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Elliptic Basis of Circular Oscillator.

L. G. MARDOYAN, G. S. POGOSYAN, A. N. SISSAKIAN
and V. M. TER-ANTONYAN

Joint Institute for Nuclear Research - P.O. Box 79, Moscow, U.S.S.R.
Department of Physics, Yerevan State University - Yerevan 375049, U.S.S.R.

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Summary. — A general scheme is developed for constructing the elliptic basis of the circular oscillator. The formulae are obtained for the calculation by computer, the behaviour of the elliptic basis is established in Cartesian and polar limits and some particular results are presented.

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Introduction.

An important problem arising in many applications of the perturbation theory with degeneration is the construction of a correct zero approximation. This problem is usually solved in the framework of secular equations; in some cases an alternative approach more powerful from a technical point of view can be used. In particular, if variables in the Schrödinger equations

$$\hat{H}_0 \psi^{(0)} = E^{(0)} \psi^{(0)} , \tag{1}$$

$$\hat{H}\psi \equiv (\hat{H}_0 + \hat{V})\psi = E\psi \tag{2}$$

may be separated in the same co-ordinates, the functions $\psi^{(0)}$ are just correct wave functions of the Hamiltonian $\hat{H}$ in zero approximation. This is possible, for instance, for a hydrogen atom either in an external homogeneous electric field or in the electric field of a pointlike fixed charge. In the first case, it is convenient to work in parabolic co-ordinates, whereas, in the second case, in the spherical co-ordinates. The greater amount of co-ordinate systems allows the solution of eq. (1) by separating the variables, the larger « chance » for the validity of the alternative approach. The great separability of the variables is allowed in the case of a hydrogen atom and an isotropic oscillator. In this connection it is important to know the wave functions (bases) of the above systems both in traditional (spherical and parabolic for a hydrogen atom and spherical and Cartesian for an isotropic oscillator) and in « exotic » co-ordinates. For a hydrogen atom and a two-dimensional hydrogen atom the latter are represented by spheroidal and elliptic co-ordinates, respectively. The corresponding bases are considered in ref. ([1-4]). This paper deals with the analysis of the problem of a quantum circular oscillator in the elliptic co-ordinates. Note that the bases we have found may be used as a zero approximation in a more complicated problem on the behaviour of a circular charged oscillator in the dipole field:

$$U = \frac{m\omega^2 r^2}{2} + \frac{\alpha}{\sqrt{y^2 + (R/2 - x)^2}} - \frac{\alpha}{\sqrt{y^2 + (R/2 + x)^2}} .$$

The problem of the quantum circular oscillator in elliptic co-ordinates cannot be solved exactly in an analytic form, as it reduces to the determination of roots of the algebraic equation of arbitrary degree. We here developed the general approach to the problem and found all the formulae for numerical calculations. The way of transformation from the elliptic to the polar and Cartesian bases is shown and elliptic wave functions of the circular oscillator are computed for the lowest quantum numbers.

1. – Elliptic co-ordinates.

Elliptic co-ordinates ξ and η to be dealt with vary within the limit $0 \leqslant \xi < \infty$, $0 \leqslant \eta \leqslant 2\pi$ and are connected with the Cartesian ones x and y by

$$x = \frac{R}{2} \cosh \xi \cos \eta , \qquad y = \frac{R}{2} \sinh \xi \sin \eta . \tag{1.1}$$

As $R \to 0$ and $R \to \infty$, the co-ordinates chosen thus degenerate into the polar

(1) C. A. Coulson and P. D. Robinson: *Proc. Phys. Soc., London*, **71**, 815 (1958).
(2) L. G. Mardoyan, G. S. Pogasyan, A. N. Sissakian and V. M. Ter-Antonyan: *Teor. Mat. Fiz.*, **61**, 99 (1984) (in Russian).
(3) L. G. Mardoyan, G. S. Pogasyan, A. N. Sissakian and V. M. Ter-Antonyan: *J. Phys. A*, **16**, 711 (1983).
(4) L. G. Mardoyan, G. S. Pogasyan, A. N. Sissakian and V. M. Ter-Antonyan: *J. Phys. A*, **18**, 455 (1985).

and Cartesian co-ordinates

$$\text{(1.2a)} \qquad \cosh\xi \to \frac{2r}{R}, \qquad \cos\eta \to \cos\varphi \qquad (R\to 0),$$

$$\text{(1.2b)} \qquad \sinh\xi \to \frac{2y}{R}, \qquad \cos\eta \to \frac{2x}{R} \qquad (R\to\infty).$$

The Laplacian and a two-dimensional volume element are given by

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{8}{R^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right).$$

$$dv = dx\,dy = \frac{R^2}{8}(\cosh 2\xi - \cos 2\eta)\,d\xi\,d\eta .$$

2. – Separation of variables.

