Rational Approximations Package for REDUCE

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1 Periodic Decimal Representation

The division of one integer by another often results in a period in the decimal part. The `rational2periodic` function in this package can recognise and represent such an answer in a periodic representation. The inverse function, `periodic2rational`, can also convert a periodic representation back to a rational number.

Periodic Representation of a Rational Number

SYNTAX: `rational2periodic(n);`

INPUT: `n` is a rational number

RESULT: `periodic({a,b} , {c1,...,cn})`

where `a/b` is the non-periodic part
and `c1,...,cn` are the digits of the periodic part.

EXAMPLE: 59/70 written as $0.8\overline{428571}$

```
1:  rational2periodic(59/70);

periodic({8,10},{4,2,8,5,7,1})
```

Rational Number of a Periodic Representation

SYNTAX: `periodic2rational(periodic({a,b},{c1,...,cn}))`
`periodic2rational({a,b},{c1,...,cn})`

INPUT:

`a`	is an integer
`b`	is 1, -1 or an integer multiple of 10
`c1,...,cn`	is a list of positive digits

RESULT: A rational number.

EXAMPLE: $0.8\overline{428571}$ written as 59/70

```
2:  periodic2rational(periodic({8,10},{4,2,8,5,7,1}));

 59
----
 70

3:  periodic2rational({8,10},{4,2,8,5,7,1});

 59
----
 70
```

Note that if `a` is zero, `b` will indicate how many places after the decimal point that the period occurs. Note also that if the answer is negative then this will be indicated by the sign of `a` (unless `a` is zero in which case it is indicated by the sign of `b`).

ERROR MESSAGE

`***** operator to be used in off rounded mode`

The periodicity of a function can only be recognised in the `off rounded` mode. This is also true for the inverse procedure.

EXAMPLES

```
4: rational2periodic(1/3);

periodic({0,1},{3})

5: periodic2rational(ws);

 1
---
 3

6: periodic2rational({0,1},{3});

 1
---
 3

7: rational2periodic(-1/6);

periodic({-1,10},{6})

8: periodic2rational(ws);

  - 1
------
  6

9: rational2periodic(6/17);

periodic({0,1},{3,5,2,9,4,1,1,7,6,4,7,0,5,8,8,2})

10: periodic2rational(ws);

 6
----
 17
```

```
11: rational2periodic(352673/3124);

periodic({11289,100},{1,4,8,5,2,7,5,2,8,8,0,9,2,1,8,9,5,0,0,6,
                      4,0,2,0,4,8,6,5,5,5,6,9,7,8,2,3,3,0,3,4,
                      5,7,1,0,6,2,7,4,0,0,7,6,8,2,4,5,8,3,8,6,
                      6,8,3,7,3,8,7,9,6,4})

12: periodic2rational(ws);

 352673
--------
  3124
```

2 Continued Fractions

A continued fraction (see [1] §4.2) has the general form

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}} \ .$$

A more compact way of writing this is as

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \ldots \ .$$

This is represented in REDUCE as

$$\texttt{contfrac}(\textit{Rational approximant}, \{\texttt{b0}, \{\texttt{a1}, \texttt{b1}\}, \{\texttt{a2}, \texttt{b2}\}, ...\})$$

SYNTAX:

```
cfrac(number);
cfrac(number,length);
cfrac(f, var);
```

```
cfrac(f, var, length);
```

INPUT:

	`number`	is any real number
	`f`	is a function
	`var`	is the function variable

Optional Argument: `length`

The `length` argument is optional. For an NON-RATIONAL function input the `length` argument specifies the number of ordered pairs, $\{a_i, b_i\}$, to be returned. It's default value is five. For a RATIONAL function input the `length` argument can only truncate the answer, it cannot return additional pairs even if the precision is increased. The default value is the complete continued fraction of the rational input. For a NUMBER input the default value is dependent on the precision of the session, and the `length` argument will only take effect if it has a smaller value than that of the number of ordered pairs which the default value would return.

