Interbasis Expansions in a Circular Oscillator.

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Summary. — Expansion coefficients for fundamental bases of a quantum circular oscillator are calculated. Formulae are found for the coefficients of expansions of the elliptic bases over the polar and Cartesian ones. The values of these coefficients are tabulated for some particular cases.

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1. – Within the method of separation of variables the problem of a quantum circular oscillator has solutions in three co-ordinate systems, Cartesian, polar and elliptic (1). The aim of this paper is to find transformations between these solutions (bases) at a fixed value of energy.

2. – Fundamental bases of the circular oscillator are eigenfunctions of the Hamiltonian (**)

$$\hat{H} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(x^2 + y^2)$$

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(**) We take the system of units $\hbar = \mu = \omega = 1$.

(1) L. G. MARDOYAN, G. S. POGOSYAN, A. N. SISSAKIAN and V. M. TER-ANTONYAN: *Teor. Math. Fis.*, **61**, 99 (1984).

and of each of the generators

$$\hat{\mathscr{P}} = \frac{1}{4}\left(x^2 - \frac{\partial^2}{\partial x^2} - y^2 + \frac{\partial^2}{\partial y^2}\right),$$

$$\hat{K} = \frac{1}{2}\left(xy - \frac{\partial^2}{\partial x \partial y}\right),$$

$$\hat{L} = \frac{1}{2i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

of O_3 group homomorphic to the SU_2 group of hidden symmetry of the circular oscillator [2]. Such bases are realized by solutions of the Schrödinger equation in the Cartesian co-ordinate system (x, y), Cartesian rotated by 45° (x', y), and polar ones. Eigenvalues of operators $\hat{\mathscr{P}}$, $\hat{K}$ and $\hat{L}$ represent the corresponding separation constants.

Let us recall the explicit form of the fundamental bases:

$$(1)\quad \begin{cases} \Psi_{J+\mu,J-\mu}(x,y) = \dfrac{(-i)^{J-\mu}}{2^J\sqrt{\pi}}\dfrac{\exp[-\frac{1}{2}(x^2+y^2)]}{\sqrt{(J+\mu)!(J-\mu)!}}H_{J+\mu}(x)H_{J-\mu}(y)\,, \\ \Psi_{J+\bar{\mu},J-\bar{\mu}}(x',y') = \dfrac{1}{2^J\sqrt{\pi}}\dfrac{\exp[-\frac{1}{2}(x'^2+y'^2)]}{\sqrt{(J+\bar{\mu})!\,(J-\bar{\mu})!}}H_{J+\bar{\mu}}(x)H_{J-\bar{\mu}}(y)\,, \\ \qquad x' = \dfrac{x-y}{\sqrt{2}}\,, \qquad y' = \dfrac{x+y}{\sqrt{2}}\,, \end{cases}$$

$$(2)\quad \begin{cases} \Psi_{2J,2\mu'}(r,\varphi) = R_{2J,2\mu'}(r)\dfrac{1}{\sqrt{2\pi}}\exp[2i\mu'\varphi]\,, \\ R_{2J,2\mu'}(r) = (-1)^{J-|\mu'|}\sqrt{\dfrac{2(J+|\mu'|)!}{(J-|\mu'|)!}}\dfrac{r^{2|\mu'|}}{(2|\mu'|)!}\cdot \\ \qquad \cdot\exp\left[-\dfrac{r^2}{2}\right]F(-J+|\mu'|, 2|\mu'|+1, r^2)\,. \end{cases}$$

The phase factors of the wave functions are chosen for convenience. Quantum numbers μ, $\bar{\mu}$ and μ' change within limits: $-J$, $-J+1, \dots,$ $J-1$, J. Number J can be both integer and half-integer and completely determines the energy spectrum $E = 2J+1$.

([2]) M. J. ENGLEFIELD: *Group Theory and the Coulomb Problem* (Wiley-Interscience, New York, N. Y., London, Sydney, Toronto, 1972).

3. – Expansions in terms of the above fundamental bases are as follows:

$$\Psi_{J+\mu,J-\mu}(x,y)=\sum_{\bar{\mu}=-J}^{J}(-i)^{J-\bar{\mu}}\,d^J_{-\mu,\bar{\mu}}\left(\frac{\pi}{2}\right)\Psi_{J+\bar{\mu},J-\bar{\mu}}(x',y')\,,\tag{3}$$

$$\Psi_{J+\mu,J-\mu}(x,y)=\sum_{\mu'=-J}^{J}d^J_{\mu',\mu}\left(\frac{\pi}{2}\right)\Psi_{2J,2\mu'}(r,\varphi)\,,\tag{4}$$

$$\Psi_{J+\bar{\mu},J-\bar{\mu}}(x,y)=\sum_{\mu'=-J}^{J}(i)^{J+\bar{\mu}-\mu'}\,d^J_{\mu',-\bar{\mu}}\left(\frac{\pi}{2}\right)\Psi_{2J,2\mu'}(r,\varphi)\,,\tag{5}$$

where

$$d^J_{\mu,\mu'}\left(\frac{\pi}{2}\right)=\frac{(-1)^{\mu-\mu'}}{2^J}\sqrt{\frac{(J+\mu)!\,(J-\mu)!}{(J+\mu')!\,(J-\mu')!}}\sum_k(-1)^k\binom{J+\mu'}{k}\binom{J-\mu'}{k+\mu-\mu'}.\tag{6}$$

$d^J_{\mu,\mu'}(\beta)$ is the Wigner function from the quantum theory of angular momentum (3).