For simplicity we shall use below the unit system in which $\hbar=\mu=\omega=1$. We start with the following Schrödinger equation, where ε is the energy of circular oscillator:

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi+\left\{\frac{\varepsilon R^2}{4}(\cosh 2\xi-\cos 2\eta)-\frac{R^4}{64}(\cosh^2 2\xi-\cos^2 2\eta)\right\}\psi=0 .$$

Upon substituting

$$\psi(\xi,\eta;R^2)=X(\xi;R^2)\,Y(\eta;R^2)$$

and introducing the separation constant $A(R^2)$, the equation splits into two ordinary differential equations:

$$\text{(2.1a)} \qquad \left(\frac{d^2}{d\xi^2}+\frac{\varepsilon R^2}{4}\cosh 2\xi-\frac{R^4}{64}\cosh^2 2\xi\right)X(\xi,R^2) = -A(R^2)X(\xi;R^2),$$

$$\text{(2.1b)} \qquad \left(\frac{d^2}{d\eta^2}-\frac{\varepsilon R^2}{4}\cos 2\eta+\frac{R^4}{64}\cos^2 2\eta\right)Y(\eta;R^2) = A(R^2)\,Y(\eta;R^2).$$

Equations (2.1a) and (2.1b) are transformed into one another by the change $\eta = i\xi$ and, as a result,

$$\text{(2.2)} \qquad \psi(\xi,\eta;R^2)=c(R^2)\,Z(i\xi;R^2)\,Z(\eta;R^2),$$

where c is the normalization constant determined by the condition

$$\text{(2.3)} \qquad \int|\psi(\xi,\eta;R^2)|^2\,dv = 1$$

and the function $Z(\zeta; R^2)$ is a solution to the equation

$$\left(\frac{d^2}{d\zeta^2}-\frac{\varepsilon R^2}{4}\cos 2\zeta+\frac{R^4}{64}\cos^2 2\zeta\right)Z(\zeta; R^2)=A(R^2)Z(\zeta; R^2). \tag{2.4}$$

It is periodic (single-valued)

$$Z(\zeta+2\pi; -R^2)=Z(\zeta; R^2) \tag{2.5}$$

and decreasing as $\zeta\to i\infty$ (finite).

3. – Recurrent relations.

We introduce the function $W(\zeta; R^2)$ according to

$$Z(\zeta; R^2)=\exp\left[-\frac{R^2}{16}\cos 2\zeta\right]W(\zeta; R^2)$$

and transform (2.4) into the Ince equation (5) (*)

$$\frac{d^2W}{d\zeta^2}+\frac{R^2}{4}\sin 2\zeta\frac{dW}{d\zeta}+\left[\frac{R^4}{64}-A(R^2)-\frac{R^2}{4}(\varepsilon-1)\cos 2\zeta\right]W=0. \tag{3.1}$$

It is known (6) that eq. (3.1) has, in the general case, two types of solutions:

$$W^{(+)}(\xi, R^2)=\sum_{k=0}^{\infty}a_k(R^2)(\cos\zeta)^k, \tag{3.2a}$$

$$W^{(-)}(\zeta; R^2)=\sum_{k=0}^{\infty}(\sin\zeta)\,b_k(R^2)(\cos\zeta)^k. \tag{3.2b}$$

The first of them is even and the second odd with respect to the change $\zeta\to-\zeta$. It may easily be verified that solutions (3.2) satisfy the condition of periodicity (2.5). Upon inserting series (3.2) into eq. (3.1) and further defining coefficients a_k and b_k by the equalities

$$a_{-1}=a_{-2}=b_{-1}=b_{-2}=0,$$

(5) E. L. INCE: *Ordinary Differential Equations* (Longman, London, 1926).
(*) The Ince equation follows also for the problem on a two-dimensional hydrogen atom in elliptic co-ordinates (2). In this problem the condition of periodicity like (2.5) selects only a certain type of solutions of the Ince equation.
(6) F. M. ARSCOTT: *Proc. R. Soc. Edinburg, Sect. A*, **67**, 265 (1967).

we arrive at the trinomial recurrent relations

$$(k+1)(k+2)a_{k+2}+\beta_k a_k+\frac{R^2}{2}(k-\varepsilon-1)a_{k-2}=0\,, \tag{3.3a}$$

$$(k+1)(k+2)b_{k+2}+\tilde\beta_k b_k+\frac{R^2}{2}(k-\varepsilon)b_{k-2}=0\,, \tag{3.3b}$$

where

$$\beta_k=-k^2+\frac{R^2}{4}(\varepsilon-2k-1)+\frac{R^4}{64}-A(R^2)\,, \tag{3.4a}$$

$$\tilde\beta_k=-(k+1)^2+\frac{R^2}{4}(\varepsilon-2k-1)+\frac{R^4}{64}-A(R^2)\,. \tag{3.4b}$$

From relations (3.3) it follows that the determination of the coefficients a_k and b_k requires four initial conditions. We take these in the form

$$a_0=a_1=b_0=b_1=1$$

and split each of relations (3.3) into two relations with even and odd k.

4. – Energy spectrum.

