Fluctuations in Canonical and Microcanonical Ensembles

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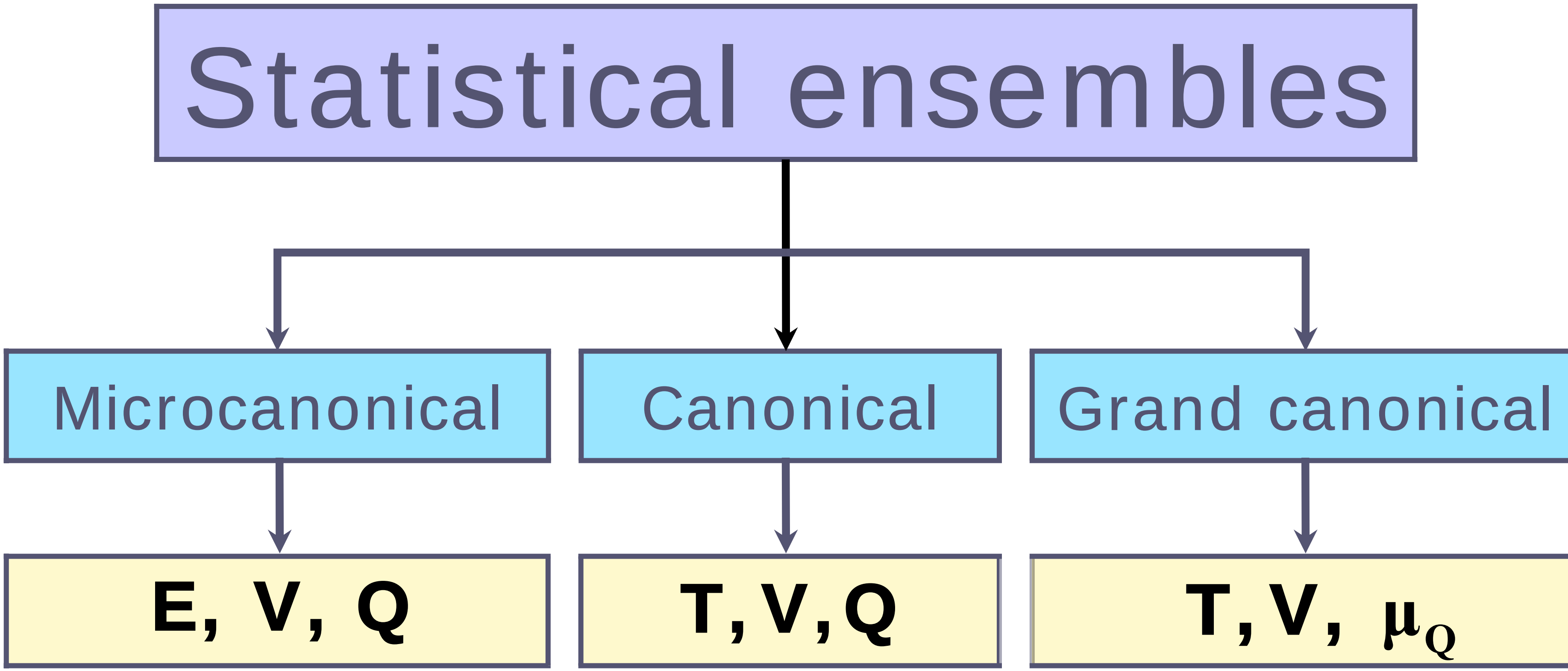
1. Grand Canonical Ensemble
2. Canonical Ensemble, $Q=0$
3. Canonical Ensemble, $Q\neq0$
4. Microcanonical Ensemble
5. Summary and Conslusions.

1. V.V. Begun, M.I. Gorenstein, M. Gaździcki, O.S.Zozulya, "Particle Number Fluctuations in a Canonical Ensemble", **Phys. Rev. C70** (2004) 034901, nucl-th/0404056.

2. V.V.Begun, M.I.Gorenstein, A.P.Kostyuk, O.S.Zozulya, "Particle Number Fluctuations in the Microcanonical Ensemble", nucl-th/0410044, **Phys. Rev. C** .

3. V.V. Begun, M.I. Gorenstein, O.S. Zozulya, "Fluctuations in the Canonical Ensemble", nucl-th/0411003, **Phys. Rev. C**.



$E \rightarrow T$ $\quad\quad$ $Q \rightarrow \mu_Q$

Partition function in g.c.e.

$$Z_{\mathrm{g.c.e.}}(V,T,\mu) = \sum_{N_{1+},N_{1-}=0}^{\infty} \cdots \sum_{N_{j+},N_{j-}=0}^{\infty} \cdots$$

$$\times \frac{(\lambda_{1+}z_1)^{N_{1+}}}{N_+!} \frac{(\lambda_{1-}z_1)^{N_{1-}}}{N_+!} \cdots \frac{(\lambda_{j+}z_j)^{N_{j+}}}{N_{j+}!} \frac{(\lambda_{j-}z_j)^{N_{j-}}}{N_{j-}!} \cdots$$

$$= \prod_j \exp(\lambda_{j+}z_j + \lambda_{j-}z_j) = \exp\left[2z\cosh(\mu/T)\right]$$

where j numerates the species, $\lambda_{j\pm} = \exp(\pm\mu/T)$

z_j is a single particle partition function:

$$z_j = \frac{g_j V}{(2\pi)^3} \int \exp[-\varepsilon_p / T]\, d^3p$$

$$= \frac{g_j V}{2\pi^2} T m_j^2\, K_2\left(\frac{m_j}{T}\right) \quad , \qquad z = \sum_j z_j$$

$$\varepsilon_p = \sqrt{p^2 + m_j^2}$$

Partition function in c.e.

$$Z_{c.e.}(V,T,\mu) = \sum_{N_{1+},N_{1-}=0}^{\infty} \cdots \sum_{N_{j+},N_{j-}=0}^{\infty} \cdots$$

$$\times \frac{(\lambda_{1+}z_1)^{N_{1+}}}{N_+!} \frac{(\lambda_{1-}z_1)^{N_{1-}}}{N_+!} \cdots \frac{(\lambda_{j+}z_j)^{N_{j+}}}{N_{j+}!} \frac{(\lambda_{j-}z_j)^{N_{j-}}}{N_{j-}!} \cdots$$

$$\times \delta\left[\left(N_{1+} + \ldots + N_{j+} + \ldots - N_{1-} - \ldots - N_{j-} - \ldots\right) - Q\right]$$

$$= I_Q(2z)$$

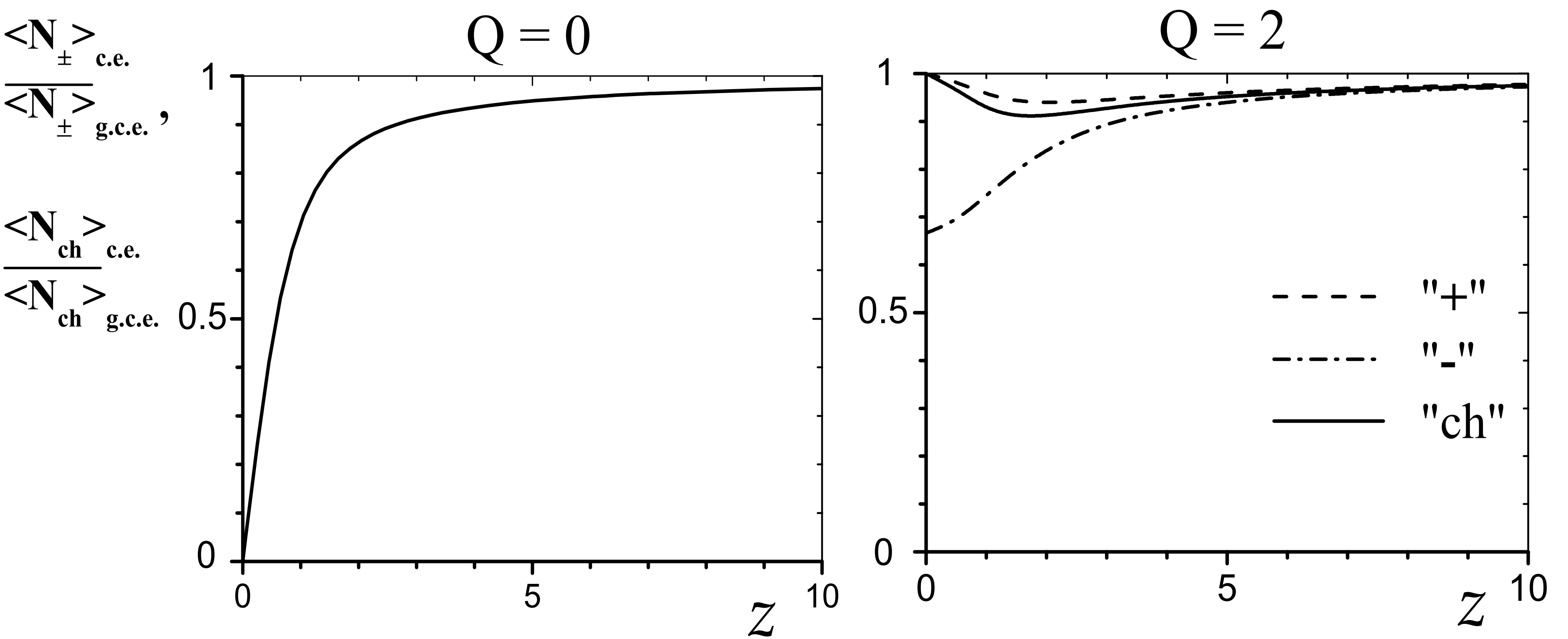
where we have used that:

$$\delta(n) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \exp(in\varphi)$$

$$I_Q(2z) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \exp[-i\,Q\varphi + 2z\cos(\varphi)]$$

note, that in c.e., in contrast to g.c.e., $\lambda_{j\pm}$ are only auxiliary parameters introduced in order to calculate mean multiplicities and fluctuations. They are set to one in the final formulas.