Expansions (3)-(5) can easily be proved. To obtain expansion coefficients of (4) and (5), one should in both their sides take the limit $r\to\infty$, use the orthonormalization of functions $(\exp[im\varphi])/\sqrt{2\pi}$, formula (6), and the symmetry relation

$$d^J_{\mu,\mu'}(\beta)=(-1)^{\mu-\mu'}\,d^J_{\mu,\mu'}(\beta)\,.$$

Expansion (3) in verified by substituting (5) into it and then reducing the result (by the addition theorem for d-functions) to expansion (4).

A group-theoretical meaning of expansions (3)-(5) can easily be understood. For what concerns each fundamental basis

$$\hat{J}^2\Psi\equiv(\hat{\mathscr{P}}^2+\hat{K}^2+\hat{L}^2)\Psi=\tfrac{1}{4}(\hat{H}^2-1)\Psi=J(J+1)\Psi\,,$$

$$\hat{\mathscr{P}}\Psi_{J+\mu,J-\mu}(x,y)=\mu\Psi_{J+\mu,J-\mu}(x,y)\,,$$

$$\hat{K}\Psi_{J+\bar{\mu},J-\bar{\mu}}(x',y')=\bar{\mu}\Psi_{J+\bar{\mu},J-\bar{\mu}}(x',y')\,,$$

$$\hat{L}\Psi_{2J,2\mu'}(r,\varphi)=\mu'\Psi_{2J,2\mu'}(r,\varphi)$$

and in view of

$$\{\hat{\mathscr{P}},\hat{K}\}=i\hat{L}\,,\qquad\{\hat{L},\hat{\mathscr{P}}\}=i\hat{K}\,,\qquad\{\hat{K},\hat{L}\}=i\hat{\mathscr{P}}$$

there is an explicit analogy which allows us to interpret transformations (3)-(5) as rotations in the three-dimensional space corresponding to the group of hidden symmetry of the circular oscillator.

(3) D. VARSHALOVICH *et al.*: *Quantum Theory of Angular Momentum* (Nauka, Leningrad, 1975) (in Russian).

4. – Let us introduce the bases with a given parity, *i.e.* eigenfunctions of operators $\hat{H}$, $\hat{\mathscr{P}}$, $\hat{P}_y$, $\hat{P}_{xy}$ and $\hat{H}$, $\hat{L}^2$, $\hat{P}_y$, $\hat{P}_{xy}$, respectively, where by definition

$$\hat{P}_y\Psi(x, y) = \Psi(x, -y), \qquad \hat{P}_y\Psi(r, \varphi) = \Psi(r, -\varphi),$$
$$\hat{P}_{xy}\Psi(x, y) = \Psi(-x, -y), \qquad \hat{P}_{xy}\Psi(r, \varphi) = \Psi(r, \varphi + \pi).$$

It is just these bases that are the limits of the elliptic bases [1].

The Cartesian bases with given parity are determined by formula (1) when

$$J + \mu = 2k, \quad J - \mu = 2n - 2k; \quad J + \mu = 2k + 1, \quad J - \mu = 2n - 2k;$$
$$J + \mu = 2k, \quad J - \mu = 2n - 2k + 1;$$
$$J + \mu = 2k + 1, \quad J - \mu = 2n - 2k + 1.$$

Here n and k are nonnegative integers, and $0 \leqslant k \leqslant n$. For the corresponding polar bases

$$\Psi^{(\pm)}_{2J,m}(r, \varphi) = R_{2J,m}(r)\frac{1}{\sqrt{\pi}}\begin{Bmatrix}\cos m\varphi \\ i \sin m\varphi\end{Bmatrix}$$

and quantum numbers J and m equal: $J = n$, $m = 2p$; $J = n + \frac{1}{2}$, $m = 2p + 1$ with sign «$+$», and $J = n + 1$, $m = 2p + 2$; $J = n + \frac{1}{2}$, $m = 2p + 1$ with sign «$-$». The same holds for the Cartesian basis $n = 0, 1, 2, \ldots$ and $0 \leqslant p \leqslant n$.

In paper [1] the method of separation of variables was used to solve the Schrödinger equation for the circular oscillator in elliptic co-ordinates

$$x = \frac{R}{2}\cosh\xi\cos\eta, \qquad y = \frac{R}{2}\sinh\xi\sin\eta,$$
$$0 \leqslant \xi < \infty, \quad 0 \leqslant \eta < 2\pi, \quad 0 < R < \infty.$$

In terms of these co-ordinates operators $\hat{P}_y$ and $\hat{P}_{xy}$ carry out transformations $\eta \to -\eta$ and $\eta \to \eta + \pi$, respectively. The elliptic bases with given parity are of the form

$$\Psi^{(c,c)} = C^{(c,c)}(R^2)\, hc^{2q}_{2n}\left(i\xi + \frac{\pi}{2}; -R^2\right) hc^{2q'}_{2n}(\eta; R^2),$$

$$\Psi^{(s,c)} = C^{(s,c)}(R^2)\, hs^{2q+1}_{2n+1}\left(i\xi + \frac{\pi}{2}; -R^2\right) hc^{2q'+1}_{2n+1}(\eta; R^2),$$

$$\Psi^{(c,s)} = C^{(c,s)}(R^2)\, hc^{2q+1}_{2n+1}\left(i\xi + \frac{\pi}{2}; -R^2\right) hs^{2q'+1}_{2n+1}(\eta; R^2),$$

$$\Psi^{(s,s)} = C^{(s,s)}(R^2)\, hs^{2q+2}_{2n+2}\left(i\xi + \frac{\pi}{2}; -R^2\right) hs^{2q'+2}_{2n+2}(\eta; R^2),$$

where

$$hc_n^m(z; R^2) = \exp\left[-\frac{R^2}{16}\cos z\right] \sum_{k=0}^{n} a_k(R^2)(\cos z)^k , \tag{7}$$

$$hs_n^m(z; R^2) = \exp\left[-\frac{R^2}{16}\cos z\right] \sin z \sum_{k=0}^{n} b_k(R^2)(\cos z)^k . \tag{8}$$

