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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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AN APPROXIMATE METHOD OF SOLVING THE QUASIPOTENTIAL EQUATIONS

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Метод приближенного решения квазипотенциальных уравнений

В работе развит метод приближенного решения квазипотенциальных уравнений для амплитуды рассеяния скалярных частиц. Приводится пример использования полученного приближенного решения.

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An Approximate Method of Solving the Quasipotential Equations

An approximate method of solving the quasipotential equations for the scalar particle scattering amplitude is developed. An example is given how the obtained approximate solution can be used.

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Introduction

The quasipotential approach in the problems of quantum field theory has lately become widely known[1]. The merits of this method have brought to life further investigations connected with its concrete applications (see, e.g.[2,3]) and modifications[4,5].

Because of the difficulties in finding the exact solutions of the quasipotential equations the utilization of various approximate methods is of considerable interest. In the present paper an approximate method of solving the quasipotential equations is suggested and its formulation is given in the first section. Let us note that the mentioned method is strongly connected with the modified perturbation theory in exponent[6] and is analogous to the one used in the paper[7] for the approximate solution of the BS equation.

It is further shown that the obtained approximate solution can be used, for example, to find the asymptotic behaviour of the scattering amplitude in the region of high energies and fixed momentum transfers.

1. Let us consider a quasipotential equation with local quasipotential for the scattering amplitude of scalar particles

$$T(\vec{p},\vec{p}\,';s)=gV(\vec{p}-\vec{p}\,';s)+g\int d\vec{q}\,K(\vec{q}^{\,2};s)V(\vec{p}-\vec{q};s)T(\vec{q},\vec{p}\,';s)\,, \qquad (1.1)$$

which in case of a suitable choice of the kernel and the quasipotential turns into one of the following well-known equations[x/]

Equation	$K(\vec{q}^{\,2};s)$	$gV(\vec{p}-\vec{p}\,';s)$
Lippmann-Schwinger	$\frac{1}{2\pi^2}\frac{1}{\vec{q}^{\,2}-k^2-i\epsilon}$	$-\frac{m}{2\pi\hbar^2}V(\vec{p}-\vec{p}\,';s)$
Logunov-Tavkhelidze	$\frac{1}{\sqrt{\vec{q}^{\,2}+m^2}}\frac{1}{\vec{q}^{\,2}-k^2-i\epsilon}$	$gV(\vec{p}-\vec{p}\,';s)$
Kadyshevsky	$\frac{1}{4\pi^2}\frac{1}{\sqrt{\vec{q}^{\,2}+m^2}}\frac{1}{\sqrt{\vec{q}^{\,2}+m^2}-\sqrt{k^2+m^2}-i\epsilon}$	$-\frac{m}{4\pi}V(\vec{p}-\vec{p}\,';s)$

To solve eq. (1.1) let us perform the Fourier transformation

$$V(\vec{p}-\vec{p}\,';s)=\frac{1}{(2\pi)^3}\int d\vec{r}\,e^{i(\vec{p}-\vec{p}\,')\vec{r}}\;V(\vec{r};s)\,, \qquad (1.2)$$

$$T(\vec{p},\vec{p}\,';s)=\int d\vec{r}\,d\vec{r}\,'\,e^{i\vec{p}\vec{r}-i\vec{p}\,'\vec{r}\,'}\;T(\vec{r},\vec{r}\,';s)\,. \qquad (1.3)$$

Substituting (1.2) and (1.3) in (1.1) we obtain

$$T(\vec{r},\vec{r}\,';s)=\frac{g}{(2\pi)^3}V(\vec{r};s)\,\delta^{(3)}(\vec{r}-\vec{r}\,')+ \qquad (1.4)$$

$$+\frac{g}{(2\pi)^3}\int d\vec{q}\,K(\vec{q}^{\,2};s)V(\vec{r};s)\,e^{-i\vec{q}\vec{r}}\int d\vec{r}\,''\,e^{i\vec{q}\vec{r}\,''}\,T(\vec{r}\,'',\vec{r}\,';s)\,.$$

[x/] Note that in the case of the Kadyshevsky equation the potential $V(\vec{p}-\vec{p}\,';s)$ is not local in terms of paper/4/.

Introducing the representation

$$T(\vec{r},\vec{r}\,';s)=\frac{g}{(2\pi)^3}V(\vec{r};s)F(\vec{r},\vec{r}\,';s) \tag{1.5}$$

we have

$$F(\vec{r},\vec{r}\,';s)=\delta^{(3)}(\vec{r}-\vec{r}\,')+\frac{g}{(2\pi)^3}\int d\vec{q}\,K(\vec{q}\,^2;s)e^{-i\vec{q}\vec{r}}\int d\vec{r}\,''e^{i\vec{q}\vec{r}\,''}V(\vec{r}\,'',s)F(\vec{r}\,'',\vec{r}\,';s) \tag{1.6}$$

It is worth noting papers [8,9] in which the connection between the quasipotential equation and the nonlocal Schrödinger equation is indicated.