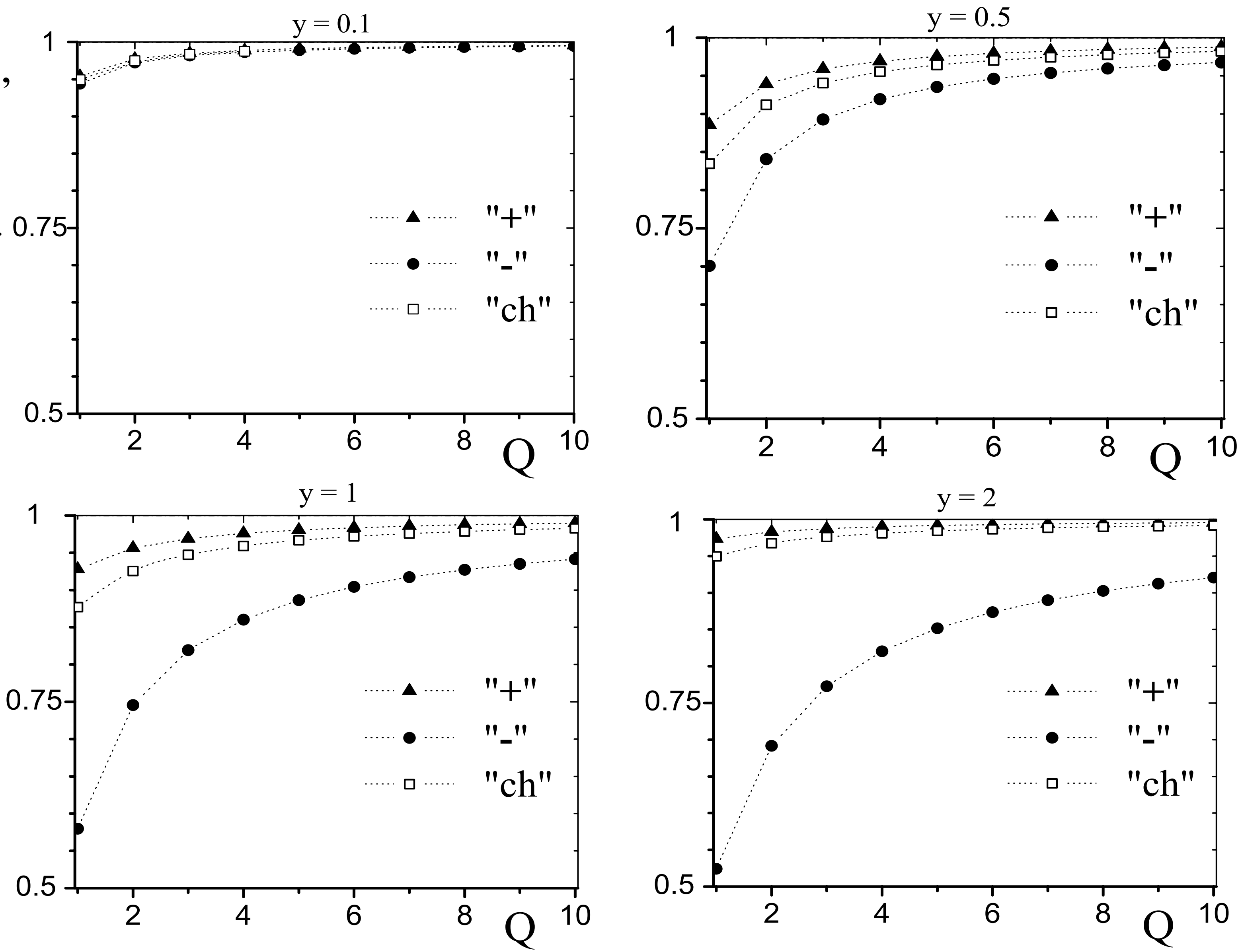
Particle number ratios.



$$< N_{j\pm} > = \lambda_{j\pm} \frac{\partial \ln Z}{\partial \lambda_{j\pm}} = a_\pm z_j ,\qquad a_\pm^{g.c.e.} = \exp(\pm\mu/T),\quad a_\pm^{c.e.} = \frac{I_{Q\mp 1}(2z)}{I_Q(2z)}.$$

$$\frac{\langle N_\pm \rangle_{c.e.}}{\langle N_\pm \rangle_{g.c.e.}}, \quad \frac{\langle N_{ch} \rangle_{c.e.}}{\langle N_{ch} \rangle_{g.c.e.}}$$

$$y = \frac{Q}{2z}$$



The scaled variances.

Variance: $$V(X) \equiv < X^2 > - < X >^2$$

Scaled variance: $$\omega^X \equiv \frac{< X^2 > - < X >^2}{< X >}$$

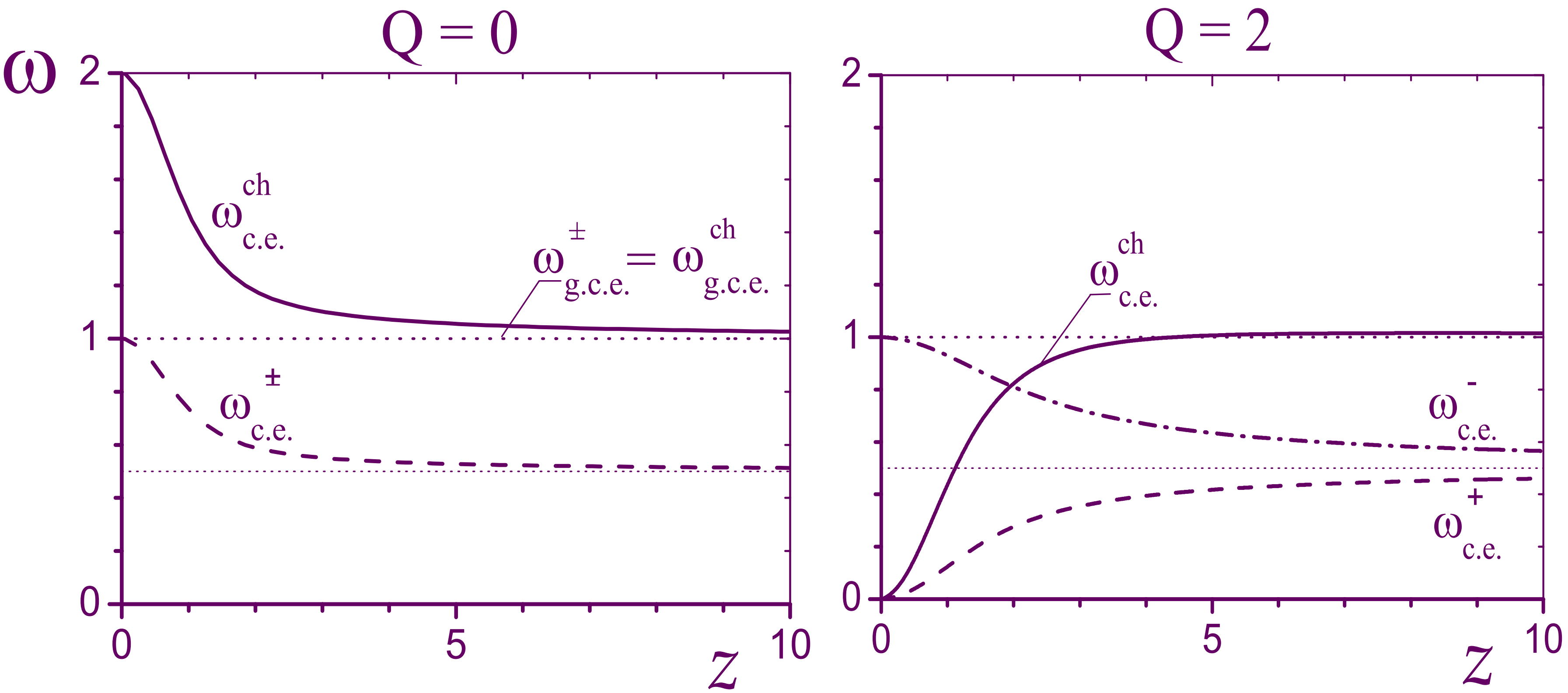
$$\omega^{\pm}_{\text{g.c.e.}} = \omega^{\text{ch}}_{\text{g.c.e.}} = 1 \; .$$