Coefficients $a_k(R^2)$ and $b_k(R^2)$ and elliptic normalization constants $C(R^2)$ are considered to be known (1). In particular $a_k(R^2)$ and $b_k(R^2)$ are defined by trinomial recurrence relations

$$(k+1)(k+2)\,a_{k+2} + \beta_k a_k + \frac{R^2}{2}(k-n-2)\,a_{k-2} = 0 ,$$

$$(k+1)(k+2)\,b_{k+2} + \tilde{\beta}_k b_k + \frac{R^2}{2}(k-n-1)\,b_{k-2} = 0 ,$$

where

$$\beta_k = -k^2 + \frac{R^2}{4}(E-2k-1) + \frac{R^4}{64} - A(R^2) ,$$

$$\tilde{\beta}_k = -(k+1)^2 + \frac{R^2}{4}(E-2k-1) + \frac{R^4}{64} - A(R^2)$$

and $A(R^2)$ is the elliptic separation constant whose eigenvalues are determined by the condition that the determinants corresponding to the recurrence relations be zero. There are also subsidiary conditions:

$$a_{-1} = a_{-2} = b_{-1} = b_{-2} = 0 , \qquad a_0 = a_1 = b_0 = b_1 = 1 .$$

Quantum numbers q and q' are integer, and $0 \leqslant q \leqslant n$, $0 \leqslant q' \leqslant n$. Number q represents the number of zeros of functions hc and hs dependent on η. When $R \to 0$ and $R \to \infty$, the elliptic bases presented above turn into the polar and Cartesian bases with given parity.

5. – Now let us expand the Cartesian bases with given parity over the polar bases

$$\Psi^{(+)}_{2k,2n-2k}(x, y) = \sum_{p=0}^{n} \omega^{(+)}_{2p} \Psi^{(+)}_{2n,2p}(r, \varphi) , \tag{9}$$

$$\Psi^{(+)}_{2k+1,2n-2k}(x, y) = \sum_{p=0}^{n} \omega^{(+)}_{2p+1} \Psi^{(+)}_{2n+1,2p+1}(r, \varphi) , \tag{10}$$

$$\Psi^{(-)}_{2k,2n-2k+1}(x, y) = \sum_{p=0}^{n} \omega^{(-)}_{2p+1} \Psi^{(-)}_{2n+1,2p+1}(r, \varphi) , \tag{11}$$

$$\Psi^{(-)}_{2k+1,2n-2k+1}(x, y) = \sum_{p=0}^{n} \omega^{(-)}_{2p+2} \Psi^{(-)}_{2n+2,2p+2}(r, \varphi) . \tag{12}$$

From expansion (4) it may easily be verified that

$$
(13)\quad \begin{cases}
\omega_0^{(+)} = \dfrac{1}{\sqrt{2}}\, d^n_{0,2k-n}\left(\dfrac{\pi}{2}\right), & \omega_{2p}^{(+)} = \sqrt{2}\, d^n_{p,2k-n}\left(\dfrac{\pi}{2}\right), \qquad p \neq 0, \\
\omega_{2p+1}^{(+)} = \sqrt{2}\, d^{n+\frac12}_{p+\frac12,2k-n+\frac12}\left(\dfrac{\pi}{2}\right), & \omega_{2p+1}^{(-)} = \sqrt{2}\, d^{n+\frac12}_{p+\frac12,2k-n+\frac12}\left(\dfrac{\pi}{2}\right), \\
\omega_{2p+2}^{(-)} = \sqrt{2}\, d^{n+1}_{p+1,2k-n}\,. &
\end{cases}
$$

The calculation can be made in another way. Taking in (9) the limit $r \to 0$ we get

$$
\omega_0^{(+)} = \frac{(-1)^{n-k}}{2^{n-\frac12}} \frac{\sqrt{(2k)!(2n-2k)!}}{k!(n-k)!}\,.
$$

The other coefficients are calculated with the use of the orthonormalization condition ([4])

$$
(14)\qquad \int\limits_0^\infty R_{2J,m}(r) R_{2J,m'}(r) \frac{\mathrm{d}r}{r} = \frac{1}{m}\, \delta_{mm'}
$$

valid for $m \neq 0$ and $m' \neq 0$ and the relation

$$
\int\limits_0^\infty \exp[-z] z^{m-1} F(a;\, b;\, z) F(c;\, d;\, z)\, \mathrm{d}z = \\
= \frac{\Gamma(m)\Gamma(d)\Gamma(d-c-m)}{\Gamma(d-c)\Gamma(d-m)}\, {}_3F_2\left\{\begin{matrix} a,\, m,\, 1+m-d \\ b,\, 1+m-c-d \end{matrix}\middle|\, 1\right\},
$$

which is verified by expansion of one confluent hypergeometric function of the integrand in powers of Z and further integration. The result is