Let us define the differential operator

$$\hat{L}_{\vec{r}}=K(-\vec{\nabla}^2_{\vec{r}}\;;s). \tag{1.7}$$

So

$$K(\vec{r};s)=\int d\vec{q}\,e^{-i\vec{q}\vec{r}}K(\vec{q}\,^2;s)=\hat{L}_{\vec{r}}(2\pi)^3\delta^{(3)}(\vec{r}). \tag{1.8}$$

Taking into account the expression (1.8), eq. (1.6) may be written in the following symbolic form

$$F(\vec{r},\vec{r}\,';s)=\delta^{(3)}(\vec{r}-\vec{r}\,')+g\,\hat{L}_{\vec{r}}[V(\vec{r};s)F(\vec{r},\vec{r}\,';s)]. \tag{1.9}$$

We shall seek the solution of this equation in the following form

$$F(\vec{r},\vec{r}\,';s)=\frac{1}{(2\pi)^3}\int d\vec{k}\,e^{W(\vec{r},\vec{k};s)}e^{-i\vec{k}(\vec{r}-\vec{r}\,')}. \tag{1.10}$$

Substituting (1.10) in (1.9), we obtain

$$\frac{1}{(2\pi)^3}\int d\vec{k}\,e^{W(\vec{r},\vec{k};s)-i\vec{k}(\vec{r}-\vec{r}\,')}=\frac{1}{(2\pi)^3}\int d\vec{k}\,e^{-i\vec{k}(\vec{r}-\vec{r}\,')}+$$

$$+\frac{g}{(2\pi)^3}\hat{L}_{\vec{r}}\int d\vec{k}\,V(\vec{r};s)\,e^{W(\vec{r},\vec{k};s)-i\vec{k}(\vec{r}-\vec{r}\,')}$$

from which follows the equation for the function $W(\vec{r},\vec{k};s)$

$$e^{W(\vec{r},\vec{k};s)} = 1 + g\hat{L}_{\vec{r}}[V(\vec{r};s)\,e^{W(\vec{r},\vec{k};s)-i\vec{k}\vec{r}}]\,e^{i\vec{k}\vec{r}} \quad . \tag{1.11}$$

Expanding the function W in powers of the coupling constant g

$$W(\vec{r},\vec{k};s) = \sum_{n=1}^{\infty} g^n W_n(\vec{r},\vec{k};s) \tag{1.12}$$

we immediately obtain from eq. (1.11) the expression for the function $W_1(\vec{r},\vec{k};s)$

$$\begin{aligned} W_1(\vec{r},\vec{k};s) &= \hat{L}_{\vec{r}}[V(\vec{r};s)\,e^{-i\vec{k}\vec{r}}]\,e^{i\vec{k}\vec{r}} = \\ &= \int d\vec{q}\,V(\vec{q};s)\hat{L}_{\vec{r}}[e^{-i(\vec{k}+\vec{q})\vec{r}}]\,e^{i\vec{k}\vec{r}} = \\ &= \int d\vec{q}\,V(\vec{q};s)\,K[(\vec{k}+\vec{q})^2;s]\,e^{-i\vec{q}\vec{r}} \quad . \end{aligned} \tag{1.13}$$

Confining ourselves to considering only W_1 instead of W in formula (1.10) we obtain from (1.10), (1.5) and (1.3) the following approximate expression for the scattering amplitude

$$T_1(\vec{p},\vec{p}\,';s) = \frac{g}{(2\pi)^3}\int d\vec{r}\,e^{i(\vec{p}-\vec{p}\,')\vec{r}}\,V(\vec{r};s)\,e^{gW_1(\vec{r},\vec{p}\,',s)} \quad . \tag{1.14}$$

The meaning of the approximation made above will be clear if we expand $T_1(\vec{p},\vec{p}\,';s)$ in powers of the coupling constant g

$$\begin{aligned} T_1^{(n+1)}(\vec{p},\vec{p}\,';s) = \frac{g^{n+1}}{n!}\int d\vec{q}_1\ldots d\vec{q}_n\,V(\vec{q}_1;s)\ldots V(\vec{q}_n;s)\,. \\ \cdot\, V(\vec{p}-\vec{p}\,'-\sum_{l=1}^{n}\vec{q}_l;s)\prod_{l=1}^{n}K[(\vec{q}_l+\vec{p}\,')^2;s] \end{aligned} \tag{1.15}$$

and compare it with the $(n+1)$-th iteration term of eq.(1.1)

$$T^{(n+1)}(\vec{p},\vec{p}\,';s) = \frac{g^{n+1}}{n!}\int d\vec{q}_1 \dots d\vec{q}_n \; V(\vec{q}_1;s)\dots V(\vec{q}_n;s)\,. \tag{1.16}$$

$$V(\vec{p}-\vec{p}\,'-\sum_{l=1}^{n}\vec{q}_l\,;s)\sum_{p} K\,[(\vec{q}_1+\vec{p}\,')^2;s]\,K[(\vec{q}_1+\vec{q}_2+\vec{p}\,')^2;s]\dots K[(\sum_{l=1}^{n}\vec{q}_l+$$

$$+\vec{p}\,')^2;s\,]\,,$$

where $\sum_p$ denote the sum over all possible permutations of momenta $q_1, \dots, q_n$.