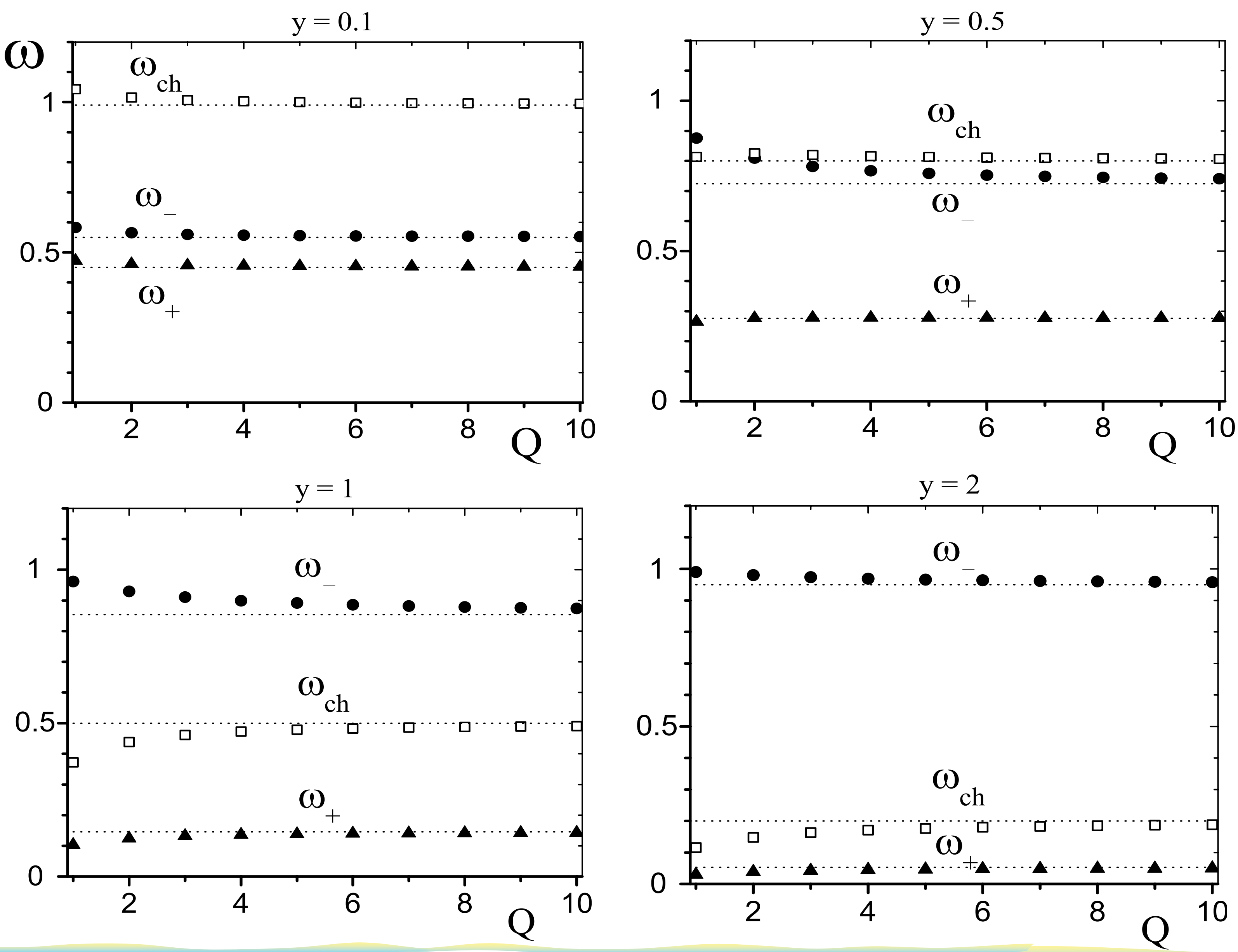
The scaled variances in c.e.

$$\omega^{\pm}_{\text{c.e.}} = 1 - z\left[\frac{I_{Q\mp1}(2z)}{I_Q(2z)} - \frac{I_{Q\mp2}(2z)}{I_{Q\mp1}(2z)}\right]$$

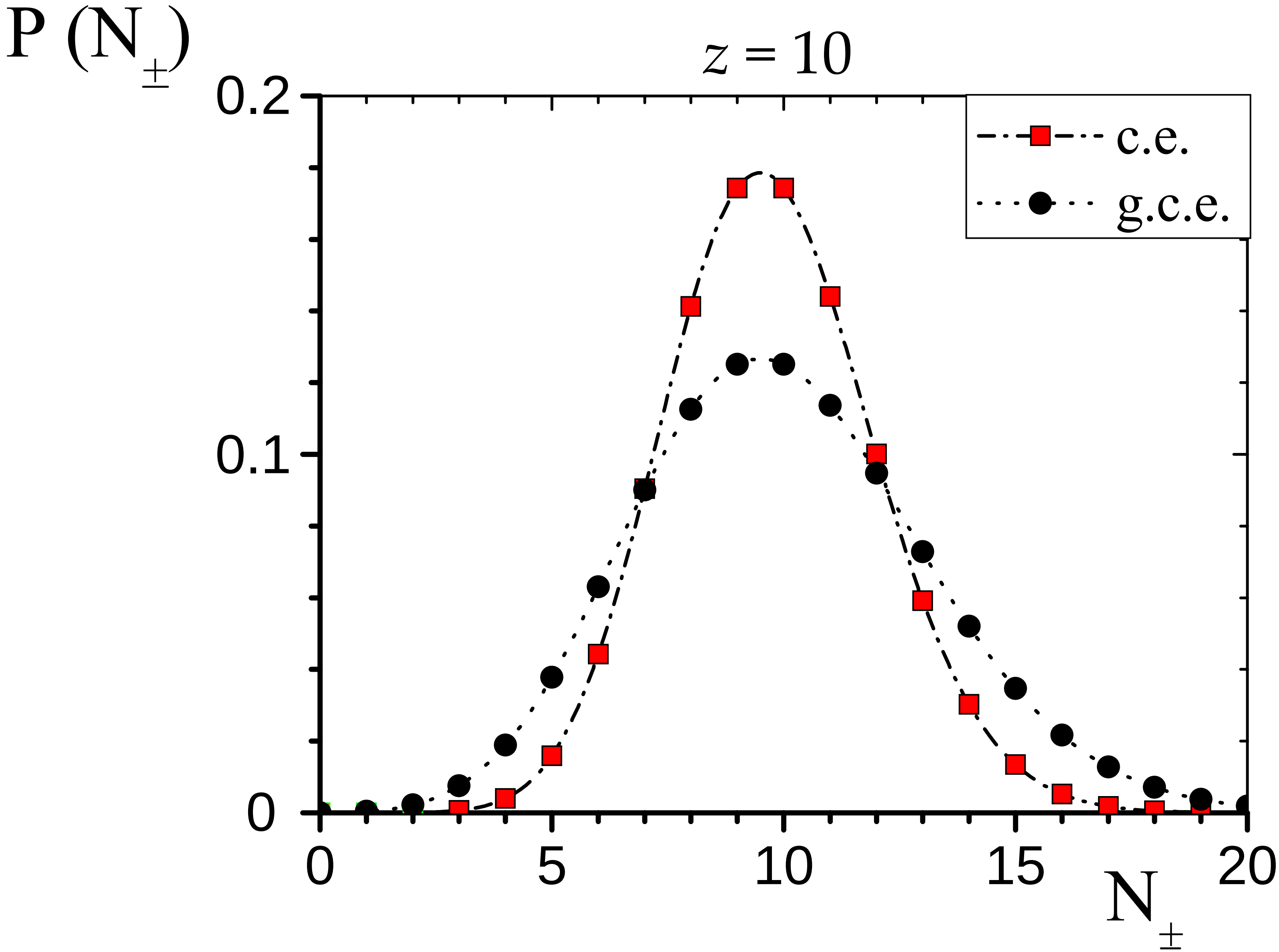
$$\omega^{\text{ch}}_{\text{c.e.}} = 1 + z\left[\frac{I_{Q-2}(2z)+I_{Q+2}(2z)+2I_Q(2z)}{I_{Q-1}(2z)+I_{Q+1}(2z)} - \frac{I_{Q-1}(2z)+I_{Q+1}(2z)}{I_Q(2z)}\right]$$

Particle number fluctuations.





y = 0.1
ω
ω_{ch}
ω_-
ω_+
Q
y = 0.5
ω_{ch}
ω_-
ω_+
Q
y = 1
ω_-
ω_{ch}
ω_+
Q
y = 2
ω_-
ω_{ch}
ω_+
Q



P (N±)
z = 10
0.2
0.1
0
0
5
10
15
20
N±
c.e.
g.c.e.