Let us examine now how the condition of the function to be finite determines the discrete energy spectrum. To be specific, let $k=2s$, and we consider the coefficients R_{2s}. At large s from (3.3a) and (3.4a), we have

$$\left(s^2+\frac{3}{2}s\right)\frac{a_{2s+2}}{a_{2s}}\frac{a_{2s}}{a_{2s-2}}-\left(s^2+\frac{R^2}{4}s\right)\frac{a_{2s}}{a_{2s-2}}+\frac{R^2}{4}s=0\,.$$

As $s^{-1}\ll 1$, the following expansions hold:

$$\frac{a_{2s+2}}{a_{2s}}\sim c_0+\frac{c_1}{s}+O\left(\frac{1}{s^2}\right),\qquad \frac{a_{2s}}{a_{2s-2}}\sim c_0+\frac{c_1}{s-1}+O\left(\frac{1}{s^2}\right)=c_0\frac{c_1}{s}+O\left(\frac{1}{s^2}\right).$$

The latter, being supplemented with the former relation, lead to the two cases:

a) $c_0=0,\ c_1=\dfrac{R^2}{4},\ a_{2s}\sim\left(\dfrac{R^2}{4}\right)^s\dfrac{l}{s!}$,

b) $c_0=1,\ c_1=-\frac{3}{2},\ a_{2s}\sim s^{-\frac{3}{2}}$.

It can easily be verified that in both cases $Z(\zeta;R^2)\to\infty$ as $\zeta\to i\,\infty$. Hence it follows that series (3.2a) should be truncated. Analogous arguments are valid also for other three recurrent relations and produce the same result. The condition for series (3.2) to be truncated results in the energy spectrum

$$\varepsilon_N = N + 1\,, \qquad N = 0, 1, 2, \ldots . \tag{4.1}$$

For the number N being even or odd the condition of finiteness of the solutions allows the following four Ince polynomials:

$$\begin{cases} c_{2n}^{2q}(\zeta;\,R^2)\sum\limits_{s=0}^{n} a_{2s}(R^2)(\cos\zeta)^{2s}\,, & N = 2n\,, \\ c_{2n+1}^{2q+1}(\zeta;\,R^2) = \sum\limits_{s=0}^{n} a_{2s+1}(R^2)(\cos\zeta)^{2s+1}\,, & N = 2n+1\,, \\ s_{2n+1}^{2q+1}(\zeta;\,R^2) = \sin\zeta\sum\limits_{s=0}^{n} b_{2s}(R^2)(\cos\zeta)^{2s}\,, & N = 2n+1\,, \\ s_{2n+2}^{2q+2}(\zeta;\,R^2) = \sin\zeta\sum\limits_{s=0}^{n} b_{2s+1}(R^2)(\cos\zeta)^{2s+1}\,, & N = 2n+2\,. \end{cases} \tag{4.2}$$

5. – Elliptic separation constant.

Once series (3.2) are truncated, each of the recurrent relations ($k = 2s$ and $k = 2s + 1$) (3.3) becomes a system of $(n + 1)$ linear homogeneous equations for the coefficients a_{2s}, a_{2s+1}, b_{2s} and b_{2s+1}. Equating the corresponding determinants to zero:

$$\begin{cases} N = 2n & N = 2n+1 \\ \begin{vmatrix} \beta_0 & 2 & 0 & & 0 \\ -nR^2 & \beta_2 & 12 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & -2R^2 & & \beta_{2n-2} & 2n(2n-1) \\ 0 & & 0 & -R^2 & \beta_{2n} \end{vmatrix} = 0 & \begin{vmatrix} \beta_1 & s & 0 & & 0 \\ -nR^2 & \beta_3 & 20 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & -2R^2 & & \beta_{2n-1} & 2n(2n+1) \\ 0 & & 0 & -R^2 & \beta_{2n+1} \end{vmatrix} = 0 \\ N = 2n+1\,, & N = 2n+2\,, \\ \begin{vmatrix} \tilde\beta_0 & 2 & 0 & & 0 \\ -nR^2 & \tilde\beta_2 & 12 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & -2R^2 & & \tilde\beta_{2n-2} & 2n(2n-1) \\ 0 & & 0 & -R^2 & \tilde\beta_{2n} \end{vmatrix} = 0 & \begin{vmatrix} \tilde\beta_1 & s & 0 & & 0 \\ -nR^2 & \tilde\beta_3 & 20 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & -2R^2 & & \tilde\beta_{2n-1} & 2n(2n+1) \\ 0 & & 0 & -R^2 & \tilde\beta_{2n+1} \end{vmatrix} = 0 \end{cases} \tag{5.1}$$

leads us to four algebraic equations of a $(n + 1)$ degree which determines eigenvalues of the separation constant $A_N^t(R^2)$. For solutions (4.2) the index t assumes the values $2q$, $2q + 1$, $2q + 1$, $2q + 2$.

6. – Elliptic basis.