$$
(15)\quad \begin{cases}
\omega_{2p}^{(+)} = \dfrac{(-1)^{n+k-p} n!}{2^{n-\frac12} k!(n-k)!} \sqrt{\dfrac{(2k)!(2n-2k)!}{(n+p)!(n-p)!}}\, {}_3F_2\left\{\begin{matrix} -k,\, -p,\, p \\ \frac12,\, -n \end{matrix}\middle|\, 1\right\}, \\
\omega_{2p+1}^{(+)} = \dfrac{(-1)^{n+k-p} n!(2p+1)}{2^n k!(n-k)!} \cdot \\
\qquad \cdot \sqrt{\dfrac{(2k+1)!(2n-2k)!}{(n+p+1)(n-p)!}}\, {}_3F_2\left\{\begin{matrix} -k,\, -p,\, p+1 \\ \frac32,\, -n \end{matrix}\middle|\, 1\right\},
\end{cases}
$$

([4]) L. G. Mardoyan, G. S. Pogosyan and V. M. Ter-Antonyan: *Izv. Akad. Nauk Arm. SSR, Ser. Fiz.*, **19**, 45 (1984).

$$
(15) \quad \begin{cases}
\omega^{(-)}_{2p+1} = \dfrac{(-1)^{n+k-p} n!}{2^n k!(n-k)!} \sqrt{\dfrac{(2k)!(2n-2k+1)!}{(n+p+1)!(n-p)!}} \, {}_3F_2 \left\{ \begin{matrix} -k, \ -p, \ p+1 \\ \frac{1}{2}, \ -n \end{matrix} \middle| 1 \right\}, \\
\omega^{(-)}_{2p+2} = \dfrac{(-1)^{n+k-p+1} n!(2p+2)}{2^{n+1} k!(n-k)!} \cdot \\
\qquad \cdot \sqrt{\dfrac{(2k+1)!(2n-2k+1)!}{(n+p+2)!(n-p)!}} \, {}_3F_2 \left\{ \begin{matrix} -k, \ -p, \ p+1 \\ \frac{3}{2}, \ -n \end{matrix} \middle| 1 \right\}.
\end{cases}
$$

The fact that coefficients in expansions (9)-(12) are expressed in terms of the generalized hypergeometric functions ${}_3F_2$ is not accidental and confirms the connection between $d^{j}_{\mu,\mu'}(\varrho)$ and ${}_3F_2(1)$ functions noticed earlier [5,6].

6. – Let us expand the elliptic bases over the polar ones

$$\Psi^{(c,c)} = \sum_{p=0}^{n} W^{(+)}_{2p}(R^2) \Psi^{(+)}_{2n,2p}(r, \varphi),$$

$$\Psi^{(s,c)} = \sum_{p=0}^{n} W^{(+)}_{2p+1}(R^2) \Psi^{(+)}_{2n+1,2p+1}(r, \varphi),$$

$$\Psi^{(c,s)} = \sum_{p=0}^{n} W^{(-)}_{2p+1}(R^2) \Psi^{(-)}_{2n+1,2p+1}(r, \varphi),$$

$$\Psi^{(s,s)} = \sum_{p=0}^{n} W^{(-)}_{2p+2}(R^2) \Psi^{(-)}_{2n+2,2p+2}(r, \varphi).$$

Taking in the first of these expressions the limit $r \to 0$ we get

$$W^{(+)}_0(R^2) = (-1)^n \sqrt{\frac{\pi}{2}} \, C^{(c,c)}(R^2).$$

To calculate the remaining coefficients, we pass, in the polar bases, from the polar to elliptic co-ordinates

$$r = \frac{R}{2}\sqrt{\cosh^2 \xi - \sin^2 \eta}, \qquad \cos\varphi = \frac{\cosh \xi \cos \eta}{\sqrt{\cosh^2 \xi - \sin^2 \eta}},$$

$$\sin\varphi = \frac{\sinh \xi \sin \eta}{\sqrt{\cosh^2 \xi - \sin^2 \eta}}$$

let η tend to $\pi/2$, and take advantage of the orthogonality (14). Then, allowing

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(5) G. I. Kuznetzov and Ya. A. Smorodinsky: *Yad. Fiz.*, **25**, 447 (1977) (English translation: *Sov. Nucl. Phys.* (1978).

(6) G. S. Pogosyan, Ya. A. Smorodinsky and V. M. Ter-Antonyan: JINR Communications P2-82-118, Dubna (1982).

for formulae (7) and (8), we obtain

$$W^{(+)}_{2p}(R^2) = \frac{(-1)^n\sqrt{2\pi}n!}{\sqrt{(n+p)!(n-p)!}} C^{(c,c)}(R^2) \sum_{s=0}^{p} \left(-\frac{4}{R^2}\right)^s a_{2s}(-R^2) \frac{(p)_s(-p)_s}{(-n)_s}, \quad p \neq 0,$$

$$W^{(+)}_{2p+1}(R^2) = \frac{(-1)^n\sqrt{2\pi}}{\sqrt{(n+p+1)!(n-p)!}} \frac{C^{(s,c)}(R^2)}{R} \sum_{s=0}^{p} \left(-\frac{4}{R^2}\right)^s b_{2s}(-k^2) \frac{(p-1)_s(-p)_s}{(-n)_s},$$

$$W^{(-)}_{2p+1}(R^2) = \frac{(-1)^{n+1}\sqrt{2\pi}(2p+1)}{\sqrt{(n+p+1)!(n-p)!}} \frac{C^{(c,s)}(R^2)}{R} \cdot \sum_{s=0}^{p} \left(-\frac{4}{R^2}\right)^s a_{2s+1}(-k^2) \frac{(p+1)_s(-p)_s}{(-n)_s},$$

