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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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INVESTIGATION OF EIKONAL APPROXIMATION IN QUANTUM FIELD THEORY BY FUNCTIONAL INTEGRATION METHOD

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Барбашов Б.М., Кулешов С.П., Матвеев В.А., Сисакян А.Н. E2-4692

Исследование эйконального приближения в квантовой теории поля методом функционального интегрирования

В работе на основе метода функционального интегрирования изучается асимптотическое поведение амплитуды рассеяния при высоких энергиях и фиксированных передачах импульса в теоретико-полевой модели $L_{вз.} = g : \psi^2(x)\phi(x):$.

Получено представление глауберовского типа для амплитуды рассеяния двух скалярных частиц в приближении лестничных и кросс-лестничных диаграмм.

Сообщения Объединенного института ядерных исследований

Дубна, 1969

Barbashov B.M., Kuleshov S.P., Matveev V.A., Sissakian A.N. E2-4692

Investigation of Eikonal Approximation in Quantum Field Theory by Functional Integration Method

The asymptotic behaviour of the scattering amplitude at high energies and fixed momentum transfers is investigated in the $L_{int} = g\psi^2\phi:$ -model by means of the functional integration method in quantum field theory. The Glauber-type representation for the scattering amplitude of two spinless particles is obtained in the approximation of ladder and cross-ladder graphs.

Communications of the Joint Institute for Nuclear Research.

Dubna, 1969

§ 1. Introduction

In the recent time the eikonal approximation of small angle scattering amplitude which is well known from nonrelativistic quantum mechanics has been intensively employed for the description of high energy hadron scattering.

In this connection the problem arises to prove the validity of the eikonal description in relativistic quantum field theory.

Recently in papers /1,2/ an approach to the study of high energy particle scattering was developed which is based on the Logunov-Tavkhelidze quasipotential equation for scattering amplitude in quantum field theory /3,4/. It was shown that under requirement of smooth behaviour of the local quasipotential the amplitude of high-energy particle scattering at small angles satisfies the eikonal or the Glauber representation /5/. Notice, that the first application of the eikonal representation in the framework of phenomenological optical potential description to the high energy scattering was given in papers of D.I.Blokhintsev et al. /6,7,8/.

Among recent works we mention papers /9,10/ where the eikonal approximation is applied to the phenomenological analysis of high energy hadron scattering on the basis of nonrelativistic Schrödinger equation with smooth effective potential. In paper /11/ the Glauber representation is generalized to the case of spin particle scattering at high energies.

It is of great interest to explore field-theoretical models to study the problem of the validity of eikonal representation in relativistic region and the structure of effective quasipotential of two particles at high energies.

We should mention papers /12,13/ where the problem of the validity of eikonal description was investigated is some lowest orders of perturbation theory.

In the present paper we demonstrate the efficacy of the functional integration method in quantum field theory /14/ in studying the problems mentioned above. As an example we consider a model of scalar "nucleons" and "mesons" with interaction lagrangian

$$L_{int} = g : \psi^2 (x) \phi(x): \tag{1.1}$$

By means of the functional integration method we have obtained for amplitude of scattering of two spinless particles or "nucleons" the closed analytic relativistic invariant expression. In obtaining the results we have neglected the vacuum polarization effects, the radiation corrections to the nucleon lines as well as the $k_i k_j$ - terms $(i \neq j)$ in the nucleon propagators in inner lines of Feinmann diagrams. The validity of these approximations at high energies will be discussed below.

The expression obtained here in the limit of high energies $s \to \infty$ at fixed momentum transfers t takes the form of the Glauber representation with effective Yukawa potential of interaction between "nucleons".

Notice, however, that the appearance of the Yukawa potential is due to the simple model we have considered. In principle by the same method one can consider a more complicated model, where the interaction between nucleons is due to the exchange of a set of mesons with various masses and spins. In the framework of such a model it is possible to construct the smooth effective potential by choosing in an appropriate manner the density of the propagators.

The plan of presentation is as follows. In § 2 the functional integration method is used to find the two-particle Green function. In the next paragraph employing the two-particle Green function the scattering amplitude is constructed and the procedure of transition to the mass shell is developed for it. Here use is made of the approximate method of calculation of functional integrals which is equivalent to the neglect in perturbation theory of the $k_i k_j$ -terms $(i \neq j)$ in the expressions for the Feynmann propagators.

§4 is devoted to the obtaining of an integral representation in the Glauber form for the scattering amplitude in the asymptotic region of high energies and fixed momentum transfers.