According to the consideration of sect. **5**, the elliptic basis (2.2) splits into the following four subbases:

$$(6.1a)\qquad \psi^{(1,1)} = c^{(1,1)} hc_{2n}^{2q}(i\xi;\, R^2)\, hc_{2n}^{2q}(\eta;\, R^2)\,, \qquad N = 2n\,, \quad \mathscr{D} = n+1 = \frac{N+2}{2}\,,$$

$$(6.1b)\qquad \psi^{(1,2)} = c^{(1,2)} hc_{2n+1}^{2q+1}(i\xi;\, R^2)\, hc_{2n+1}^{2q+1}(\eta;\, R^2), \qquad N = 2n+1\,, \quad \mathscr{D} = n+1 = \frac{N+1}{2}\,,$$

$$(6.1c)\qquad \psi^{(2,1)} = c^{(2,1)} hs_{2n+1}^{2q+1}(i\xi;\, R^2)\, hs_{2n+1}^{2q+1}(\eta;\, R^2)\,, \qquad N = 2n+1\,, \quad \mathscr{D} = n+1 = \frac{N+1}{2}\,,$$

$$(6.1d)\qquad \psi^{(2,2)} = c^{(2,2)} hs_{2n+2}^{2q+2}(i\xi;\, R^2)\, hs_{2n+2}^{2q+2}(\eta;\, R^2)\,, \qquad N = 2n+2\,, \quad \mathscr{D} = n+1 = \frac{N}{2}\,.$$

Here $\mathscr{D}$ is the number of states at a given N, and hc and hs stand for polynomials (4.2) multiplied by the factor $\exp[-(R^2/16)\cos 2\zeta]$.

Wave functions (6.1) as eigenfunctions of the Hamiltonian are at $N \neq N'$ orthogonal:

$$\int \psi_{N,t}^{*(i,j)}\, \psi_{N',t}^{(i,j)}\, \mathrm{d}V = 0\,.$$

The Sturm-Liouville theory yields the equalities

$$\int_0^\infty hc_j^{*t}(i\xi;\, R^2)\, hc_j^{t'}(i\xi;\, R^2)\, \mathrm{d}\xi = \int_0^\infty hs_j^{*t}(i\xi;\, R^2)\, hs_j^{t'}(i\xi;\, R^2)\, \mathrm{d}\xi = 0\,,$$

$$\int_0 hc_j^{*t}(\eta;\, R^2)\, hc_j^{t'}(\eta;\, R^2)\, \mathrm{d}\eta = \int_0^{2\pi} hs_j^{*t}(\eta;\, R^2)\, hs_j^{t'}(\eta;\, R^2)\, \mathrm{d}\eta = 0\,,$$

which allow us to prove that, for $t \neq t'$,

$$\int \psi_{N,t}^{*(i,j)}\, \psi_{N,t'}^{(i,j)}\, \mathrm{d}V = 0\,.$$

Equation (2.4) represents the eigenvalue problem for the separation constant

$A(R^2)$ and the corresponding eigenfunctions. As the operator in the left-hand side of eq. (2.4) is invariant under the transformation

$$\zeta \to \zeta + \frac{\pi}{2}, \quad R^2 \to -R^2, \tag{6.2}$$

the substitution $R^2 \to -R^2$ does not change the set of eigenvalues of the separation constant $A(R^2)$, but changes only their numeration:

$$A_N^t(R^2) = A_N^{t'}(-R^2).$$

As a result, transformation (6.2) transforms solutions (4.2) into each other. ARSCOTT [7] has established the following rule of correspondence:

$$\left|\begin{aligned} &hc_{2n}^{2q}\left(\zeta + \frac{\pi}{2}; -R^2\right) \to hc_{2n}^{2q'}(\zeta; R^2), \\ &hc_{2n+1}^{2q+1}\left(\zeta + \frac{\pi}{2}; -R^2\right) \to hs_{2n+1}^{2q+1}(\zeta; R^2), \\ &hs_{2n+1}^{2q+1}\left(\zeta + \frac{\pi}{2}; -R^2\right) \to hc_{2n+1}^{2q'+1}(\zeta; R^2), \\ &hs_{2n+2}^{2q+2}\left(\zeta + \frac{\pi}{2}; -R^2\right) \to hs_{2n+2}^{2q'+2}(\zeta; R^2). \end{aligned}\right. \tag{6.3}$$

From (6.3) we conclude that the elliptic basis (6.1) can be represented in the form

$$\psi^{(1,1)} = c^{(1,1)}\, hc_{2n}^{2q}\left(i\xi + \frac{\pi}{2}; -R^2\right) hc_{2n}^{2q'}(\eta; R^2), \tag{6.4a}$$

$$\psi^{(1,2)} = c^{(1,2)}\, hs_{2n+1}^{2q+1}\left(i\xi + \frac{\pi}{2}; -R^2\right) hc_{2n+1}^{2q'+1}(\eta; R^2), \tag{6.4b}$$

$$\psi^{(2,1)} = c^{(2,1)}\, hc_{2n+1}^{2q+1}\left(i\xi + \frac{\pi}{2}; -R^2\right) hs_{2n+1}^{2q'+1}(\eta; R^2), \tag{6.4c}$$

$$\psi^{(2,2)} = c^{(2,2)}\, hs_{2n+2}^{2q+2}\left(i\xi + \frac{\pi}{2}; -R^2\right) hs_{2n+2}^{2q'+1}(\eta; R^2). \tag{6.4d}$$

(7) A. ERDÉLYE and H. BATEMAN: *Higher Transcendental Functions*, Vol. 1 (McGraw-Hill Book Co., Inc., New York, N. Y., 1953).