Energy fluctuations.

One-particle energy:

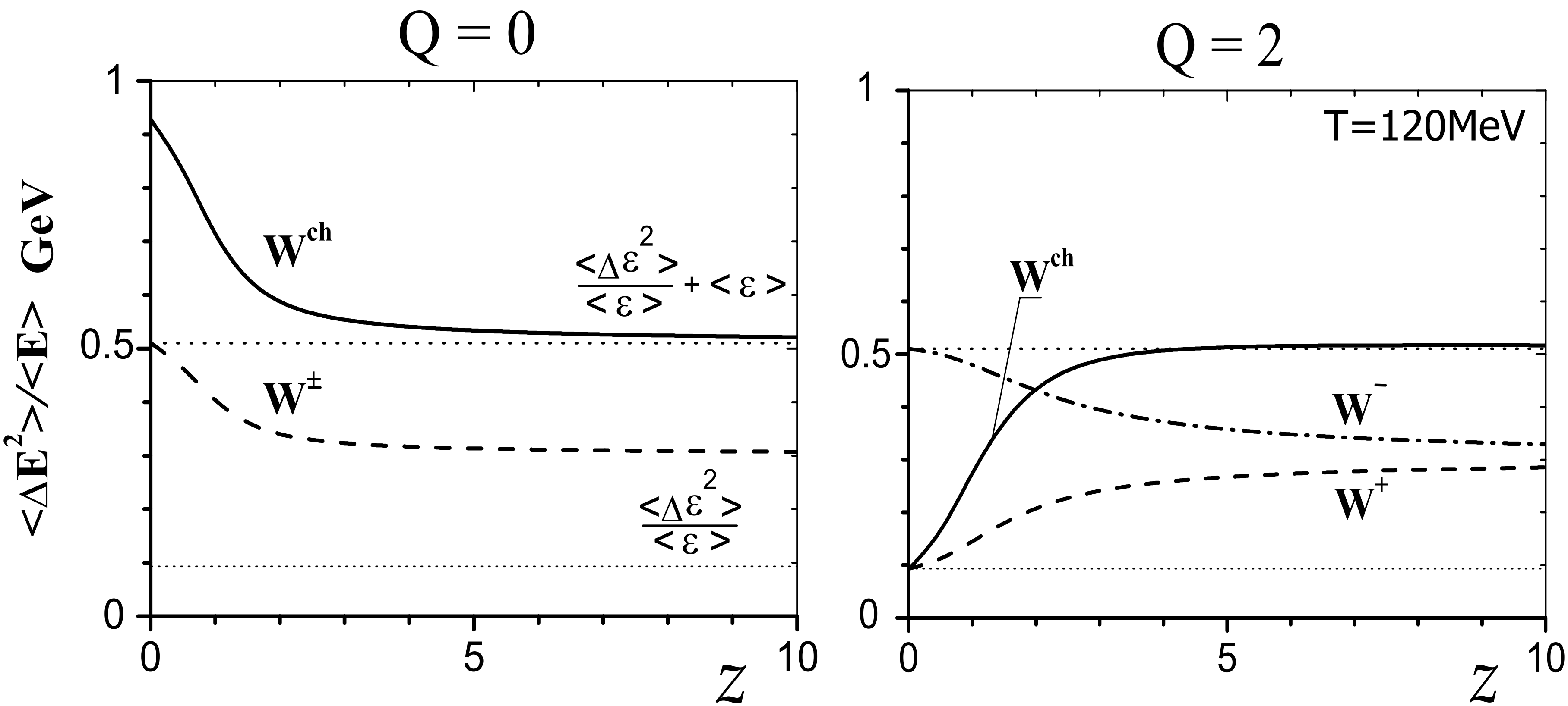
$$<\varepsilon_j> = -\frac{\partial z_j / \partial \beta}{z_j}, \qquad (\beta = 1/T)$$

$$<\varepsilon_j^2> = \frac{\partial z_j^2 / \partial \beta^2}{z_j},$$

Energy fluctuations:

$$W^{j\pm} \equiv \frac{<E_{j\pm}^2> - <E_{j\pm}>^2}{<E_{j\pm}>}$$

$$= \frac{<\varepsilon_j^2> - <\varepsilon_j>^2}{<\varepsilon_j>} + <\varepsilon_j> \omega^{j\pm}$$



$$E_{\pm} \equiv \sum_j E_{j\pm} \ , \qquad E_{ch} \equiv \sum_j (E_{j+} + E_{j-}) \ ,$$

$$< \varepsilon > \equiv \sum_j z_j < \varepsilon_j > / z \ , \qquad < \varepsilon^2 > \equiv \sum_j z_j < \varepsilon_j^2 > / z \ ,$$

Quantum statistics effects.

$$< n_p^{\pm} >_{g.c.e.} = \frac{1}{\exp\left[\left(\sqrt{p^2+m^2} \mp \mu\right)/T\right] - \gamma}\,, \qquad \omega^{\alpha} = \frac{\sum_{p,k} < \Delta n_p^{\alpha} \Delta n_k^{\alpha} >}{\sum_p < n_p^{\alpha} >}\,,$$

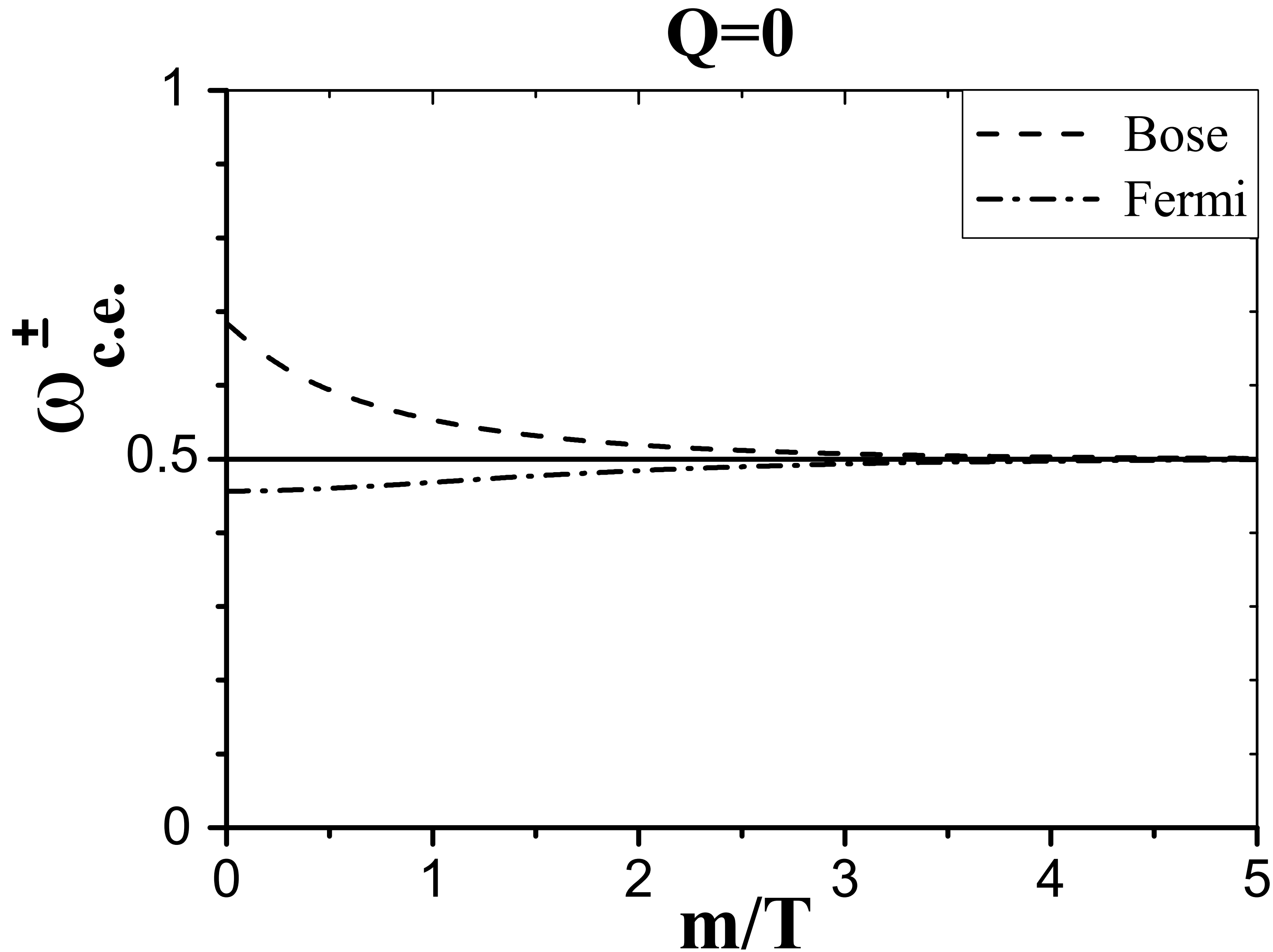
$$< \Delta n_p^{\pm 2} >_{g.c.e.} \equiv < (n_p^{\pm})^2 >_{g.c.e.} - < n_p^{\pm} >_{g.c.e.}^2 = < n_p^{\pm} >_{g.c.e.} (1 + \gamma < n_p^{\pm} >_{g.c.e.})$$

$$\mu = 0, \quad m/T \to 0 :$$

$$\omega_{g.c.e.}^{\pm Bose} = \frac{\pi^2}{6\zeta(3)} \approx 1.368, \qquad \omega_{g.c.e.}^{\pm Fermi} = \frac{\pi^2}{9\zeta(3)} \approx 0.912,$$