§2. Two-Particle Green Function in the Model $L_{int} = g : \psi^2(x)\phi(x) :$

The one-particle Green function of the quantum field $\psi(x)$ in the external field $\phi(x)$ satisfies the equation[14]:

$$[i^2 \partial_\mu^2 - m^2 + g\phi(x)]\, G(xy|\phi) = -\delta^4(x-y), \tag{2.1}$$

the solution for which can be written in the form of the functional integral[15]

$$G(xy|\phi) = i\int_0^\infty ds\, e^{-ism^2} C_\nu \int \delta^4\nu \exp\{-i\int_0^s d\xi\, [\nu_\mu^2(\xi) - \\ - g\phi(x - 2\int_\xi^s \nu(\eta)d\eta)]\}\, \delta^4(x - y - 2\int_0^s \nu(\eta)d\eta). \tag{2.2}$$

The quantum two-particle Green function is connected with the one-particle function as follows[16]

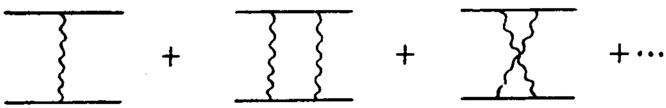
$$G(x_1 x_2|x_3 x_4) = C_\phi \int \delta\phi \exp\{-\frac{i}{2}\int \phi(q) D^{-1}(q)\phi(-q)dq\} \times \\ \times [G(x_1 x_3|\phi)G(x_2 x_4|\phi) + G(x_1 x_4|\phi)G(x_2 x_4|\phi)]\, S_0(\phi), \tag{2.3}$$

where $S_0(\phi)$ is the average of the S-matrix over the $\psi(x)$-field vacuum. As was already mentioned, we do not take into account the closed loops of the field $\psi(x)$ and put $S_0(\phi)=1$.

Inserting (2.2) in (2.3) and performing the functional integration over ϕ which is reduced in this case to simple Gaussian quadratures we get[17]

$$G(x_1 x_2 | x_3 x_4) = i^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \, e^{-im^2(s_1+s_2)} \, C_\nu \int \delta\nu_1 \int \delta\nu_2 \times$$
$$\times \exp\{-i \int_0^{s_1} \nu_1^2(\xi)\, d\xi - i \int_0^{s_2} \nu_2^2(\xi)\, d\xi + i g^2 \int_0^{s_1} d\xi_1 \int_0^{s_2} d\xi_2 \times \tag{2.4}$$
$$\times D[x_1 - x_2 + 2 \int_{\xi_2}^{s_2} \nu_2(\eta)\, d\eta - 2 \int_{\xi_1}^{s_1} \nu_1(\eta)\, d\eta]\} \times$$
$$\times \delta^4(x_1 - x_3 - 2 \int_0^{s_1} \nu_1(\eta) d\eta)\, \delta^4(x_2 - x_4 - 2 \int_0^{s_2} \nu_2(\eta) d\eta) + (x_3 \leftrightarrow x_4).$$

We note that eq. (2.4) allows to take into account only diagrams of the following type



since the terms resulting in radiational corrections to each of the both nucleon lines have been neglected in receiving eq. (2.4).

Going over to the momentum space

$$G(q_1 q_2 | p_1 p_2) = \frac{1}{(2\pi)^8} \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \; G(x_1 x_2 | x_3 x_4)\, e^{-iq_1x_1 - iq_2x_2 + ip_1x_3 + ip_2x_4} \tag{2.5}$$

we carry out easily an integration over x_3, x_4 taking into account the δ^4 -functions in (2.4). By replacing then the functional variables

$$\nu_1(\eta) \to \nu_1(\eta) - p_1 \ ; \quad \nu_2(\eta) \to \nu_2(\eta) - p_2 \tag{2.6}$$

we have

$$G(q_1 q_2 \mid p_1 p_2) = \frac{i^2}{(2\pi)^8} \int d^4x_1 \int d^4x_2 \, e^{ix_1(p_1-q_1)+ix_2(p_2-q_2)} \times$$

$$\times \int_0^\infty ds_1 \int_0^\infty ds_2 \, e^{is_1(p_1^2-m^2)+is_2(p_2^2-m^2)} \, C_\nu \int \delta\nu_1 \int \delta\nu_2 \times$$

$$\times \exp\{-i\int_0^{s_1} \nu_1^2(\eta)d\eta - i\int_0^{s_2} \nu_2^2(\eta)d\eta + ig^2\int_0^{s_1} d\xi_1 \int_0^{s_2} d\xi_2 \times \tag{2.7}$$

$$\times D[x_1 - x_2 + 2p_1(s_1-\xi_1) - 2p_2(s_2-\xi_2) + 2\int_{\xi_2}^{s_2}\nu_2(\eta)d\eta - 2\int_{\xi_1}^{s_1}\nu_1(\eta)d\eta]\} +$$

$$+ (p_1 \leftrightarrow p_2).$$

After passing to the variables

$$y = x_1 + x_2 \ ; \quad x = x_1 - x_2 \tag{2.8}$$

in eq. (2.7), an integration over y can be performed which gives the δ^4 -function ensuring the four-momentum conservation law.