$$W^{(-)}_{2p+2}(R^2) = \frac{(-1)^{n+1}\sqrt{2\pi}n!(2p+2)}{\sqrt{(n+p+2)!(n-p)!}} \frac{C^{(s,s)}(R^2)}{R^2} \cdot \sum_{s=0}^{p} \left(-\frac{4}{R^2}\right)^s b_{2s+1}(-k^2) \frac{(p+2)_s(-p)_s}{(-n)_s},$$

where

$$(\alpha)_s = \Gamma(\alpha+s)/\Gamma(\alpha).$$

Next, let us expand the elliptic bases over the Cartesian ones:

$$\Psi^{(c,c)} = \sum_{k=0}^{n} U^{(+)}_{2k}(R^2)\Psi^{(+)}_{2k,2n-2k}(x,y),$$

$$\Psi^{(s,c)} = \sum_{k=0}^{n} U^{(+)}_{2k+1}(R^2)\Psi^{(+)}_{2k+1,2n-2k}(x,y),$$

$$\Psi^{(c,s)} = \sum_{k=0}^{n} U^{(-)}_{2k}(R^2)\Psi^{(-)}_{2k,2n-2k+1}(x,y),$$

$$\Psi^{(s,s)} = \sum_{k=0}^{n} U^{(-)}_{2k+1}(R^2)\Psi^{(-)}_{2k+1,2n-2k+1}(x,y).$$

Then we take the limit $\eta \to \pi/2$ and make use of the orthogonality of the Hermite polynomials and formulae (7), (8):

$$U^{(+)}_{2k}(R^2) = \frac{(-1)^k 2^n k!\sqrt{\pi}}{\sqrt{(2k)!(2n-2k)!}} \frac{C^{(c,c)}(k^2)}{(R^2)^{n-k}} \sum_{s=0}^{k} a_{2s+2n-2k}(-k^2) \frac{(2s+2n-2k)!}{s!(-R^2)^s},$$

$$U^{(+)}_{2k+1}(R^2) = \frac{(-1)^k 2^n k!\sqrt{\pi}}{\sqrt{(2k+1)!(2n-2k)!}} \frac{C^{(s,c)}(R^2)}{R(R^2)^{n-k}} \sum_{s=0}^{k} b_{2s+2n-2k}(-R^2) \frac{(2s+2n-2k)!}{s!(-R^2)^s},$$

n	k	$U_{2k}^{(+)}(R^2)$	$\lambda^{(c,c)}$
0	0	1	$\lambda^{(c,c)} = 0$
1	0	$\frac{1}{\sqrt{2}}\left\{\frac{\lambda^{(c,c)}+2+R^2/2}{\lambda^{(c,c)}+2}\right\}^{1/2}$	$\lambda^{(c,c)}(\lambda^{(c,c)}+4) = \frac{R^4}{4}$
1	1	$-\frac{1}{2\sqrt{2}}\left(\lambda^{(c,c)}+2-\frac{R^2}{2}\right)\left\{\frac{\lambda^{(c,c)}+2+R^2/2}{\lambda^{(c,c)}+2}\right\}^{1/2}$	
2	0	$\left\{1+\frac{1}{24}(\lambda^{(c,c)}+4-R^2)^2+\left(\frac{\lambda^{(c,c)}+4-R^2}{\lambda^{(c,c)}+4+R^2}\right)^2\right\}^{-1/2}$	$\lambda^{(c,c)}(\lambda^{(c,c)}+4)(\lambda^{(c,c)}+16) = R^4(\lambda^{(c,c)}+12)$
2	1	$-\frac{1}{2\sqrt{6}}(\lambda^{(c,c)}+4-R^2)\left\{1+\frac{1}{24}(\lambda^{(c,c)}+4-R^2)^2+\left(\frac{\lambda^{(c,c)}+4-R^2}{\lambda^{(c,c)}+4+R^2}\right)^2\right\}^{-1/2}$	
2	2	$\frac{\lambda^{(c,c)}+4-R^2}{\lambda^{(c,c)}+4+R^2}\left\{1+\frac{1}{24}(\lambda^{(c,c)}+4-R^2)+\left(\frac{\lambda^{(c,c)}+4-R^2}{\lambda^{(c,c)}+4-R}\right)^2\right\}^{-1/2}$	

n	p	$W_{2p+1}^{(-)}(R^2)$	$\lambda^{(c,s)}$
0	0	1	$\lambda^{(c,s)} = -1+\frac{R^2}{4}$
1	0	$\left\{\frac{\lambda^{(c,s)}+9}{2\lambda^{(c,s)}+10-R^2/2}\right\}^{1/2}$	$(\lambda^{(c,s)}+1)(\lambda^{(c,s)}+9) = \frac{R^2}{2}(\lambda^{(c,s)}+9)+\frac{3R^4}{16}$
1	1	$-\frac{4}{\sqrt{3}R^2}(\lambda^{(c,s)}+1-R^2/2)\left\{\frac{\lambda^{(c,s)}+9}{2\lambda^{(c,s)}+10-R^2/2}\right\}^{1/2}$	
2	0	$\left\{1+\frac{2}{R^4}(\lambda^{(c,s)}+1-3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(c,s)}+1-3R^2/4}{\lambda^{(c,s)}+25}\right)^2\right\}^{-1/2}$	$(\lambda^{(c,s)}+1)(\lambda^{(c,s)}+9)(\lambda^{(c,s)}+25) = \frac{3R^2}{4}(\lambda^{(c,s)}+9)(\lambda^{(c,s)}+25)+\frac{R^4}{16}(13\lambda^{(c,s)}+205)-\frac{15R^6}{64}$
2	1	$-\frac{\sqrt{2}}{R^2}(\lambda^{(c,s)}+1-3R^2/4)\left\{1+\frac{2}{R^4}(\lambda^{(c,s)}+1-3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(c,s)}+1-3R^2/4}{\lambda^{(c,s)}+25}\right)^2\right\}^{1/2}$	
2	2	$\frac{\sqrt{10}}{4}\frac{\lambda^{(c,s)}+1-3R^2/4}{\lambda^{(c,s)}+25}\left\{1+\frac{2}{R^4}(\lambda^{(c,s)}+1-3R^2/4)^2+\frac{5}{8}\left(\frac{\lambda^{(c,s)}+1-3R^2/4}{\lambda^{(c,s)}+25}\right)^2\right\}^{1/2}$	

