

Rational Approximations Package for REDUCE

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1 Periodic Decimal Representation

The division of one integer by another often results in a period in the decimal part. The `rational2periodic` function in this package can recognise and represent such an answer in a periodic representation. The inverse function, `periodic2rational`, can also convert a periodic representation back to a rational number.

Periodic Representation of a Rational Number

SYNTAX: `rational2periodic(n);`

INPUT: `n` is a rational number

RESULT: `periodic({a,b} , {c1,...,cn})`

where `a/b` is the non-periodic part
and `c1, ..., cn` are the digits of the periodic part.

EXAMPLE: `59/70` written as `0.8428571`

`1: rational2periodic(59/70);`

`periodic({8,10},{4,2,8,5,7,1})`

Rational Number of a Periodic Representation

SYNTAX: `periodic2rational(periodic({a,b},{c1,...,cn}))`
`periodic2rational({a,b},{c1,...,cn})`

INPUT: `a` is an integer
 `b` is 1, -1 or an integer multiple of 10
 `c1,...,cn` is a list of positive digits

RESULT: A rational number.

EXAMPLE: $0.8\overline{428571}$ written as $59/70$

2: `periodic2rational(periodic({8,10},{4,2,8,5,7,1}));`

```

59
----
70

```

3: `periodic2rational({8,10},{4,2,8,5,7,1});`

```

59
----
70

```

Note that if `a` is zero, `b` will indicate how many places after the decimal point that the period occurs. Note also that if the answer is negative then this will be indicated by the sign of `a` (unless `a` is zero in which case it is indicated by the sign of `b`).

ERROR MESSAGE

***** operator to be used in off rounded mode

The periodicity of a function can only be recognised in the `off rounded` mode. This is also true for the inverse procedure.

EXAMPLES

4: rational2periodic(1/3);

periodic({0,1},{3})

5: periodic2rational(ws);

$$\begin{array}{r} 1 \\ \text{---} \\ 3 \end{array}$$

6: periodic2rational({0,1},{3});

$$\begin{array}{r} 1 \\ \text{---} \\ 3 \end{array}$$

7: rational2periodic(-1/6);

periodic({-1,10},{6})

8: periodic2rational(ws);

$$\begin{array}{r} - 1 \\ \text{-----} \\ 6 \end{array}$$

9: rational2periodic(6/17);

periodic({0,1},{3,5,2,9,4,1,1,7,6,4,7,0,5,8,8,2})

10: periodic2rational(ws);

$$\begin{array}{r} 6 \\ \text{---} \\ 17 \end{array}$$

```
11: rational2periodic(352673/3124);
```

```
periodic({11289,100},{1,4,8,5,2,7,5,2,8,8,0,9,2,1,8,9,5,0,0,6,
                    4,0,2,0,4,8,6,5,5,5,6,9,7,8,2,3,3,0,3,4,
                    5,7,1,0,6,2,7,4,0,0,7,6,8,2,4,5,8,3,8,6,
                    6,8,3,7,3,8,7,9,6,4})
```

```
12: periodic2rational(ws);
```

```
  352673
-----
  3124
```

2 Continued Fractions

A continued fraction (see [1] §4.2) has the general form

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}$$

A more compact way of writing this is as

$$b_0 + \frac{a_1|}{|b_1|} + \frac{a_2|}{|b_2|} + \frac{a_3|}{|b_3|} + \dots$$

This is represented in REDUCE as

```
contfrac(Rational approximant, {b0, {a1, b1}, {a2, b2}, .....})
```

SYNTAX: cfrac(number);
 cfrac(number, length);
 cfrac(f, var);

```
cfrac(f, var, length);
```

INPUT:

| | |
|--------|--------------------------|
| number | is any real number |
| f | is a function |
| var | is the function variable |

Optional Argument: length

The `length` argument is optional. For an NON-RATIONAL function input the `length` argument specifies the number of ordered pairs, $\{a_i, b_i\}$, to be returned. Its default value is five. For a RATIONAL function input the `length` argument can only truncate the answer, it cannot return additional pairs even if the precision is increased. The default value is the complete continued fraction of the rational input. For a NUMBER input the default value is dependent on the precision of the session, and the `length` argument will only take effect if it has a smaller value than that of the number of ordered pairs which the default value would return.

EXAMPLES

```
13: cfrac(23.696);
```

$$\text{contfrac}\left(\frac{2962}{125}, \{23, \{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 2\}, \{1, 5\}\}\right)$$

```
14: cfrac(23.696,3);
```

$$\text{contfrac}\left(\frac{237}{10}, \{23, \{1, 1\}, \{1, 2\}, \{1, 3\}\}\right)$$

```
15: cfrac pi;
```

$$\text{contfrac}\left(\frac{1146408}{364913}, \{3, \{1, 7\}, \{1, 15\}, \{1, 1\}, \{1, 292\}, \{1, 1\}, \{1, 1\}, \{1, 1\}, \{1, 2\}, \{1, 1\}\}\right)$$

16: $\text{cfrac}(\pi, 3);$

$$\text{contfrac}\left(\frac{355}{113}, \{3, \{1, 7\}, \{1, 15\}, \{1, 1\}\}\right)$$

17: $\text{cfrac}(\pi * e * \sqrt{2}, 4);$

$$\text{contfrac}\left(\frac{10978}{909}, \{12, \{1, 12\}, \{1, 1\}, \{1, 68\}, \{1, 1\}\}\right)$$