$$G(q_1 q_2 \mid p_1 p_2) = \frac{i^2}{(2\pi)^4} \delta^4(p_1 + p_2 - q_1 - q_2) \int_0^\infty ds_1 \int_0^\infty ds_2 \times$$

$$\times e^{is_1(p_1^2-m^2)+is_2(p_2^2-m^2)} \int d^4x \, e^{ix(p_1-q_1)} \, C_\nu \int \delta\nu_1 \int \delta\nu_2 \times \tag{2.9}$$

$$\times \exp\{-i\int_0^{s_1}\nu_1^2(\eta)d\eta - i\int_0^{s_2}\nu_2^2(\eta)d\eta + i g^2\int_0^{s_1}d\xi_1\int_0^{s_2}d\xi_2 \times$$

$$\times D[x + 2p_1\xi_1 + 2p_2\xi_2 + 2\int_{s_2-\xi_2}^{s_2}\nu_2(\eta)d\eta - 2\int_{s_1-\xi_1}^{s_1}\nu_1(\eta)d\eta]\} +$$

$$+ (p_1 \leftrightarrow p_2)$$

Let us discuss eq. (2.9) in more detail.

Making an expansion in the coupling constant g^2 and carrying out the functional integration over ν which by the Fourier transformation reduces to simple Gaussian quadratures, we get the perturbation series for $G(q_1 q_2 | p_1 p_2)$. In doing so, the integration over the functional variable ν in the argument of the D-function leads to a quadratic dependence on the meson momentum k.

The elimination of ν from the D-function argument in (2.9) thus means, in the language of the Feynmann graphs, the neglect of the quadratic dependence on k in the nucleon propagator, i.e.

$$\frac{1}{(p+\sum_i k_i)^2 - m^2} \to \frac{1}{2p\sum_i k_i}.$$

Such an approximation is, as is known[15,16,17], valid for the infrared asymptotics in quantum electrodynamics. Its validity for the study of the high energy behaviour of the scattering amplitude is, however, not proven. Therefore we shall use the approximate method of calculation of the integrals over ν which allows one to retain the quadratic dependence of the propagators on k_i.

§3. The Scattering Amplitude

Using the expression for the two-particle Green function (2.9) we now find the two-nucleon scattering amplitude by the well-known formula:

$$(2\pi)^4\,\delta^4(p_1+p_2-q_1-q_2)\,f(q_1 q_2|p_1 p_2) =$$

$$= \lim_{q_1^2,\,q_2^2,\,p_1^2,\,p_2^2\to m^2} (q_1^2-m^2)(q_2^2-m^2)(p_1^2-m^2)(p_2^2-m^2)\,i\,G(q_1 q_2|p_1 p_2). \tag{3.1}$$

Inserting (2.9) into (3.1) we get

$$\begin{aligned}
(2\pi)^4 f(q_1 q_2|p_1 p_2) = {} & \lim_{q_1^2,\,q_2^2,\,p_1^2,\,p_2^2\to m^2} (q_1^2-m^2)(q_2^2-m^2)(p_1^2-m^2)(p_2^2-m^2)\times \\
& \times \frac{i^3}{(2\pi)^4}\int_0^\infty ds_1 \int_0^\infty ds_2\; e^{is_1(p_1^2-m^2)+is_2(p_2^2-m^2)} \int d^4x\, e^{i(p_1-q_1)x}\, C_\nu \int\delta\nu_1 \int\delta\nu_2 \times \\
& (-ig^2)\int_0^{s_1} d\eta_1 \int_0^{s_2} d\eta_2 \int \frac{d^4k}{k^2-\mu^2+i\epsilon} \exp\{ikx + 2ik[p_1\eta_1 - p_2\eta_2 - \\
& - \int_{s_1-\eta_1}^{s_1} \nu_1(\eta)\,d\eta + \int_{s_2-\eta_2}^{s_2} \nu_2(\eta)\,d\eta\,]\}\cdot \int_0^1 d\lambda \exp\{-ig^2\lambda \int_0^{s_1} d\xi_1 \int_0^{s_2} d\xi_2 \times \\
& \times D(x + 2p_1\xi_1 - 2p_2\xi_2 - 2\int_{s_1-\xi_1}^{s_1} \nu_1(\eta)\,d\eta + 2\int_{s_2-\xi_2}^{s_2} \nu_2(\eta)\,d\eta)\} + \\
& + (p_1 \leftrightarrow p_2).
\end{aligned} \tag{3.2}$$

In this expression the operation of subtraction of the unity from the exponential whose exponent contains D -function is performed by the formula

$$e^{gD(x)} - 1 = g \int_0^1 d\lambda \; D(x)\, e^{\lambda g D(x)} . \tag{3.3}$$

It results in the exclusion of the terms corresponding to the propagation of the both particles wihtout interaction.

We make in (3.2) the expansion of the last exponential in a series in g^2 which permits us, integrating over x to get $(2\pi)^4 \delta^4 (p_1 - q_1 + k + \sum_{i=1}^{n} k_i)$.