7. – Polar and Cartesian limits.

We shall analyse how formulae (6.4) in the limits $R \to 0$ and $R \to \infty$ lead to polar and Cartesian wave functions of the circular oscillator. From (3.3), (3.4) and (5.1) it follows that, when $R \to 0$, each of four trinomial recurrent relations (3.3) ($k = 2s$; $k = 2s + 1$) splits into two binomial relations:

$$(7.1a) \qquad \left\{\begin{aligned} &(2s+1)(2s+2)\,a_{2s+2} + 4(q+s)(q-s)\,a_{2s} = 0\,,\\ &(2s+2)(2s+3)\,a_{2s+3} + 4(q+s+1)(q-s)\,a_{2s+1} = 0\,,\\ &(2s+1)(2s+2)\,b_{2s+2} + 4(q+s+1)(q-s)\,b_{2s} = 0\,,\\ &(2s+2)(2s+3)\,b_{2s+3} + 4(q+s+2)(q-s)\,b_{2s+1} + 0\,, \end{aligned}\right. \qquad 0 \leqslant s \leqslant q-1\,,$$

$$(7.1b) \qquad \left\{\begin{aligned} &4(q+s)(q-s)\,a_{2s} + R^2(s-n-1)\,a_{2s-2} = 0\,,\\ &4(q+s+1)(q-s)\,a_{2s+1} + R^2(s-n-1)\,a_{2s-1} = 0\,,\\ &4(q+s+1)(q-s)\,b_{2s} + R^2(s-n-1)\,b_{2s-2} = 0\,,\\ &4(q+s+2)(q-s)\,b_{2s+1} + R^2(s-n-1)\,b_{2s-1} = 0\,. \end{aligned}\right. \qquad q+1 \leqslant s \leqslant n\,,$$

Solving (7.1) we have

$$(7.2a) \qquad \left\{\begin{aligned} &a_{2s}(0) = \frac{(q)_s(-q)_s}{s!(\frac{1}{2})_s}\,, \qquad a_{2s+1}(0) = \frac{(q+1)_s(-q)_s}{s!(\frac{3}{2})_s}\,,\\ &b_{2s}(0) = \frac{(q+1)_s(-q)_s}{s!(\frac{1}{2})_s}\,, \qquad b_{2s+1}(0) = \frac{(q+2)_s(-q)_s}{s!(\frac{3}{2})_s}\,,\\ &a_{2s}(-R^2) \xrightarrow{R\to 0} (-1)^q \frac{2q^{-1}(q-n)_{s-q}}{(s-q)!(2q+1)_{s-q}}\left(-\frac{R^2}{4}\right)^{s-q}, \end{aligned}\right. \qquad 0 \leqslant S \leqslant q\,,$$

$$(7.2b) \qquad \left\{\begin{aligned} &a_{2s+1}(-R^2) \xrightarrow{R\to 0} \frac{(-1)^q 2^{2q}}{2q+1}\,\frac{(q-n)_{s-q}}{(s-q)!(2q+2)_{s-q}}\left(-\frac{R^2}{4}\right)^{s-q},\\ &b_{2s}(-R^2) \xrightarrow{R\to 0} \frac{(-1)^q 2^{2q}}{(s-q)!}\,\frac{(q-h)_{s-q}}{(2q+2)_{s-q}}\left(-\frac{R^2}{4}\right)^{s-q},\\ &b_{2s+1}(-R^2) \xrightarrow{R\to 0} \frac{(-1)^q 2^{2q}}{q+1}\,\frac{(q-n)_{s-q}}{(s-q)!(2q+3)_{s-q}}\left(-\frac{R^2}{4}\right)^{s-q}. \end{aligned}\right. \qquad q+1 \leqslant S \leqslant n\,,$$

Making use of formulae (1.2a), (7.2) and of the relations [7]

$$\cos az = F\left(\frac{a}{2}, -\frac{a}{2}; \frac{1}{2}; \sin^2 z\right) = \cos z\, F\left(\frac{1}{2} + \frac{a}{2}, \frac{1}{2} - \frac{a}{2}; \frac{1}{2}; \sin^2 z\right),$$

$$\sin az = a \sin z F\left(\frac{1}{2}+\frac{a}{2}, \frac{1}{2}-\frac{a}{2}; \frac{3}{2}; \sin^2 z\right) =$$

$$= a \sin z \cos z F\left(1+\frac{a}{2}, 1-\frac{a}{2}; \frac{3}{2}; \sin^2 z\right),$$

we arrive at the following behaviour of functions hc and hs when $R \to 0$:

$$hc_{2n}^{2q}(\eta, 0) = (-1)^q \cos 2q\varphi\,, \qquad hc_{2n+1}^{2q+1}(\eta, 0) = \frac{(-1)^q}{2q+1} \cos(2q+1)\varphi\,,$$