n	p	$W^{(+)}_{2p+1}(R^2)$	$\lambda^{(s,c)}$
0	0	1	$\lambda^{(s,c)} = -1 - \frac{R^2}{4}$
1	0	$\left\{\frac{\lambda^{(s,c)} + 9}{2\lambda^{(s,c)} + 10 + R^2/2}\right\}^{\frac{1}{2}}$	$(\lambda^{(s,c)} + 1)(\lambda^{(s,c)} + 9) =$
1	1	$-\frac{4}{\sqrt{3}R^2}(\lambda^{(s,c)} + 1 + R^2/2)\left\{\frac{\lambda^{(s,c)} + 9}{2\lambda^{(s,c)} + 10 + R^2/2}\right\}^{\frac{1}{2}}$	$= -\frac{R^2}{2}(\lambda^{(s,c)} + 9) + \frac{3R^4}{16}$
2	0	$\left\{1 + \frac{2}{R^4}(\lambda^{(s,c)} + 1 + 3R^2/4)^2 + \frac{5}{8}\left(\frac{\lambda^{(s,c)} + 1 + 3R^2/4}{\lambda^{(s,c)} + 25}\right)^2\right\}^{-\frac{1}{2}}$	$\lambda(^{(s,c)} + 1)(\lambda^{(s,c)} + 9)(\lambda^{(s,c)} + 25) =$
2	1	$-\frac{\sqrt{2}}{R^2}(\lambda^{(s,c)} + 1 + 3R^2/4)\left\{1 + \frac{2}{R^4}(\lambda^{(s,c)} + 1 + 3R^2/4)^2 + \frac{5}{8}\left(\frac{\lambda^{(s,c)} + 1 + 3R^2/4}{\lambda^{(s,c)} + 25}\right)^2\right\}^{-\frac{1}{2}}$	$= -\frac{3R^2}{4}(\lambda^{(s,c)} + 9)(\lambda^{(s,c)} + 25) +$
2	2	$\frac{\sqrt{10}}{4}\frac{\lambda^{(s,c)} + 1 + 3R^2/4}{\lambda^{(s,c)} + 25}\left\{1 + \frac{2}{R^4}(\lambda^{(s,c)} + 1 + 3R^2/4)^2 + \frac{5}{8}\left(\frac{\lambda^{(s,c)} + 1 + 3R^2/4}{\lambda^{(s,c)} + 25}\right)^2\right\}^{\frac{1}{2}}$	$+ \frac{R^4}{16}(13\lambda^{(s,c)} + 205) + \frac{15R^6}{64}$

n	p	$W^{(+)}_{2p}(R^2)$	$\lambda^{(c,c)}$
0	0	1	$\lambda^{(c,c)} = 0$
1	0	$\left(\frac{\lambda^{(c,c)} + 4}{3\lambda^{(c,c)} + 4}\right)^{\frac{1}{2}}$	$\lambda^{(c,c)}(\lambda^{(c,c)} + 4) = \frac{R^4}{4}$
1	1	$-\frac{2\sqrt{2}}{R^2}\lambda^{(c,c)}\left(\frac{\lambda^{(c,c)} + 4}{3\lambda^{(c,c)} + 4}\right)^{\frac{1}{2}}$	
2	0	$\left\{1 + \frac{8}{3}\left(\frac{\lambda^{(c,c)}}{R^2}\right)^2 + \frac{2}{3}\left(\frac{\lambda^{(c,c)}}{\lambda^{(c,c)} + 16}\right)^2\right\}^{-\frac{1}{2}}$	$\lambda^{(c,c)}(\lambda^{(c,c)} + 4)(\lambda^{(c,c)} + 16) =$ $= R^4(\lambda^{(c,c)} + 12)$
2	1	$-\frac{4}{\sqrt{6}}\frac{\lambda^{(c,c)}}{R^2}\left\{1 + \frac{8}{3}\left(\frac{\lambda^{(c,c)}}{R^2}\right)^2 + \frac{2}{3}\left(\frac{\lambda^{(c,c)}}{\lambda^{(c,c)} + 16}\right)^2\right\}^{-\frac{1}{2}}$	
2	2	$\frac{2}{\sqrt{6}}\frac{\lambda^{(c,c)}}{\lambda^{(c,c)} + 16}\left\{1 + \frac{8}{3}\left(\frac{\lambda^{(c,c)}}{R^2}\right)^2 + \frac{2}{3}\left(\frac{\lambda^{(c,c)}}{\lambda^{(c,c)} + 16}\right)^2\right\}^{-\frac{1}{2}}$	