Now it is easy to integrate the obtained expression over k. Then reducing again the series to the exponential and making the change of the functional variables

$$\begin{aligned} \nu_1(\eta) &\to \nu_1(\eta) + (p_1 - q_1)\, \theta(\eta - (s_1 - \eta_1)) \\ \nu_2(\eta) &\to \nu_2(\eta) + (p_2 - q_2)\, \theta(\eta - (s_2 - \eta_2)) \end{aligned} \tag{3.4}$$

we get the following expression

$$\begin{aligned} f(q_1 q_2 | p_1 p_2) = {} & \frac{i}{(2\pi)^4} \lim_{q_1^2, q_2^2, p_1^2, p_2^2 \to m^2} (q_1^2 - m^2)(q_2^2 - m^2)(p_1^2 - m^2)(p_2^2 - m^2) \times \\ & \times \int_0^\infty ds_1 \int_0^\infty ds_2\, e^{is_1(p_1^2 - m^2) + is_2(p_2^2 - m^2)} \; C_\nu \int \delta\nu \int \delta\nu_2 \exp\{ -i \int_0^{s_1} \nu_1^2(\eta)\, d\eta - i \int_0^{s_2} \nu_2^2(\eta)\, d\eta \} \times \\ & \times (ig^2) \int_0^{s_1} d\eta_1 \int_0^{s_2} d\eta_2 \int d^4x\, D(x)\, e^{-ix(p_1 - q_1) + i\eta_1(q_1^2 - p_1^2) + i\eta_2(q_2^2 - p_2^2)} \times \\ & \times \int_0^1 d\lambda \, \exp\{ ig^2\lambda \int_0^{s_1} d\xi_1 \int_0^{s_2} d\xi_2 \, D[-x + 2p_1(\xi_1 - \eta_1) - 2p_2(\xi_2 - \eta_2) - \\ & - 2(p_1 - q_1)(\xi_1 - \eta_1)\theta(\eta_1 - \xi_1) - 2(p_1 - q_1)(\xi_2 - \eta_2)\theta(\eta_2 - \xi_2) - 2 \int_{s_1 - \xi_1}^{s_1 - \eta_1} \nu_1(\eta)\, d\eta + \\ & + 2 \int_{s_2 - \xi_2}^{s_2 - \eta_2} \nu_2(\eta)\, d\eta]\} + (p_1 \leftrightarrow p_2) . \end{aligned} \tag{3.5}$$

In order to pass to the mass shell it is necessary to pick out the pole terms cancelling the zeros $(p_1^2 - m^2)$ and $(q_1^2 - m^2)$.

Now it is easy to remark that the integration limits in eq. (3.5) may be changed in the following manner

$$\int_0^\infty ds_1 \int_0^{s_1} d\eta_1 \int_0^\infty ds_2 \int_0^{s_2} d\eta_2 \rightarrow \int_0^\infty d\eta_1 \int_{\eta_1}^\infty ds_1 \int_0^\infty d\eta_2 \int_{\eta_2}^\infty ds_2 \ . \tag{3.6}$$

Making the change of the variables

$$s_1 \rightarrow s_1 + \eta_1 ; \qquad s_2 \rightarrow s_2 + \eta_2 \tag{3.7}$$

we get the expression

$$f(q_1 q_2 | p_1 p_2) = \frac{i}{(2\pi)^4} \lim_{q_1^2, q_2^2, p_1^2, p_2^2 \to m^2} (q_1^2 - m^2)(q_2^2 - m^2)(p_1^2 - m^2)(p_2^2 - m^2) \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^{s_1} d\eta_1 \int_0^{s_2} d\eta_2 \times$$

$$\times e^{is_1(p_1^2 - m^2) + is_2(p_2^2 - m^2) + i\eta_1(q_1^2 - m^2) + i\eta_2(q_2^2 - m^2)} C_\nu \int \delta\nu_1 \int \delta\nu_2 \quad \exp\{-i \int_0^{s_1+\eta_1} \nu_1^2(\eta) d\eta -$$

$$-i \int_0^{s_2+\eta_2} \nu_2^2(\eta) d\eta \} (ig^2) \int d^4x \ D(x) e^{-ix(p_1 - q_1)} \int_0^1 d\lambda \exp\{ig^2\lambda \int_0^{s_1+\eta_1} d\xi_1 \int_0^{s_2+\eta_2} d\xi_2 \times$$

$$\times D[-x + 2p_1(\xi_1 - \eta_1) - 2(p_1 - q_1)(\xi_1 - \eta_1)\theta(\eta_1 - \xi_1) - 2p_2(\xi_2 - \eta_2) - 2(p_1 - q_1)(\xi_2 - \eta_2)\theta(\eta_2 - \xi_2) .$$

$$-2 \int_{s_1+\eta_1-\xi_2}^{s_1} \nu_1(\eta) d\eta + 2 \int_{s_2+\eta_2-\xi_2}^{s_2} \nu_2(\eta) d\eta]\} + (p_1 \leftrightarrow p_2) . \tag{3.8}$$