$$hs_{2n+1}^{2q+1}(\eta, 0) = (-1)^q \sin(2q+1)\varphi\,, \qquad hs_{2n+2}^{2q+2}(\eta, 0) = \frac{(-1)^q}{2q+2} \sin(2q+2)\varphi\,,$$

$$hc_{2n}^{2q}\left(i\xi+\frac{\pi}{2}; -R^2\right) \xrightarrow{R\to 0} \frac{2^{4q-1}}{R^{2q}} \exp\left[-\frac{r^2}{2}\right] r^{2q} F(q-n; 2q+1; r^2) + O\left(\frac{1}{R^{2q-2}}\right),$$

$$hc_{2n+1}^{2q+1}\left(i\xi+\frac{\pi}{2}; -R^2\right) \xrightarrow{R\to 0}$$

$$\xrightarrow{R\to 0} -i\, \frac{2^{4q+1}}{R^{2q+1}} \frac{1}{2q+1} \exp\left[-\frac{r^2}{2}\right] r^{2q+1} F(q-n; 2q+2; r^2) + O\left(\frac{1}{R^{2q-1}}\right),$$

$$hs_{2n+1}^{2q+1}\left(i\xi+\frac{\pi}{2}; -R^2\right) \xrightarrow{R\to 0}$$

$$\xrightarrow{R\to 0} \frac{2^{4q+1}}{R^{2q+1}} \exp\left[-\frac{r^2}{2}\right] r^{2q+1} F(q-n; 2q+2; r^2) + O\left(\frac{1}{R^{2q-1}}\right),$$

$$hs_{2n+2}^{2q+2}\left(i\xi+\frac{\pi}{2}; -R^2\right) \xrightarrow{R\to 0}$$

$$\xrightarrow{R\to 0} -i\, \frac{2^{4q+3}}{R^{2q+2}} \exp\left[-\frac{r^2}{2}\right] r^{2q+2} F(q-n; 2q+3; r^2) + O\left(\frac{1}{R^{2q-1}}\right).$$

From these formulae and normalization condition (2.3) it is seen that the elliptic basis (6.4), when $R \to 0$, becomes the polar basis $(0 \leqslant m \leqslant N)$

$$\psi_{Nm}^{(\pm)}(r, \varphi) \sim r^m \exp\left[-\frac{r^2}{2}\right] F\left(-\frac{N-m}{2}; m+1; r^2\right) \begin{Bmatrix} \cos m\varphi \\ i \sin m\varphi \end{Bmatrix}$$

and in that limit

$$c^{(1,1)}(R^2) \sim (R)^{2a}\,, \qquad c^{(1,2)}(R^2) \sim c^{(2,1)}(R^2) \sim R^{2q+1}\,, \qquad c^{(2,2)}(R^2) \sim R^{2q+2}\,.$$

Let us take in (3.4) and (5.1) the limit $R \to \infty$. Then instead of (3.3) we get

$$
(7.3) \quad \left\{
\begin{aligned}
&(2s+1)(2s+2)\,a_{2s+2} + R^2(q-s)\,a_{2s} = 0\,,\\
&\qquad (q-s)\,a_{2s} + (s-n-1)\,a_{2s-2} = 0\,,\\
&(2s+2)(2s+3)\,a_{2s+3} + R^2(q-s)\,a_{2s+1} = 0\,,\\
&\qquad (q-s)\,a_{2s} + (s-n-1)\,a_{2s-2} = 0\,,\\
&(2s+1)(2s+2)\,b_{2s+2} + R^2(q-s)\,l_{2s} = 0\,,\\
&\qquad (q-s)\,b_{2s} + (s-n-1)\,b_{2s-2} = 0\,,\\
&(2s+2)(2s+3)\,b_{2s+3} + R^2(q-s)\,b_{2s+1} = 0\,,\\
&\qquad (q-s)\,b_{2s+1} + (s-n-1)\,b_{2s-1} = 0\,,
\end{aligned}
\right.
$$

and eight recurrent relations following from (7.3) upon the change $q \to q'$, $R^2 \to -R^2$. Solving these recurrent relation and making use of (1.2b), we arrive at the formulae ($R \to \infty$)

$$
hc_{2n}^{2q}(\eta;\, R^2) = hs_{2n+1}^{2q+1}(\eta;\, k^2) \to F(-q;\, \tfrac{1}{2};\, x^2)\,,
$$

$$
hc_{2n+1}^{2q+1}(\eta;\, R^2) = hs_{2n+2}^{2q+2}(\eta,\, R^2) = \frac{2x}{R}\, F(-q;\, \tfrac{3}{2};\, x^2)\,,
$$

$$
hc_{2n}^{2q'}\left(i\xi + \frac{\pi}{2};\, -R^2\right) = hs_{2n+2}^{2q'+1}\left(i\xi + \frac{\pi}{2};\, -R^2\right) \to F(-q';\, \tfrac{1}{2};\, y^2)\,,
$$

$$
hc_{2n+1}^{2q'+1}\left(i\xi + \frac{\pi}{2};\, -R^2\right) = hs_{0n+2}^{2q'+2}\left(i\xi + \frac{\pi}{2};\, -R^2\right) \to -\,i\,\frac{2y}{R}\, F(-q';\, \tfrac{3}{2};\, y^2)\,,
$$

from which and the normalization condition (2.3) it is seen that the elliptic basis (6.4) transforms, when $R \to \infty$, into the Cartesian basis

$$
\psi_{n_1 n_2}(x, y) \sim \exp\left[-\frac{x^2 + y^2}{2}\right] H_{n_1}(x) H_{n_2}(y)
$$

and that in the same limit

$$
c^{(1,1)}(R^2) \sim \text{const}\,, \qquad c^{(1,2)}(R^2) \sim c^{(2,1)}(R^2) \sim R\,, \qquad c^{(2,2)}(R^2) \sim R^2\,.
$$

8. – Particular cases.