n	p	$W^{(-)}_{2p+2}(R^2)$	$\lambda^{(s,s)}$
0	0	1	$\lambda^{(s,s)} = -4$
1	0	$\frac{1}{\sqrt{2}}\left\{\frac{\lambda^{(s,s)}+16}{\lambda^{(s,s)}+10}\right\}^{1/2}$	$(\lambda^{(s,s)}+4)(\lambda^{(s,s)}+16) = \frac{R^4}{4}$
1	1	$-\frac{\sqrt{2}}{R^2}(\lambda^{(s,s)}+4)\left\{\frac{\lambda^{(s,s)}+16}{\lambda^{(s,s)}+10}\right\}^{1/2}$	
2	0	$\left\{1+\frac{8}{5R^4}(\lambda^{(s,s)}+4)^2+\frac{3}{5}\left(\frac{\lambda^{(s,s)}+4}{\lambda^{(s,s)}+36}\right)^2\right\}^{-1/2}$	$(\lambda^{(s,s)}+4)(\lambda^{(s,s)}+16)\times$ $\times\lambda(^{(s,s)}+36) = R^4(\lambda^{(s,s)}+24)$
2	1	$-\frac{4}{\sqrt{10}R^2}(\lambda^{(s,s)}+4)\left\{1+\frac{8}{5R^4}(\lambda^{(s,s)}+4)^2+\frac{3}{5}\left(\frac{\lambda^{(s,s)}+4}{\lambda^{(s,s)}+36}\right)^2\right\}^{-1/2}$	
2	2	$\sqrt{\frac{3}{5}}\frac{\lambda^{(s,s)}+4}{\lambda^{(s,s)}+36}\left\{1+\frac{8}{5R^4}(\lambda^{(s,s)}+4)^2+\frac{3}{5}\left(\frac{\lambda^{(s,s)}+4}{\lambda^{(s,s)}+36}\right)^2\right\}^{-1/2}$	

n	k	$U^{(+)}_{2k+1}(R^2)$	$\lambda^{(s,c)}$
0	0	1	$\lambda^{(s,c)} = -1-\frac{R^2}{4}$
1	0	$\left\{\frac{\lambda^{(s,c)}+3+3R^2/4}{2\lambda^{(s,c)}+10+R^2/2}\right\}^{1/2}$	$(\lambda^{(s,c)}+1)(\lambda^{(s,c)}+9) =$ $= -\frac{R^2}{2}(\lambda^{(s,c)}+9)+\frac{3R^4}{16}$
1	1	$-\frac{1}{2\sqrt{3}}(\lambda^{(s,c)}+7-R^2/4)\left\{\frac{\lambda^{(s,c)}+3+3R^2/4}{2\lambda^{(s,c)}+10+R^2/2}\right\}^{1/2}$	
2	0	$\left\{1+\frac{1}{72}(\lambda^{(s,c)}+13-3R^2/4)^2+\frac{5}{9}\left(\frac{\lambda^{(s,c)}+13-3R^2/4}{\lambda^{(s,c)}+5+5R^2/4}\right)^2\right\}^{-1/2}$	$(\lambda^{s,c}+1)(\lambda^{(s,c)}+9)(\lambda^{(s,c)}+25) =$ $= -\frac{3R^2}{4}(\lambda^{(s,c)}+9)(\lambda^{(s,c)}+25)+$ $+\frac{R^4}{16}(13\lambda^{(s,c)}+205)+\frac{15R^6}{64}$
2	1	$-\frac{\lambda^{(s,c)}+13-3R^2/4}{6\sqrt{2}}\left\{1+\frac{1}{72}(\lambda^{(s,c)}+13-3R^2/4)^2+\frac{5}{9}\left(\frac{\lambda^{(s,c)}+13-3R^2/4}{\lambda^{(s,c)}+5+5R^2/4}\right)^2\right\}^{-1/2}$	
2	2	$\frac{\sqrt{5}}{3}\frac{\lambda^{(s,c)}+13-3R^2/4}{\lambda^{(s,c)}+5+5R^2/4}\left\{1+\frac{1}{72}(\lambda^{(s,c)}+13-3R^2/4)^2+\frac{5}{9}\left(\frac{\lambda^{(s,c)}+13-3R^2/4}{\lambda^{(s,c)}+5+5R^2/4}\right)^2\right\}^{-1/2}$	

n	k	$U^{(-)}_{2k+1}(R^2)$	$\lambda^{(s,s)}$
0	0	1	$\lambda^{(s,s)} = -4$
1	0	$\frac{1}{\sqrt{2}}\left\{\frac{\lambda^{(s,s)} + 10 + R^2/2}{\lambda^{(s,s)} + 10}\right\}^{\frac{1}{2}}$	$(\lambda^{(s,s)} + 4)(\lambda^{(s,s)} + 16) = \frac{R^4}{4}$
1	1	$-\frac{1}{6\sqrt{2}}(\lambda^{(s,s)} + 10 - R^2/2)\left\{\frac{\lambda^{(s,s)} + 10 + R^2/2}{\lambda^{(s,s)} + 10}\right\}^{\frac{1}{2}}$	
2	0	$\left\{1 + \frac{(\lambda^{(s,s)} + 16 - R^2)^2}{120} + \left(\frac{\lambda^{(s,s)} + 16 - R^2}{\lambda^{(s,s)} + 16 + R^2}\right)^2\right\}^{-\frac{1}{2}}$	$(\lambda^{(s,s)} + 4)(\lambda^{(s,s)} + 16) \times$ $\times(\lambda^{(s,s)} + 36) = R^4(\lambda^{(s,s)} + 24)$
2	1	$-\frac{\lambda^{(s,s)} + 16 - R^2}{2\sqrt{30}}\left\{1 + \frac{1}{120}(\lambda^{(s,s)} + 16 - R^2)^2 + \left(\frac{\lambda^{(s,s)} + 16 - R^2}{\lambda^{(s,s)} + 16 + R^2}\right)^2\right\}^{-\frac{1}{2}}$	
2	2	$\frac{\lambda^{(s,s)} + 16 - R^2}{\lambda^{(s,s)} + 16 + R^2}\left\{1 + \frac{1}{120}(\lambda^{(s,s)} + 16 - R^2)^2 + \left(\frac{\lambda^{(s,s)} + 16 - R^2}{\lambda^{(s,s)} + 16 + R^2}\right)^2\right\}^{-\frac{1}{2}}$	