Now after obvious substitutions

$$\xi_1 \rightarrow \xi_1 + \eta_1 ; \qquad \xi_2 \rightarrow \xi_2 + \eta_2 \tag{3.9}$$

we introduce new variables r_i $(i = 1,2,3,4)$

$$s_1 = \frac{r_1}{p_1^2 - m^2}; \quad s_2 = \frac{r_3}{p_2^2 - m^2};$$

$$\eta_1 = \frac{r_3}{q_1^2 - m^2}; \quad \eta_2 = \frac{r_4}{q_2^2 - m^2}, \tag{3.10}$$

the integration over which will separate the pole terms needed. Performing then the transition to the mass shell we get the amplitude

$$f(q_1 q_2 | p_1 p_2) = c_\nu \int \delta\nu_1 \int \delta\nu_2 \exp\{-i \int_0^{A+A} \nu_1^2(\eta)\, d\eta - i \int_0^{A+A} \nu_2^2(\eta)\, d\eta\} \times$$

$$\times \frac{(ig)^2}{(2\pi)^4} \int d^4x\, D(x)\, e^{-ix(p_1 - q_1)} \int_0^1 d\lambda \exp\{ig^2\lambda \int_{-A}^{A} d\xi_1 \int_{-A}^{A} d\xi_2 \times \tag{3.11}$$

$$\times D[-x + 2\xi_1 (p_1\theta(\xi_1) + q_1\theta(-\xi_1)) - 2\xi_2 (p_2\theta(\xi_2) + q_2\theta(-\xi_2)) -$$

$$-2 \int_{A-\xi_1}^{A} \nu_1(\eta)\, d\eta + 2 \int_{A-\xi_2}^{A} \nu_2(\eta)\, d\eta]\} + (p_1 \leftrightarrow p_2),$$

where $A \to \infty$ at $p_{1,2}^2 ; q_{1,2}^2 \to m^2$.

The transition to the limit $A \to \infty$ should be performed only after the functional integration over $\nu_1(\eta)$ and $\nu_2(\eta)$ and the integration over ξ_1, ξ_2 .

An exact functional integration over $\nu_1(\eta)$ and $\nu_2(\eta)$ does not appear to be possible, therefore this is done approximately using the method developed in ref./15/. This method is based on the formula

$$C_\nu \int \delta\nu \exp\{-i \int_0^s \nu^2(\eta)\, d\eta\} e^{g^2 F(y,\nu)} =$$

$$= e^{\bar{F}(y)} \int \delta\nu \exp\{-i \int_0^s \nu^2(\eta)\, d\eta\} \sum_{n=0}^{\infty} \frac{(g^2)^n (F - \bar{F})^n}{n!}, \tag{3.12}$$

where

$$\bar{F}(y) = C_\nu \int \delta\nu \ \exp\left\{-i \int^{s} \nu^2(\eta)\, d\eta \right\} F(y) \ . \tag{3.13}$$

If we restrict ourselves in the sum over n to the first term $n=0$ then this approximation

$$C_\nu \int \delta\nu \ \exp\left\{-i \int_0^{s} \nu^2(\eta)\, d\eta \right\} e^{g^2 F(y,\nu)} \approx e^{g^2 \bar{F}(y)} \tag{3.14}$$

means that the exponential exponents in (3.11) depending on should be replaced by their average value over the Gaussian measure, according to (3.13). It is easily seen that such an integration of the D-function leads to the appearance of the quadratic dependence on the momentum k in its Fourier transform.

Indeed, since we have

$$D\left(x + \int_{\xi_2}^{\xi_1} \nu(\eta)\, d\eta\right) = \frac{1}{(2\pi)^4} \int D(k)\, e^{ikx + ik\int_{\xi_2}^{\xi_1} \nu(\eta)\, d\eta}\, d^4k \ , \tag{3.15}$$

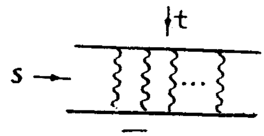
in (3.13) the integration may be performed

$$C_\nu \int \delta\nu\, e^{-i\int_0^s \nu^2(\eta)\, d\eta}\, D\left(x + \int_{\xi_2}^{\xi_1} \nu(\eta)\, d\eta\right) = \frac{1}{(2\pi)^4} \int D(k)\, e^{ikx + ik^2|\xi_1 - \xi_2|}\, d^4k \ . \tag{3.16}$$

When integrating further over ξ_1 and ξ_2 the k^2- dependence appears in the denominator of the nucleon propagator $\dfrac{1}{p^2 + 2p\sum_i k_i + \sum_i k_i^2 - m^2}$. However, in this approximation we get in the propagators no terms of the type $k_i k_j$ where k_i and k_j are the momenta of different mesons. The subsequent terms in the sum (3.12) take into account corrections to the approximation[x/] $k_i k_j = 0$ $(i \neq j)$.

x/ This approximation is discussed for the infrared region in quantum electrodynamics in refs./15,18/.