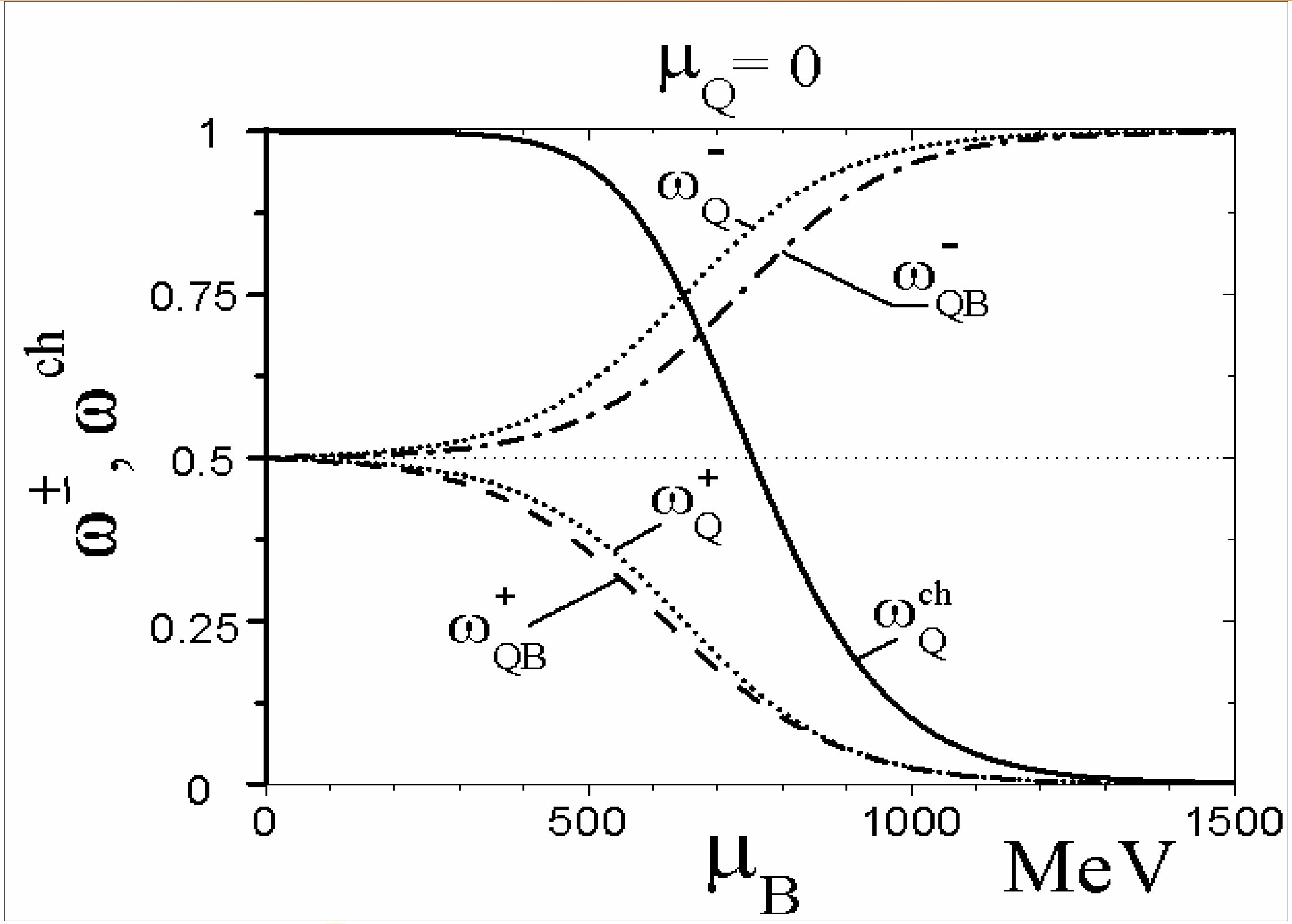
$$\mathbf{Q = 0 :}$$

$$\omega_{c.e.}^{\pm Bose} = \frac{\pi^2}{12\zeta(3)} \approx \mathbf{0.684}, \qquad \omega_{c.e.}^{\pm Fermi} = \frac{\pi^2}{18\zeta(3)} \approx \mathbf{0.456},$$



Q=0
Bose
Fermi
$\omega^{\pm}_{c.e.}$
1
0.5
0
0
1
2
3
4
5
m/T

A system with two conserved charges (p,n,π-gas).



The microcanonical partition function

$$W_N(E,V) = \frac{1}{N!}\left[\frac{gV}{2\pi^3}\right]^N \int d^3p^{(N)} \ldots \int d^3p^{(1)}$$

$$\times\ \delta(E - \sum_{k=1}^{N} |\vec{p}^{(k)}|) = \frac{1}{E}\frac{x^N}{(3N-1)!N!}\ ,$$

$$W(E,V) \equiv \sum_{N=1}^{\infty} W_N(E,V) = \frac{x}{2E}\ {}_0F_3\left(;\frac{4}{3},\frac{5}{3},2;\frac{x}{27}\right)$$

where $x \equiv \dfrac{gVE^3}{\pi^2}$.

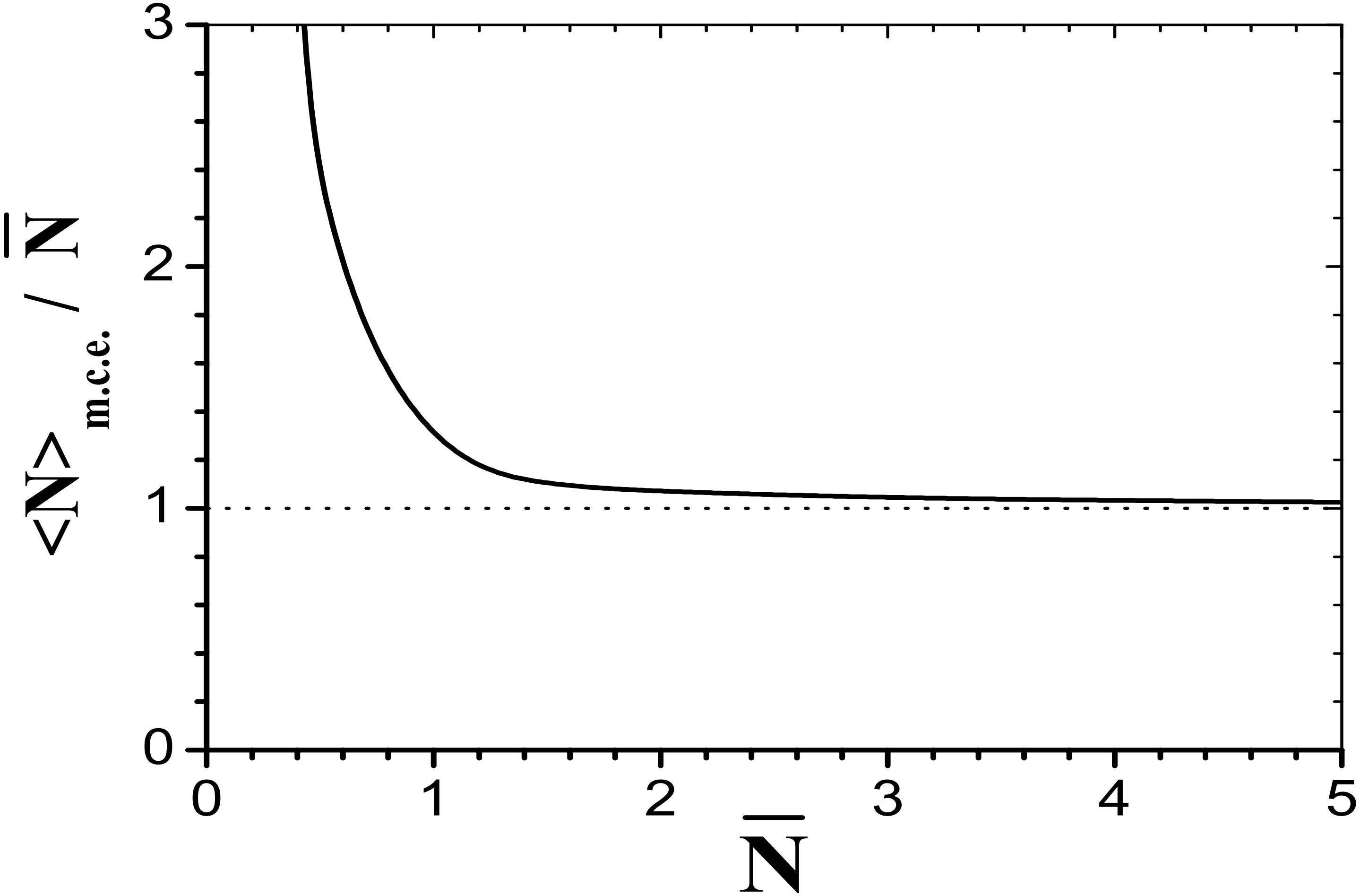
Average number of particles

We compare the results of the MCE and GCE at equal volumes V and energies $<E>_{g.c.e.} = E$. It follows:

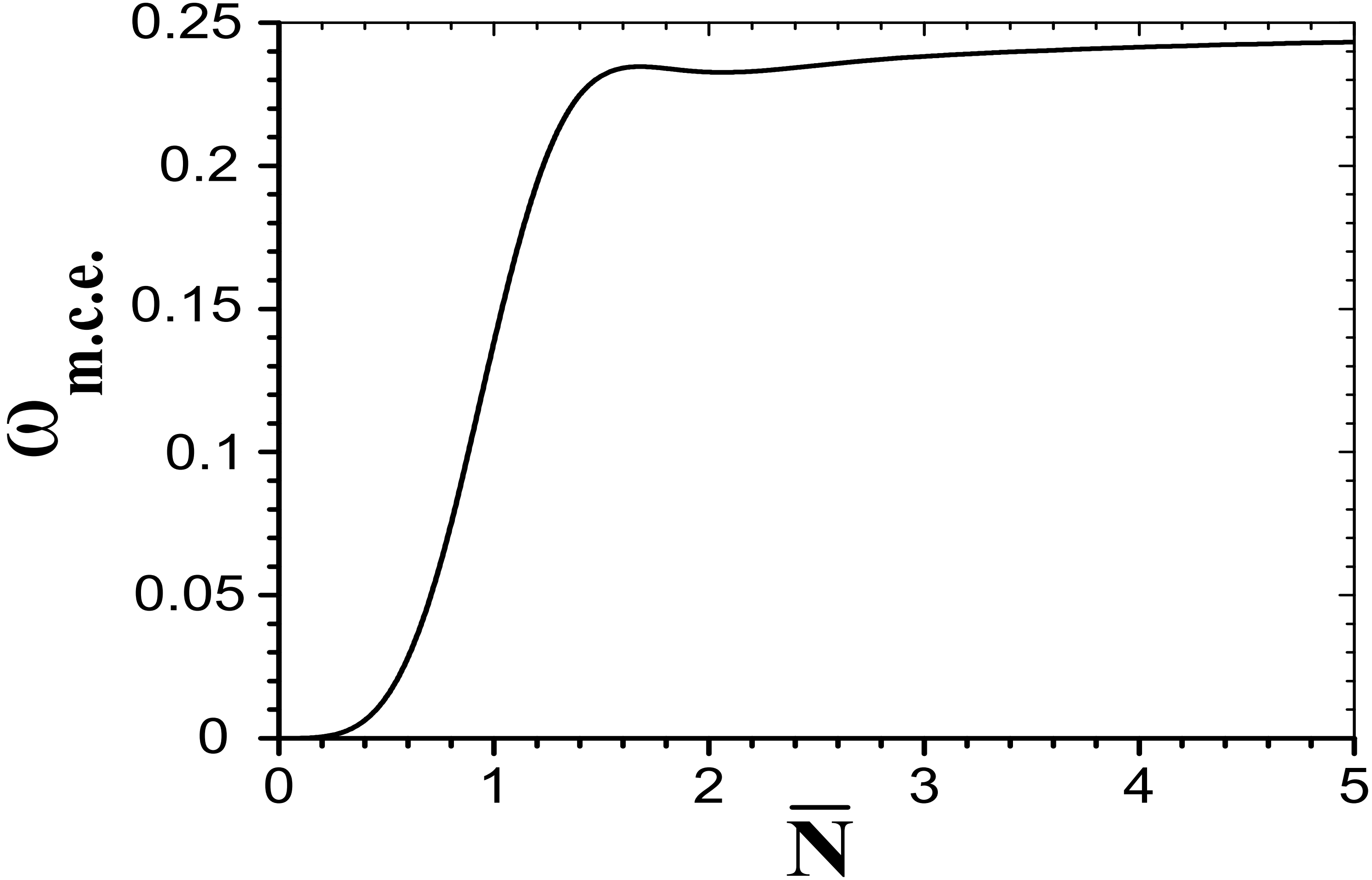
$$< N >_{g.c.e.} = \frac{gV\,T^3}{\pi^2}, \qquad < E >_{g.c.e.} = \frac{3gV\,T^4}{\pi^2} \quad \Rightarrow \quad < N >_{g.c.e.} \equiv \overline{N} = \left(\frac{x}{27}\right)^{1/4}$$

The average number of particles in the m.c.e. equals to:

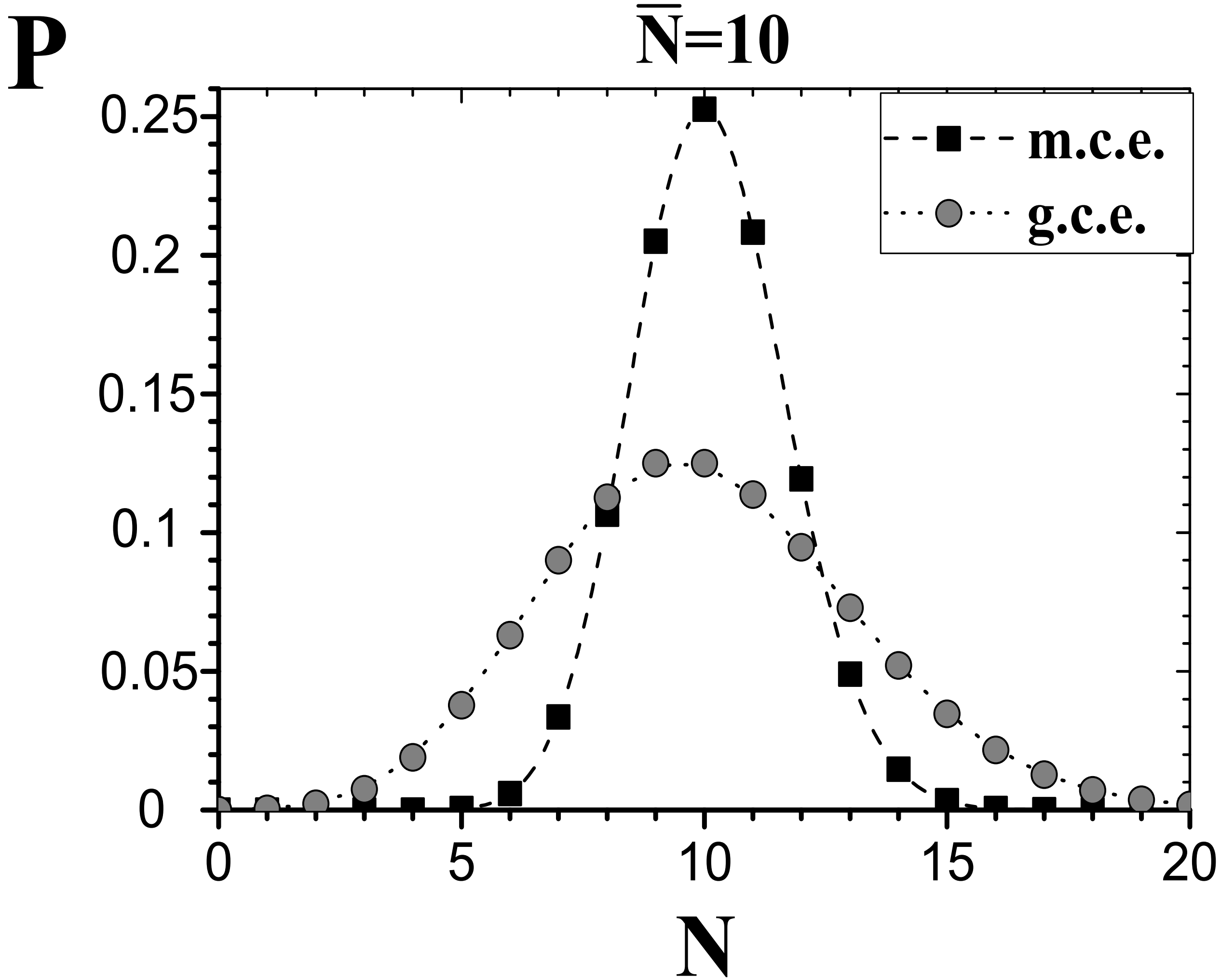
$$< N >_{m.c.e.} \equiv \frac{1}{W(E,V)} \sum_{N=1}^{\infty} N\, W_N(E,V) = \frac{{}_0F_3\left(;1,\frac{4}{3},\frac{5}{3};\frac{x}{27}\right)}{{}_0F_3\left(;\frac{4}{3},\frac{5}{3},2;\frac{x}{27}\right)}\,.$$