Equations (5.1) in the general case cannot be solved analytically. In table I we present information on the orthonormalized elliptic basis for lowest quantum numbers.

TABLE I.

N	$\psi^{(2,1)}(\xi,\eta;R^2)$	A'
1	$i\frac{R}{\sqrt{2\pi}}\sin\eta\sinh\xi\exp\left[-\frac{R^2}{16}(\cosh 2\xi+\cos 2\eta)\right]$	$A'=-1+\frac{R^2}{4}$
3	$-i\frac{R}{\sqrt{\pi}}\left\{1+\frac{3}{16}\left(\frac{R^2}{A'+9}\right)^2\right\}^{-\frac{1}{2}}\sin\eta\sinh\xi\exp\left[-\frac{R^2}{16}(\cosh 2\xi+\cos 2\eta)\right]\times$ $\times\left\{1+\frac{1}{2}\left(A'+1-\frac{3R^2}{4}\right)\cos^2\eta\right\}\left\{1-\frac{1}{6}\left(A'+1+\frac{R^2}{4}\right)\sinh^2\xi\right\}$	$(A'+1)(A'+9)=$ $=\frac{R^2}{2}(A'+9)+\frac{3R^4}{16}$
5	$i\sqrt{\frac{3}{2\pi}}R\left\{1+2\left(\frac{A'+1}{R^2}-\frac{3}{4}\right)^2+\frac{5}{2}\left(\frac{R^2}{A'+25}\right)^2\left(\frac{A'+1}{R^2}-\frac{3}{4}\right)^2\right\}^{-\frac{1}{2}}\times$ $\times\sin\eta\sinh\xi\exp\left[-\frac{R^2}{16}(\cosh 2\xi+\cos 2\eta)\right]\left\{1+\frac{1}{2}\left(A'+1-\frac{5R^2}{4}\right)\cos^2\eta+\right.$ $\left.+\frac{1}{24}\left[\left(A'+1-\frac{5R^2}{4}\right)\left(A'+9-\frac{R^2}{4}\right)+4R^2\right]\cos^4\eta\right\}\times$ $\times\left\{1-\frac{1}{6}\left(A'+1+\frac{3R^2}{4}\right)\sinh^2\xi+\frac{1}{120}\left[\left(A'+1-\frac{3R^2}{4}\right)\left(A'+9-\frac{R^2}{4}\right)-12R^2\right]\sinh^4\xi\right\}$	$(A'+1)(A'+9)(A'+25)=$ $=\frac{3R^2}{4}(A'+9)(A'+25)+$ $+\frac{R^4}{16}(13A'+205)-\frac{15R^6}{64}$
N	$\psi^{(1,1)}(\xi,\eta;R^2)$	A'
0	$\frac{1}{\sqrt{\pi}}\exp\left[-\frac{R^2}{16}(\cosh 2\xi+\cos 2\eta)\right]$	$A'=0$
2	$-\frac{1}{\sqrt{2\pi}}\left\{\frac{A'+4}{A'+2}\right\}^{-\frac{1}{2}}\exp\left[-\frac{R^2}{16}(\cosh 2\xi+\cos 2\eta)\right]\times$ $\times\left\{1+\frac{1}{2}\left(A'-\frac{R^2}{2}\right)\cos^2\eta\right\}\left\{1-\frac{1}{2}\left(A'+\frac{R^2}{2}\right)\sinh^2\xi\right\}$	$A'(A'+4)=\frac{R^2}{4}$
4	$\frac{1}{\sqrt{\pi}}\left\{1+\frac{4}{3}\left(\frac{A'}{R}\right)^2+\frac{1}{3}\left(\frac{A'}{A'+16}\right)^2\right\}^{-\frac{1}{2}}\exp\left[-\frac{R^2}{16}(\cosh 2\xi+\cos 2\eta)\right]\times$ $\times\left\{1+\frac{1}{2}(A'-R^2)\cos^2\eta+\frac{A}{24}(A'+4-R^2)\cos\eta\right\}\times$ $\times\left\{1-\frac{1}{[illegible]}(A'+R^2)\sinh^2\xi+\frac{A'}{[illegible]}(A'+4+R^2)\sinh^2\xi\right\}$	$A'(A'+4)(A'+16)=$ $=R^4(A'+12)$

