^{(\alpha,\beta)}(x)$ - is the Jacobi polinomial :

$$\tilde{Y}_{jk_1k_2}(\alpha,\beta,\gamma) = \frac{1}{\pi}\left\{\frac{(j+1)(j_o+\sigma_1)!(j_o-\sigma_1)!}{2(j_o+\sigma_2)!\,(j_o-\sigma_2)!}\right\}^{1/2}(\sin\alpha)^{|k_1|}\cdot$$

$$(\cos\alpha)^{|k_2|}\cdot P_{j_o-\sigma_1}^{(|k_1|,|k_2|)}(\cos 2\alpha)\exp\{ik_1\beta+ik_2\gamma\}$$

We used the notation : $\bar{m}_1=(2m_1+1)/2$, $\bar{m}_2=(m_2-|m_2|)/2$, $j_o=j/2$, $\sigma_1=(|k_1|+|k_2|)/2$, $\sigma_2=(|k_1|-|k_2|)/2$.

2. The double polar basis

$$f_{tq}(x) = \left\{\frac{(t+|q|)!}{t!}\right\}^{1/2}\frac{e^{-x^2/2}}{|q|!}\cdot x^{|q|/2}\,F(-t;|q|+1;x)$$

3. The spherical basis of a hydrogen atom

$$R_{nl}(r) = \frac{2\alpha^{3/2}}{\sqrt{n}}\,\frac{(2\alpha r)^l}{(2l+1)!}\left\{\frac{(n+l)!}{(n-l-1)!}\right\}^{1/2}e^{-\alpha r}.$$

$$\cdot F\,(l-n+1;2l+2;2\alpha r)$$

4. The parabolic basis of a hydrogen atom

$$f_{p\,m}(x) = \left\{\frac{(p+|m|)!}{p!}\right\}^{1/2} \frac{e^{-x/2}}{|m|!} \cdot x^{|m|/2} F\,(-p;|m|+1;x)$$

All the bases are normalized to unity.

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