$$< N >_{m.c.e.} \approx \overline{N}\left(1 + \frac{1}{8\,\overline{N}} + \frac{35}{1152\,\overline{N}^2} + \ldots\right), \qquad N \to \infty.$$



$$\omega_{\text{m.c.e.}} \approx \frac{1}{4}\left(1 - \frac{1}{8\,\overline{N}} + \ldots\right), \qquad N \to \infty.$$



$$P_{g.c.e.}(E,V,N) = \exp(-\overline{N})\ \frac{\overline{N}^N}{N!}.$$

The MCE for charged particles

$$W(E,V,Q=0) = \sum_{N_+=1}^{\infty}\sum_{N_-=1}^{\infty} W_{N_+,N_-}(E,V,Q=0)$$

$$= \sum_{N_+=1}^{\infty}\sum_{N_-=1}^{\infty} \frac{1}{N_+!}\frac{1}{N_-!}\left[\frac{gV}{2\pi^3}\right]^{N_+}\left[\frac{gV}{2\pi^3}\right]^{N_-} \int d^3p^{(N_+)} \ldots \int d^3p^{(1)} \int d^3q^{(N_-)} \ldots \int d^3q^{(1)}$$

$$\times\ \delta\left[E - \sum_{k=1}^{N_+} |\vec{p}^{(k)}| - \sum_{k=1}^{N_-} |\vec{q}^{(k)}|\right]\ \delta(N_+ - N_-)$$

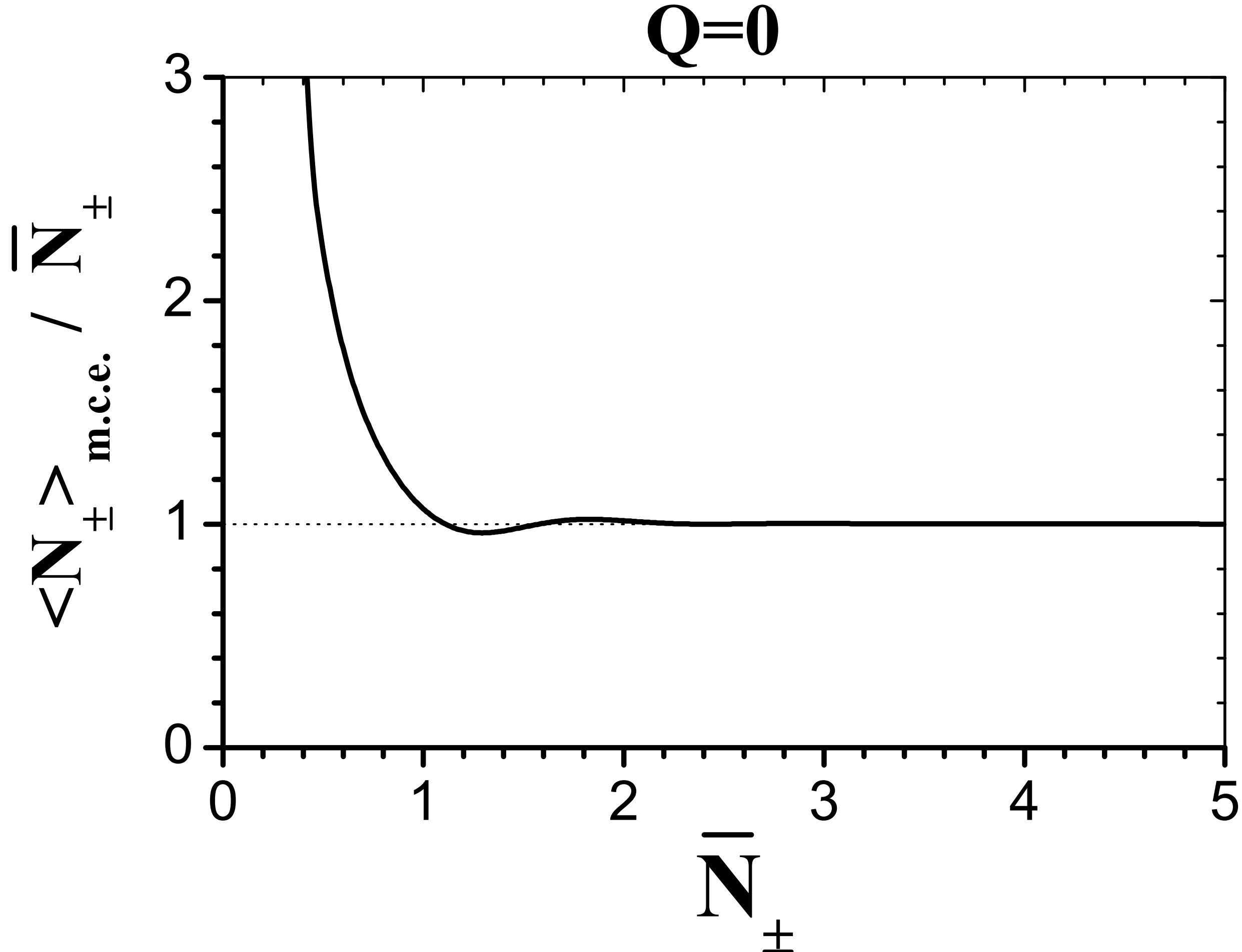
$$= \sum_{N_+=1}^{\infty}\sum_{N_-=1}^{\infty} \int_0^{\infty} dE_+ \int_0^{\infty} dE_-\ W_{N_+}(E_+,V)\, W_{N_-}(E_-,V)\ \delta[E - E_+ - E_-]\ \delta(N_+ - N_-)$$

$$= \frac{x^2}{120\,E}\ {}_0F_7\left(;\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3},\frac{11}{6},2,2;\left(\frac{x}{216}\right)^2\right)$$