N	$\psi^{(2,2)}(\xi, \eta; R^2)$	A'
2	$\frac{i}{8}\sqrt{\frac{2}{\pi}}\, R^2 \sin 2\eta \sinh 2\xi \exp\left[-\frac{R^2}{16}(\cosh 2\xi + \cos 2\eta)\right]$	$A' = -4$
4	$-i\sqrt{\frac{6}{\pi}}\,\frac{R}{16}\left\{\frac{A'+16}{A'+10}\right\}^{-\frac{1}{2}} \sin 2\eta \sinh 2\xi \exp\left[-\frac{R^2}{16}(\cosh 2\xi + \cos 2\eta)\right] \times$ $\times\left\{1+\frac{1}{6}\left(A'+4-\frac{R^2}{2}\right)\cos^2\eta\right\}\left\{1-\frac{1}{6}\left(A'+4+\frac{R^2}{2}\right)\sinh^2\xi\right\}$	$(A'+4)(A'+16) = \frac{R^4}{4}$
6	$i\sqrt{\frac{3}{2\pi}}\,\frac{R}{4}\left\{1+\frac{8}{5}\left(\frac{A'+4}{R}\right)^2+\frac{3}{5}\left(\frac{A'+4}{A^{(+)\prime}+36}\right)^2\right\}^{-\frac{1}{2}} \sin 2\eta \sinh 2\xi \times$ $\times \exp\left[-\frac{R^2}{16}(\cosh 2\xi + \cos 2\eta)\right]\left\{1+\frac{1}{6}(A'+4-R^2)\cos^2\eta+\frac{A'+4}{120}(A'+16-R^2)\cos^4\eta\right\}\times$ $\times\left\{1-\frac{1}{6}(A'+4+R^2)\sinh^2\xi+\frac{A'+4}{120}(A'+16+R^2)\sinh^4\xi\right\}$	$(A'+4)(A'+16)(A'+36) =$ $= R^4(A'+24)$

N	$\psi^{(1,2)}(\xi, \eta; R^2)$	A'
1	$\frac{R}{\sqrt{2\pi}}\cos\eta \cosh\xi \exp\left[-\frac{R^2}{16}(\cosh 2\xi + \cos 2\eta)\right]$	$A' = -1 - \frac{R^2}{4}$
3	$\frac{4}{\sqrt{3\pi}}\,\frac{A'+9}{R}\left\{1+\frac{16}{3}\left(\frac{A'+9}{R^2}\right)^2\right\}^{-\frac{1}{2}} \cosh\xi \cos\eta \exp\left[-\frac{R^2}{16}(\cosh 2\xi + \cos 2\eta)\right]\times$ $\times\left\{1+\frac{1}{6}\left(A'+1-\frac{R^2}{4}\right)\cos^2\eta\right\}\left\{1-\frac{1}{2}\left(A'+1+\frac{3R^2}{4}\right)\sinh^2\xi\right\}$	$(A'+1)(A'+9) =$ $= -\frac{R^2}{2}(A'+9)+\frac{3R^4}{16}$
5	$\sqrt{\frac{3}{2\pi}}\,R\left\{1+2\left(\frac{A'+1}{R^2}+\frac{3}{4}\right)^2+\frac{5}{8}\left(\frac{R^2}{A^{(-)\prime}+25}\right)^2\left(\frac{A'+1}{R^2}+\frac{3}{4}\right)^2\right\}^{-\frac{1}{2}}\cdot$ $\cdot\cos\eta \cosh\xi \exp\left[-\frac{R^2}{16}(\cosh 2\xi + \cos 2\eta)\right]\left\{1+\frac{1}{6}\left(A'+1-\frac{3R^2}{4}\right)\cos^2\eta+\right.$ $\left.+\frac{1}{120}\left[\left(A'+1-\frac{3R^2}{4}\right)\left(A'+9+\frac{R^2}{4}\right)+12R^2\right]\cos^4\eta\right\}\times$ $\times\left\{1-\frac{1}{2}\left(A'+1+\frac{5R^2}{4}\right)\sinh^2\xi+\frac{1}{24}\left[\left(A'+1+\frac{5R^2}{4}\right)\left(A'+9+\frac{R^2}{4}\right)-4R^2\right]\sinh^4\xi\right\}$	$(A'+1)(A'+9)(A'+25) =$ $= -\frac{3R^2}{4}(A'+9)(A'+25)+$ $+\frac{R^4}{16}(13A'+205)+\frac{15R^6}{64}$

* * *

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● RIASSUNTO (*)

Si sviluppa uno schema generale per costruire la base ellittica dell'oscillatore circolare. Si ottengono le formule per il calcolo col computer, si stabilisce il comportamento della base ellittica nei limiti cartesiani e polari e si presentano alcuni risultati particolari.

(*) *Traduzione a cura della Redazione.*

Эллиптический базис кругового осциллятора.

Резюме. — Развита общая схема построения эллиптического базиса кругового осциллятора. Получены формулы, на основе которых следует проводить дальнейшие вычисления на ЭВМ; выявлено поведение эллиптического базиса в полярном и декартовом пределах; выписаны некоторые частные результаты.