The applicability of this approach in the region of high energies[18] s at fixed momentum transfers t may be cleared up in perturbation theory. It may be shown in particular that neglecting the terms $k_i k_j$ $(i \neq j)$ in nucleon propagators in the case of ladder diagrams



does not change the asymptotics at high energies, which at n -mesons exchange has the form $\frac{\ell n\, s}{s^{n-1}}$. However it is necessary to note that this approach sharply effects the asymptotic of the Feynmann graphs in momentum transfers at $t \to \infty$ and s -fixed.

So, making integration over ξ_1 and ξ_2, we get the relativistic invariant expression of a closed form.

$$f(q_1\, q_2 \,|\, p_1\, p_2) = \frac{(ig)^2}{(2\pi)^4} \int d^4x\, D(x)\, e^{-ix(p_1-q_1)} \int_0^1 d\lambda\, e^{-i\lambda \chi(x;\,q_1 q_2;\,p_1,p_2)} + \tag{3.17}$$

$$+ (p_1 \leftrightarrow p_2),$$

where

$$\chi(x;q_1,q_2;p_1,p_2) = \frac{g^2}{(2\pi)^4} \int \frac{d^4k\, e^{-ikx}}{k^2-\mu^2+i\epsilon} \Big[\frac{1}{(k^2+2kp_1)(k^2-2kp_2)} + \tag{3.18}$$

$$+ \frac{1}{(k^2-2kq_1)(k^2-2kp_2)} + \frac{1}{(k^2+2kp_1)(k^2+2kq_2)} + \frac{1}{(k^2-2kq_1)(k^2+2kq_2)} \Big].$$

§4. Eikonal Approximation for the Scattering Amplitude.

We shall consider the scattering amplitude (3.17) asymptotic behaviour in the region of high energies and fixed momentum transfers. It is convenient for the further consideration to go over to the c.m.s.

$$\vec{p}_1 = -\vec{p}_2 , \quad \vec{q}_1 = -\vec{q}_2 , \quad p_{10} = p_{20} = q_{10} = q_{20} \tag{4.1}$$

in which the Mandelstam variables s, t, u have the form

$$\begin{aligned} s &= (p_1 + p_2)^2 = 4(\vec{p}_1^{\,2} + m^2) \\ t &= T^2 = -2\vec{p}_1^{\,2}(1 - \cos\theta) \\ u &= U^2 = -2\vec{p}_1^{\,2}(1 + \cos\theta), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} T &= (p_1 - q_1) = (q_2 - p_2) \\ U &= (p_2 - q_1) = (q_2 - p_1) . \end{aligned} \tag{4.3}$$

At high energies s and fixed momentum transfers t it is not difficult to show that the momentum transfers T is perpendicular to p_1 and p_2 /19/

$$T = \frac{t}{s}(p_2 - p_1) + \Delta , \tag{4.4}$$

where

$$(\Delta \cdot p_1) = (\Delta \cdot p_2) = 0 . \tag{4.5}$$

We choose the direction of the momentum $\vec{p}$ along the z -axis

$$p_1 = (p_0, 0, 0, p_z)$$
$$p_2 = (p_0, 0, 0, -p_z) \tag{4.6}$$

and get

$$\Lambda = (0, \vec{\Delta}_{\perp}, 0) \quad . \tag{4.7}$$

For studying the scattering amplitude asymptotic behaviour in the considered region we shall make an analysis of the phase function (3.18), which we shall represent in the form :

$$\chi(x; q_1, q_2; p_1, p_2) = \chi_1 + \chi_2 \quad , \tag{4.8}$$

where

$$\chi_1 = \frac{g^2}{(2\pi)^4} \int \frac{d^4k\, e^{-ikx}}{k^2-\mu^2+i\epsilon} \left[\frac{1}{(k^2+2kp_1)(k^2-2kp_2)} + \frac{1}{(k^2-2kq_1)(k^2+2kq_2)} \right] \tag{4.9}$$

$$\chi_2 = \frac{g^2}{(2\pi)^4} \int \frac{d^4k\, e^{-ikx}}{k^2-\mu^2+i\epsilon} \left[\frac{1}{(k^2-2kq_1)(k^2-2kp_2)} + \frac{1}{(k^2+2kp_1)(k^2+2kq_2)} \right] \tag{4.10}$$

It is not difficult to see that the account of only χ_1 corresponds to the Feynmann ladder diagrams in perturbation theory, but the function χ_2 is responsible for the appearance of the nonplane graphs with crossed mesons lines.