n	k	$U^{(-)}_{2k}(R^2)$	$\lambda^{(c,s)}$
0	0	1	$\lambda^{(c,s)} = -1 + \frac{R^2}{4}$
1	0	$\left\{\frac{\lambda^{(c,s)} + 7 + R^2/4}{2\lambda^{(c,s)} + 10 - R^2/4}\right\}^{\frac{1}{2}}$	$(\lambda^{(c,s)} + 1)(\lambda^{(c,s)} + 9) =$ $= \frac{R^2}{2}(\lambda^{(c,s)} + 9) + \frac{3R^4}{16}$
1	1	$-\frac{1}{2\sqrt{3}}(\lambda^{(c,s)} + 3 - 3R^2/4)\left\{\frac{\lambda^{(c,s)} + 7 + R^2/4}{2\lambda^{(c,s)} + 10 - R^2/4}\right\}^{\frac{1}{2}}$	
2	0	$\left\{1 + \frac{1}{40}(\lambda^{(c,s)} + 5 - 5R^2/4)^2 + \frac{3}{5}\left(\frac{\lambda^{(c,s)} + 5 - 5R^2/4}{\lambda^{(c,s)} + 13 + 3R^2/4}\right)^2\right\}^{-\frac{1}{2}}$	$(\lambda^{(c,s)} + 1)(\lambda^{(c,s)} + 9)(\lambda^{(c,s)} + 25) =$ $= \frac{3R^2}{4}(\lambda^{(c,s)} + 9)(\lambda^{(c,s)} + 25) +$ $+ \frac{R^4}{16}(13\lambda^{(c,s)} + 205) - \frac{15R^6}{16}$
2	1	$-\frac{1}{2\sqrt{10}}(\lambda^{(c,s)} + 5 - 5R^2/4)\left\{1 + \frac{1}{40}(\lambda^{(c,s)} + 5 - 5R^2/4)^2 + \frac{3}{5}\left(\frac{\lambda^{(c,s)} + 5 - 5R^2/4}{\lambda^{(c,s)} + 13 + 3R^2/4}\right)^2\right\}^{-\frac{1}{2}}$	
2	2	$\sqrt{\frac{3}{5}}\,\frac{\lambda^{(c,s)} + 5 - 5R^2/4}{\lambda^{(c,s)} + 13 + 3R^2/4}\left\{1 + \frac{1}{40}(\lambda^{(c,s)} + 5 - 5R^2/4)^2 + \frac{3}{5}\left(\frac{\lambda^{(c,s)} + 5 - 5R^2/4}{\lambda^{(c,s)} + 13 + 3R^2/4}\right)^2\right\}^{-\frac{1}{2}}$	

$$U_{2k}^{(-)}(R^2) = \frac{(-1)^{k+1} 2^n k! \sqrt{2\pi}}{\sqrt{(2k)!(2n-2k+1)!}} \frac{C^{(c,s)}(R^2)}{R(R^2)^{n-k}} \cdot$$

$$\cdot \sum_{s=0}^{k} a_{2s+2n-2k+1}(-R^2) \frac{(2s+2n-2k+1)!}{s!(-R^2)^s},$$

$$U_{2k+1}^{(-)}(R^2) = \frac{(-1)^k 2^{n+1} k! \sqrt{\pi}}{\sqrt{(2k+1)!(2n-2k+1)!}} \frac{C^{(s,s)}(R^2)}{R^2(R^2)^{n-k}} \cdot$$

$$\cdot \sum_{s=0}^{k} b_{2s+2n-2k+1}(-R^2) \frac{(2s+2n-2k+1)!}{s!(-R^2)^s}.$$

For completeness we also present the tables with particular values of coefficients $W(R^2)$ and $U(R^2)$. (There the notation $\lambda = A(R^2) - R^4/64$ is used.)

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● RIASSUNTO (*)

Si calcolano i coefficienti di sviluppo per le basi fondamentali di un oscillatore circolare quantistico. Si trovano formule per i coefficienti di sviluppo delle basi su quelle polari e cartesiane. Si tabulano i valori di questi coefficienti per alcuni casi particolari.

(*) *Traduzione a cura della Redazione.*

Межбазисные разложения в круговом осцилляторе.

Резюме. — Вычислены коэффициенты, определяющие разложения в фундаментальных базисах квантового кругового осциллятора. Получены формулы для коэффициентов в разложениях эллиптических базисов по полярным и декартовым. Составлены табличные значения этих коэффициетов для некоторых частных случаев.