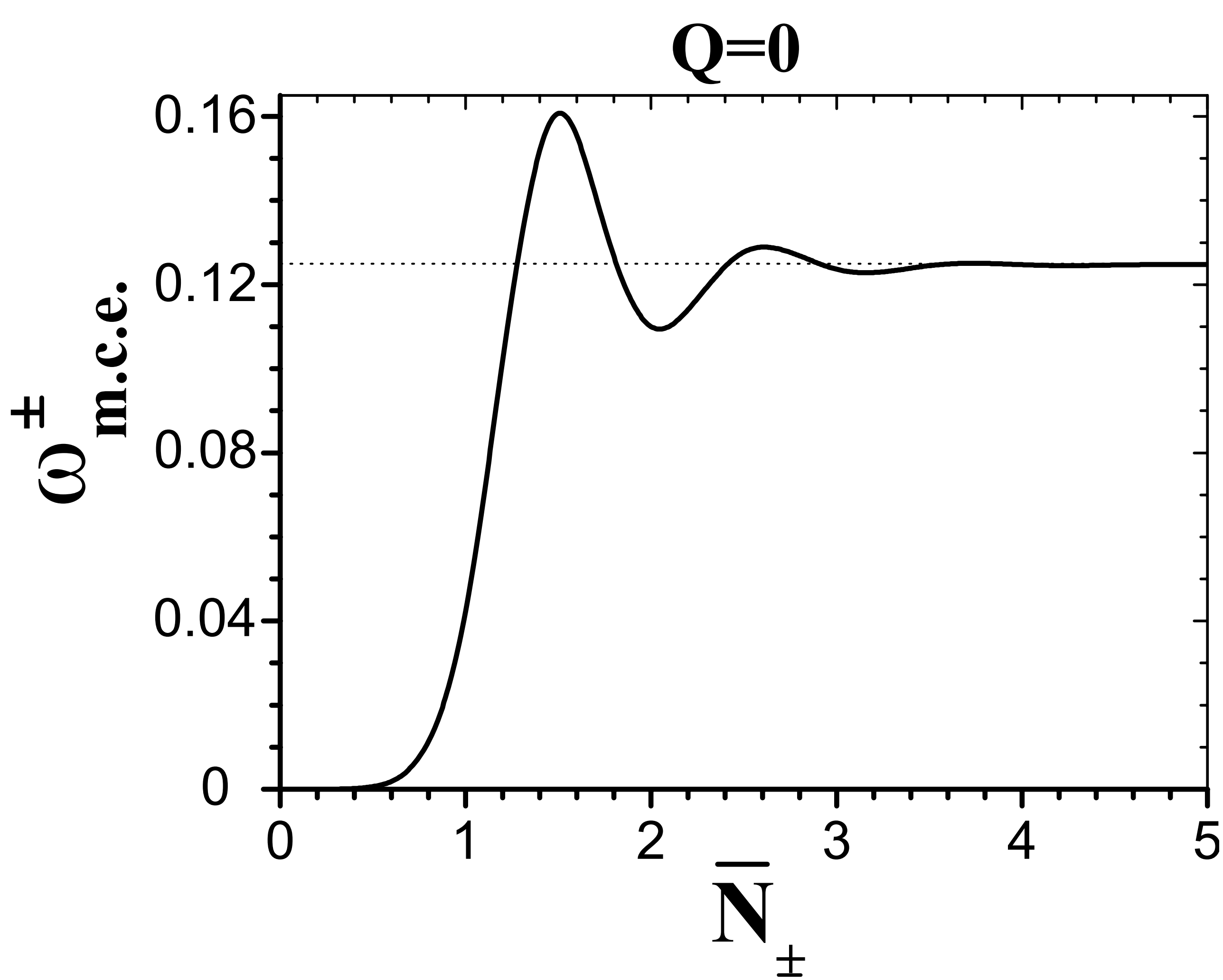
Average number of particles

$$<N_\pm>_{g.c.e.} = \frac{gVT^3}{\pi^2}, \quad <E>_{g.c.e.} = \frac{6gVT^4}{\pi^2} \quad \Rightarrow <N_\pm>_{g.c.e.} \equiv \overline{N}_\pm = \left(\frac{x}{216}\right)^{1/4}$$

$$<N_\pm>_{m.c.e.} = \frac{{}_0F_7\left(;1,\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3},\frac{11}{6},2;\left(\frac{x}{216}\right)^2\right)}{{}_0F_7\left(;\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3},\frac{11}{6},2,2;\left(\frac{x}{216}\right)^2\right)}.$$



$$< N_\pm >_{m.c.e.} \approx \overline{N}_\pm \left(1 + \frac{49}{2304\ \overline{N}_\pm^{\,2}} + \ldots \right), \qquad N \to \infty.$$



$$\omega^{\pm}_{m.c.e.} \approx \frac{1}{8}\left(1 - \frac{49}{1152\ \overline{N}^{2}_{\pm}} + \ldots\right), \qquad N \to \infty .$$

Quantum statistic effects

$$< n_p^{\pm} >_{g.c.e.} = \frac{1}{\exp\left[\left(\sqrt{p^2 + m^2} \mp \mu\right)/T\right] - \gamma} \,,$$

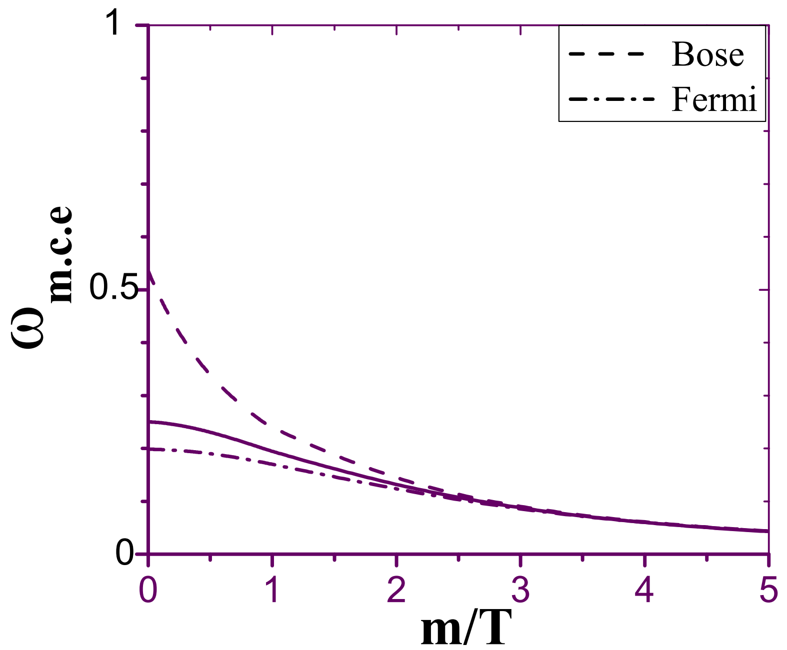
$$< \Delta n_p^{\pm\,2} >_{g.c.e.} \equiv < (n_p^{\pm})^2 >_{g.c.e.} - < n_p^{\pm} >^2_{g.c.e.} = < n_p^{\pm} >_{g.c.e.} (1 + \gamma < n_p^{\pm} >_{g.c.e.}) \equiv \upsilon_p^{\pm\,2}$$

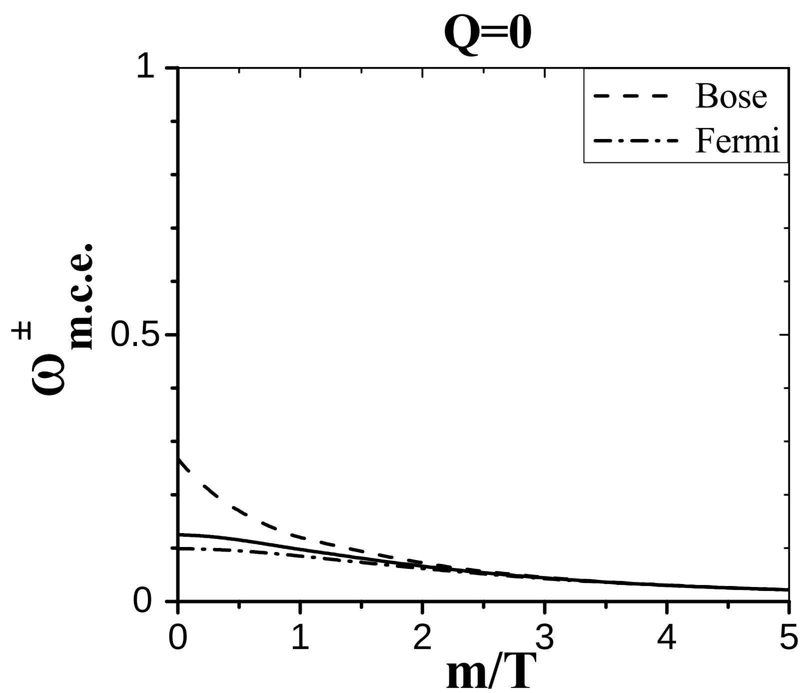
$$W_{g.c.e.}(\{n_p\}) = \prod_p w_p(n_p) \,, \qquad w_p(n_p) = \exp\left[- \frac{(\Delta n_p)^2}{2\upsilon_p^2} \right],$$

$$W_{m.c.e.}(\{n_p\}) = \prod_p w_p(n_p) \;\; \delta\left(E - \sum_p \varepsilon_p n_p \right),$$

$$W_{m.c.e.}(\{n_p\}, Q = 0) = \prod_p w_p(n_p) \;\; \delta\left(E - \sum_{p,\alpha} \varepsilon_p n_p^{\alpha} \right) \; \delta\left(\sum_p n_p^{+} - \sum_p n_p^{-} \right).$$

Quantum statistic effects





$$\omega^{\pm B}_{\mathrm{m.c.e.}} \approx 0.535\,,$$

$$\omega^{\pm}_{\mathrm{m.c.e.}} \approx \mathbf{1/4} = 0.25\,,$$

$$\omega^{\pm F}_{\mathrm{m.c.e.}} \approx 0.198\,,$$

$$\omega^{\pm B}_{\mathrm{m.c.e.}}(Q=0) \approx 0.268\,,$$

$$\omega^{\pm}_{\mathrm{m.c.e.}}(Q=0) \approx \mathbf{1/8} = 0.125\,,$$

$$\omega^{\pm F}_{\mathrm{m.c.e.}}(Q=0) \approx 0.099\,.$$

Conclusions.

1. Statistical ensembles are not equivalent for particle number fluctuations,
2. An exact charge conservation reduces fluctuations in thermodynamic limit,
3. At the nonzero net charge c.e. predicts a difference of N_+ and N_- fluctuations,
4. Energy fluctuations are mainly determined by particle number fluctuations,
5. "ω" changes from 0.5 to 5/9 for single and double charged particles,
6. The second charge conservation law lead to additional suppression of fluctuations.
7. E=const, $\Rightarrow$ $\omega=1/4$,
8. E=const, Q=0 $\Rightarrow$ $\omega=1/8$,
9. Quantum statistics effects lead to Bose enhancement (Fermi suppression) of fluctuations.