It is easy to see from the expressions (1.15) and (1.16) that in case of the Lippmann-Schwinger equation the performed approximation coincides with the so-called "$q_i\, q_j = 0$ approximation".

Let us note that the suggested approximate solution may prove to be useful for not very small interaction $gW_1 \approx 1$ in case of the following corrections to the scattering amplitude being small

$$W_2 \ll W_1^2\,, \text{ etc.}$$

2. In this section we consider the case when the obtained approximate expressions for the scattering amplitude can be used to find the asymptotics when s tends to infinity and t is fixed. We shall derive the well-known Molière-Glauber representation taking as an example the Logunov-Tavkhelidze equation. Thus we have

$$W_1 = \int d\vec{q}\;\; \frac{V(\vec{q}\,;s)}{\sqrt{(\vec{q}+\vec{p}\,')^2+m^2}}\;\; \frac{e^{-i\vec{q}\vec{r}}}{(\vec{q}+\vec{p}\,')^2+m^2-\frac{s}{4}-i\epsilon} \tag{2.1}$$

where $\vec{p}^{\,2}=\vec{p}\,'^2=k^2=\frac{s}{4}-m^2$ on the mass shell.

Let us choose the z axis along the $(\vec{p}+\vec{p}\,')$ vector. Then

$$\vec{p}-\vec{p}\,'=\vec{\Delta}_\perp\,,\quad \vec{\Delta}_\perp\vec{n}_z=0\,,\quad t=-\vec{\Delta}_\perp^{\,2}$$

and when $s\to\infty$

$$p_z'\approx\frac{\sqrt{s}}{2}\,,\quad \vec{p}_\perp^{\,\prime 2}\approx-\frac{t}{4}\ll p_z^{\prime 2}$$

and

$$W_1(\vec{r},\vec{p}\,';s)\approx\frac{2}{s}\,\frac{1}{(2\pi)^3}\int d\vec{r}\,'\,V(\vec{r}\,';s)\int d^2\vec{q}_\perp e^{-i\vec{q}_\perp(\vec{r}_\perp-\vec{r}_\perp')}\,\cdot$$

$$\cdot\int_{-\infty}^{+\infty}dq_z\,\frac{e^{iq_z(z'-z)}}{q_z-i\epsilon}=\frac{2i}{s}\int_z^{\infty}dz'\,V(\sqrt{\vec{r}_\perp^{\,2}+z'^2};s).\tag{2.2}$$

Substituting (2.2) in (1.14) we obtain

$$T_1(\vec{p},\vec{p}\,';s)=\frac{g}{(2\pi)^3}\int d^2r_\perp e^{i\vec{r}_\perp\vec{\Delta}_\perp}\int_{-\infty}^{+\infty}dz\,V(\sqrt{\vec{r}_\perp^{\,2}+z^2};s)\,\cdot$$

$$\cdot\exp\{\frac{2ig}{s}\int_z^{\infty}dz'\,V(\sqrt{\vec{r}_\perp^{\,2}+z'^2};s)=\tag{2.3}$$

$$=-\frac{s}{(2\pi)^3 2i}\int d^2\vec{r}_\perp e^{i\vec{r}_\perp\vec{\Delta}_\perp}\int_{-\infty}^{+\infty}dz\,\frac{d}{dz}\{\exp[\frac{2ig}{s}\int_z^{\infty}dz'\,V(\sqrt{r_\perp^2+z'^2};s)]\}$$

from which the final formula for smooth potentials follows

$$T(\vec{p},\vec{p}\,';s)\approx\frac{s}{(2\pi)^3}\int d^2\vec{r}_\perp\,e^{i\vec{r}_\perp\vec{\Delta}_\perp}\,\frac{e^{2i\chi}-1}{2i}\,,\tag{2.4}$$

where

$$\chi = \chi(\vec{r}_\perp; s) = \frac{g}{s} \int_{-\infty}^{+\infty} dz\, V(\sqrt{\vec{r}_\perp^{\,2} + z^2}\,; s). \tag{2.5}$$

It can be shown that in the region of high energies and fixed momentum transfers the corrections to the above mentioned approximation do not change the asymptotic of the scattering amplitude, because

$$\frac{\delta W_1}{W_1} = O\left(\frac{1}{\sqrt{s}}\right)$$

$$\frac{W_n}{W_1} = O\left(\frac{1}{s^{n-1}\sqrt{s}}\right), \quad n \geq 2. \tag{2.6}$$

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