Let us consider in more detail the phase function χ_1 . It may be found , in principle, with the help of residue theory. However, it is more easy to make use of the analytical properties of the function χ_1 in the variable s . Its discontinuity on the cut which goes along the real positive s -semi-axis is determined according to the well-known rules/20/ and is equal

$$\Delta_s \chi_1 = \frac{-g^2}{(2\pi)^2} \int \frac{d^4 k\, e^{-ikx}}{k^2-\mu^2+i\epsilon} [\delta(k^2+2kp_1)\delta(k^2-2kp_2)+$$

$$+\delta(k^2-2kq_1)\delta(k^2+2kq_1)] = \tag{4.11}$$

$$= \frac{g^2}{4(2\pi)^4 p_0} \int \frac{d^2\vec{k}_\perp e^{i\vec{k}_\perp \vec{x}_\perp}}{\sqrt{p_0^2 - \vec{k}_\perp^2}} \left\{ \frac{e^{-ix_z \frac{\vec{k}^2}{2p_0}}}{\mu^2 + \vec{k}_\perp^2 + \left(\frac{\vec{k}_\perp^2}{2p_0}\right)^2} + \frac{e^{-2ix_z p_0}}{\mu^2 + \vec{k}_\perp^2 + (2p_0)^2} \right\}.$$

We note that eq. (4.11) for the χ_1 function discontinuity on the cut does not depend on x_0 and therefore does not contain the retardation.

In the high energy limit, when $p_0 = \frac{\sqrt{s}}{2} \to \infty$, eq. (4.11) as $x_\perp$ non-zero has the following form

$$\Delta_s \chi_1 = \frac{g^2}{(2\pi)^2 s} \int \frac{d^2\vec{k}_\perp e^{i\vec{k}_\perp \vec{x}_\perp}}{\vec{k}_\perp^2 + \mu^2} \doteq \frac{ig^2}{2\pi s} K_0(\mu |\vec{x}_\perp|), \tag{4.12}$$

where $K_0(z)$ is the Kelwin function of zero order.

Using a dispersion relation without subtraction one may restore the phase function χ_1 at high energies

$$\chi_1 = \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{ds' \Delta_{s'} \chi_1}{(s'-s)} = -\frac{g^2 K_0(\mu |\vec{x}_\perp|)}{2\pi^2 s} \ln\left(-\frac{s}{s_0}\right). \tag{4.13}$$

The phase function χ_2 may be investigated in a similar way. The χ_2 function discontinuity on the cut, which goes along the real negative s -semi-axis, is equal to

$$\Delta_s \chi_2 = -\frac{g^2}{(2\pi)^2}\int \frac{d^4 k\, e^{-ikx}}{k^2-\mu^2+i\epsilon}[\delta(k^2-2kq_1)\delta(k^2-2kp_2)+$$

$$+\delta(k^2+2kp_1)\delta(k^2+2kq_2)] = \tag{4.14}$$

$$=\frac{g^2}{4(2\pi)^2 p_0}\int \frac{d^2\vec{k}_\perp e^{i\vec{k}_\perp \vec{x}_\perp}}{\sqrt{p_0^2-\vec{k}_\perp^2}}\left[\frac{e^{ix_0\frac{\vec{k}_\perp^2}{2p_0}}}{\vec{k}_\perp^2+\mu^2-\left(\frac{\vec{k}_\perp^2}{2p_0}\right)^2}+\frac{e^{-2ix_0p_0}}{\vec{k}_\perp^2+\mu^2-(2p_0)^2}\right].$$

Eq. (4.14) contains, generally speaking, the dependence on x_0 , however it may be neglected in the high energy limit and at non-zero $x_\perp$. As a result we get

$$\Delta_s \chi_2 = \frac{g^2}{(2\pi)^2 s}\int \frac{d^2\vec{k}_\perp e^{i\vec{k}_\perp\vec{x}_\perp}}{\vec{k}_\perp^2+\mu^2} = \frac{ig^2}{2\pi s}K_0(\mu|\vec{x}_\perp|)\ . \tag{4.15}$$

Now, using a dispersion relation without subtraction we get the following expression for the phase function χ_2 at high energies

$$\chi_2 = \frac{1}{2\pi i}\int_{-\infty}^{-s_0}\frac{ds'\Delta_{s'}\chi_2}{(s'-s)} = \frac{g^2 K_0(\mu|\vec{x}_\perp|)}{2\pi^2 s}\ln\frac{s}{s_0}\ . \tag{4.16}$$

Thus the full phase function at high energies and for $\vec{x}_\perp \neq 0$ has the form

$$\chi = \chi_1 + \chi_2 = \frac{g^2}{2\pi s}K_0(\mu|\vec{x}_\perp|)\ . \tag{4.17}$$

It is interesting to note that in the sum of the both parts of the phase function (4.13) and (4.16) the terms, containing logarithmical dependence on energy, concealled out. In the given case this fact is due to the crossing symmetry of eq. (3.18).

The phase function behaviour at impact distances smaller than the particle wave length

$$x_\perp \lesssim \lambda = \frac{1}{p_0} \tag{4.18}$$

can be determined from eqs. (4.11) and (4.14).