18: $\text{cfrac}((x+2/3)^2/(6*x-5), x, 1);$

$$\text{contfrac}\left(\frac{9x^2 + 12x + 4}{54x - 45}, \left\{\frac{6x + 13}{36}, \left\{1, \frac{24x - 20}{9}\right\}\right\}\right)$$

19: $\text{cfrac}((x+2/3)^2/(6*x-5), x, 10);$

$$\text{contfrac}\left(\frac{9x^2 + 12x + 4}{54x - 45}, \left\{\frac{6x + 13}{36}, \left\{1, \frac{24x - 20}{9}\right\}\right\}\right)$$

20: $\text{cfrac}(e^x, x);$

$$\text{contfrac}\left(\frac{x^3 + 9x^2 + 36x + 60}{3x^2 - 24x + 60}, \{1, \{x, 1\}, \{-x, 2\}, \{x, 3\}, \{-x, 2\}, \{x, 5\}\}\right)$$

21: `cfrac(x^2/(x-1)*e^x,x);`

$$\text{contfrac}\left(\frac{x^6 + 3x^4 + x^2}{3x^4 - x^2 - 1}, \{0, \{-x^2, 1\}, \{-2x^2, 1\}, \{x^2, 1\}, \{x^2, 1\}, \{x^2, 1\}\}\right)$$

22: `cfrac(x^2/(x-1)*e^x,x,2);`

$$\text{contfrac}\left(\frac{x^2}{2x^2 - 1}, \{0, \{-x^2, 1\}, \{-2x^2, 1\}\}\right)$$

3 Padé Approximation

The Padé approximant represents a function by the ratio of two polynomials. The coefficients of the powers occurring in the polynomials are determined by the coefficients in the Taylor series expansion of the function (see [1]). Given a power series

$$f(x) = c_0 + c_1(x - h) + c_2(x - h)^2 \dots$$

and the degree of numerator, n , and of the denominator, d , the `pade` function finds the unique coefficients a_i, b_i in the Padé approximant

$$\frac{a_0 + a_1x + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_dx^d}.$$

SYNTAX: `pade(f, x, h, n, d);`

INPUT:

- `f` is the function to be approximated
- `x` is the function variable
- `h` is the point at which the approximation is evaluated
- `n` is the (specified) degree of the numerator
- `d` is the (specified) degree of the denominator

RESULT: Padé Approximant, ie. a rational function.

ERROR MESSAGES

***** not yet implemented

The Taylor series expansion for the function, `f`, has not yet been implemented in the REDUCE Taylor Package.

***** no Pade Approximation exists

A Padé Approximant of this function does not exist.

***** Pade Approximation of this order does not exist

A Padé Approximant of this order (ie. the specified numerator and denominator orders) does not exist but one of a different order may exist.

EXAMPLES

23: pade(sin(x),x,0,3,3);

$$\frac{x^2(-7x^2 + 60)}{3(x^2 + 20)}$$

24: pade(tanh(x),x,0,5,5);

$$\frac{x^4(x^2 + 105x^2 + 945)}{15(x^4 + 28x^2 + 63)}$$

25: pade(atan(x),x,0,5,5);

$$\frac{x^4(64x^2 + 735x^2 + 945)}{15(15x^4 + 70x^2 + 63)}$$

26: pade(exp(1/x),x,0,5,5);

***** no Pade Approximation exists

27: pade(factorial(x),x,1,3,3);

***** not yet implemented

28: pade(asech(x),x,0,3,3);

$$\frac{-3\log(x)^2x^2 + 8\log(x)^2 + 3\log(2)^2x^2 - 8\log(2)^2 + 2x^2}{3x^2 - 8}$$

29: taylor(ws-asech(x),x,0,10);

$$\log(x) \cdot (0 + 0(x^{11})) + \left(\frac{13}{768}x^6 + \frac{43}{2048}x^8 + \frac{1611}{81920}x^{10} + 0(x^{11}) \right)$$

30: pade(sin(x)/x^2,x,0,10,0);

***** Pade Approximation of this order does not exist

31: pade(sin(x)/x^2,x,0,10,2);

$$\frac{(-x^{10} + 110x^8 - 7920x^6 + 332640x^4 - 6652800x^2 + 39916800)/(39916800x)}$$

32: pade(exp(x),x,0,10,10);

$$\begin{aligned}
& (x^{10} + 110x^9 + 5940x^8 + 205920x^7 + 5045040x^6 \\
& + 90810720x^5 + 1210809600x^4 + 11762150400x^3 \\
& + 79394515200x^2 + 335221286400x + 670442572800) / \\
& (x^{10} - 110x^9 + 5940x^8 - 205920x^7 + 5045040x^6 \\
& - 90810720x^5 + 1210809600x^4 \\
& - 11762150400x^3 + 79394515200x^2 \\
& - 335221286400x + 670442572800)
\end{aligned}$$

33: `pade(sin(sqrt(x)),x,0,3,3);`

$$\begin{aligned}
& (\text{sqrt}(x) * \\
& (56447x^3 - 4851504x^2 + 132113520x - 885487680)) \backslash \\
& (7*(179x^3 - 7200x^2 - 2209680x - 126498240))
\end{aligned}$$

References

- [1] Baker(Jr.), George A. and Graves-Morris, Peter:
Padé Approximants, Part I: Basic Theory, (Encyclopedia of mathematics and its applications, Vol 13, Section: Mathematics of physics), Addison-Wesley Publishing Company, Reading, Massachusetts, 1981.