EXAMPLES

```
13: cfrac(23.696);

          2962
contfrac(------,{23,{1,1},{1,2},{1,3},{1,2},{1,5}})
          125


14: cfrac(23.696,3);

          237
contfrac(-----,{23,{1,1},{1,2},{1,3}})
          10

15: cfrac pi;
```

```
          1146408
contfrac(---------,
          364913

         {3,{1,7},{1,15},{1,1},{1,292},{1,1},{1,1},{1,1},{1,2},{1,1}})

16: cfrac(pi,3);

          355
contfrac(-----,{3,{1,7},{1,15},{1,1}})
          113

17: cfrac(pi*e*sqrt(2),4);

          10978
contfrac(-------,{12,{1,12},{1,1},{1,68},{1,1}})
           909

18: cfrac((x+2/3)^2/(6*x-5),x,1);

             2
          9*x  + 12*x + 4    6*x + 13      24*x - 20
contfrac(-----------------,{----------,{1,-----------}})
             54*x - 45          36             9

19: cfrac((x+2/3)^2/(6*x-5),x,10);

             2
          9*x  + 12*x + 4    6*x + 13      24*x - 20
contfrac(-----------------,{----------,{1,-----------}})
             54*x - 45          36             9


20: cfrac(e^x,x);
```

```
             3      2
            x  + 9*x  + 36*x + 60
contfrac(-----------------------,{1,{x,1},{ - x,2},{x,3},{ - x,2},{x,5}})
                 2
              3*x  - 24*x + 60

21: cfrac(x^2/(x-1)*e^x,x);

             6      4    2
            x  + 3*x  + x
contfrac(----------------,{0,
               4    2
            3*x  - x  - 1

                  2             2       2       2       2
              { - x ,1}, { - 2*x ,1}, {x ,1}, {x ,1}, {x ,1}})

22: cfrac(x^2/(x-1)*e^x,x,2);

               2
               x              2            2
contfrac(----------,{0,{ - x ,1},{ - 2*x ,1}})
               2
            2*x  - 1
```

3 Padé Approximation

The Padé approximant represents a function by the ratio of two polynomials. The coefficients of the powers occuring in the polynomials are determined by the coefficients in the Taylor series expansion of the function (see [1]). Given a power series

$$f(x) = c_0 + c_1(x - h) + c_2(x - h)^2 \dots$$

and the degree of numerator, n, and of the denominator, d, the `pade` function finds the unique coefficients a_i, b_i in the Padé approximant

$$\frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_d x^d} .$$

SYNTAX: `pade(f, x, h, n, d);`

INPUT:

- `f` is the funtion to be approximated
- `x` is the function variable
- `h` is the point at which the approximation is evaluated
- `n` is the (specified) degree of the numerator
- `d` is the (specified) degree of the denominator

RESULT: Padé Approximant, ie. a rational function.

ERROR MESSAGES

`***** not yet implemented`

The Taylor series expansion for the function, f, has not yet been implemented in the REDUCE Taylor Package.

`***** no Pade Approximation exists`

A Padé Approximant of this function does not exist.

```
***** Pade Approximation of this order does not exist
```

A Padé Approximant of this order (ie. the specified numerator and denominator orders) does not exist but one of a different order may exist.

EXAMPLES

```
23: pade(sin(x),x,0,3,3);

          2
 x*( - 7*x  + 60)
------------------
       2
   3*(x  + 20)

24: pade(tanh(x),x,0,5,5);

     4        2
 x*(x  + 105*x  + 945)
-----------------------
      4       2
 15*(x  + 28*x  + 63)

25: pade(atan(x),x,0,5,5);

        4        2
 x*(64*x  + 735*x  + 945)
--------------------------
         4       2
 15*(15*x  + 70*x  + 63)

26: pade(exp(1/x),x,0,5,5);

***** no Pade Approximation exists
```

```
27: pade(factorial(x),x,1,3,3);

***** not yet implemented

28: pade(asech(x),x,0,3,3);

            2                        2                  2
- 3*log(x)*x  + 8*log(x) + 3*log(2)*x  - 8*log(2) + 2*x
---------------------------------------------------------
                           2
                       3*x  - 8

29: taylor(ws-asech(x),x,0,10);

               11
log(x)*(0 + O(x  ))

     13    6     43    8    1611    10       11
 + (-----*x  + ------*x  + -------*x   + O(x  ))
     768        2048        81920

30: pade(sin(x)/x^2,x,0,10,0);

***** Pade Approximation of this order does not exist

31:  pade(sin(x)/x^2,x,0,10,2);

     10         8         6           4            2
( - x   + 110*x  - 7920*x  + 332640*x  - 6652800*x

  + 39916800)/(39916800*x)

32: pade(exp(x),x,0,10,10);
```

```
  10        9         8           7            6
(x   + 110*x  + 5940*x  + 205920*x  + 5045040*x

              5               4                3
  + 90810720*x  + 1210809600*x  + 11762150400*x

                 2
  + 79394515200*x  + 335221286400*x + 670442572800)/

      10        9         8           7            6
    (x   - 110*x  + 5940*x  - 205920*x  + 5045040*x

                     5               4
         - 90810720*x  + 1210809600*x

                        3                2
         - 11762150400*x + 79394515200*x

         - 335221286400*x + 670442572800)

33: pade(sin(sqrt(x)),x,0,3,3);

(sqrt(x)*
            3            2
  (56447*x  - 4851504*x  + 132113520*x - 885487680))\

              3         2
    (7*(179*x  - 7200*x  - 2209680*x - 126498240))
```

References

[1] Baker(Jr.), George A. and Graves-Morris, Peter:
Padé Approximants, Part I: Basic Theory, (Encyclopedia of mathematics and its applications, Vol 13, Section: Mathematics of physics), Addison-Wesley Publishing Company, Reading, Massachusetts, 1981.