Fixing $\lambda = \frac{1}{p_0}$ and letting $\vec{x}_\perp$ tend to zero, we get

$$\chi \,|_{\vec{x}_\perp \to 0} \to \chi_0(s). \tag{4.19}$$

The $\chi_0(s)$ value is finite and has the following asymptotic behaviour at high energies

$$\chi_0(s) \sim \frac{1}{s} \ln^2 \frac{s}{\mu^2}. \tag{4.20}$$

Singling out in the $x_\perp$ - plane the small ϵ -vicinity of the zero point it is possible to show that the contribution of this region to the scattering amplitude vanishes at $\epsilon \to 0$.

In view of the fact that the phase function (4.17) does not contain x_0- and x_z -dependence in high energy limit and using the formula

$$\int dx_0\, dx_z\, D^c(x)\, e^{-i\left(\frac{t}{\sqrt{s}}\right)x_z} = \frac{1}{(2\pi)^2} \int \frac{d^2\vec{k}_\perp\, dk_z\, e^{i\vec{k}_\perp \vec{x}_\perp}}{\vec{k}_\perp^2 + k_z^2 + \mu^2}\, \delta\left(k_z - \frac{t}{\sqrt{s}}\right) = \tag{4.21}$$

$$= \frac{1}{(2\pi)^2} \int \frac{d^2\vec{k}_\perp\, e^{i\vec{k}_\perp \vec{x}_\perp}}{\vec{k}_\perp^2 + \mu^2 + \left(\frac{t}{\sqrt{s}}\right)^2},$$

for the first term of the amplitude (3.17) at small scattering angles $\frac{t}{s} \to 0$ we get the expression

$$f_1(s,t) = \lim_{\epsilon \to 0} - \frac{is}{(2\pi)^4} \int_{|x_\perp| \geq \epsilon} d^2 \vec{x}_\perp \, e^{i \vec{x}_\perp \vec{\Delta}_\perp} \left(e^{-\frac{ig^2}{2\pi s} K_0(\mu |\vec{x}_\perp|)} - 1 \right), \tag{4.22}$$

where

$$\vec{\Delta}_\perp^2 = t .$$

The second term of the scattering amplitude obtained with the aid of the replacements $p_1 \leftrightarrow p_2$ or $T \leftrightarrow U$ has the form

$$f_2(s,u) = - \frac{g^2}{(2\pi)^4} \int d^4 x \, D(x) e^{ixU} \int_0^1 d\lambda \, e^{-i\lambda \chi (p_1 \leftrightarrow p_2)} . \tag{4.23}$$

At small scattering angles and high energies eq. (4.23) containes in the integrand a rapidly oscillating exponential e^{ixU} and decreases more quickly than $f_1(s,t)$ by one degree of $\frac{1}{s}$.

Thus, we have got for the high energies and small angles scattering amplitude the integral representation (4.22) which coincides with the quantum-mechanical Glauber type representation with the eikonal function:

$$\chi(s, \vec{x}_\perp) = - \frac{g^2}{2\pi s} K_0 (\mu |\vec{x}_\perp|) = \frac{1}{s} \int_{-\infty}^{\infty} V \left(\sqrt{x_z^2 + \vec{x}_\perp^2} \right) dx_z , \tag{4.24}$$

where

$$V(s, |\vec{x}|) = - \frac{g^2}{4\pi} \frac{e^{-\mu |\vec{x}|}}{|\vec{x}|}, \tag{4.25}$$

is the Yukawa two-particle interaction potential.

§5. Conclusion

Using the functional integration method we have obtained the closed relativistically invariant and cross-symmetrical analytic expression for the scattering amplitude of two spinless "nucleons" in the model $L_{int} = g : \psi^2 \phi :$. In the limit of high energies $s \to \infty$ and fixed momentum transfers t this expression takes the form of the Glauber representation (4.22) with the eikonal function , which corresponds to the Yukawa interaction potential between the "nucleons" (4.24). However, we have taken into account only the usual ladder graphs and generalized ladder graphs with crossed meson lines neglecting the vacuum polarization effects, radiation corrections to the nucleon lines and the so-called $k_i k_j$ -terms $(i \neq j)$ in the nucleon propagators.

The result obtained means essentially that in the framework of the approximations used the retardation effects disappear in the limit of high energies at small angles.

Notice, that the n -th term of an expansion of the scattering amplitude (4.22) in powers of g^2 has the asymptotic behaviour $\frac{1}{s^{n-1}}$. Such an asymptotic behaviour is an agreement with the asymptotic behaviour of the sum of the corresponding Feynmann graphs up to the sixth order, as it is shown in paper /12/.

The expression (4.22) coincides with the result of paper /21/, where the eikonal approximation for the amplitude of scattering of two particles is investigated by the method of Schwinger variational derivatives. But in that paper the terms k^2 are eliminated from the nucleon propagators what makes covergence of the integrals on the upper limit worse. Besides the problem of the disappearance of the retardation effects in the final result remains unclear.

It should be noted, however, that the study of the importance of the retardation effects for high energy particle appears to be rather interesting.

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