SQUID: Basic equations and characteristics

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1 Introduction

Superconducting **QU**antum **I**nterference **D**evice **(SQUID)** is a superconducting loop containing one or more Josephson Junctions. Two physical phenomena are combined in SQUIDs; flux quantization in a superconducting loops, and the Josephson effect. SQUIDs are the most sensitive detectors of magnetic flux known. The double junction interferometer (dc-SQUID) consists of two junctions connected in parallel on a superconducting loop, as it's shown in fig 1.

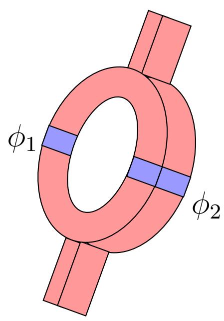

Figure 1: dc-SQUID

Majorana bound states have been predicted to be hosted in Josephson junctions [1–4] and superconducting quantum interference devices (SQUIDs) with topologically nontrivial barriers [5], these SQUIDs are expected to be used as a real quantum gates that are topologically decoupled from local sources of decoherance.

2 Trivial DC-SQUID

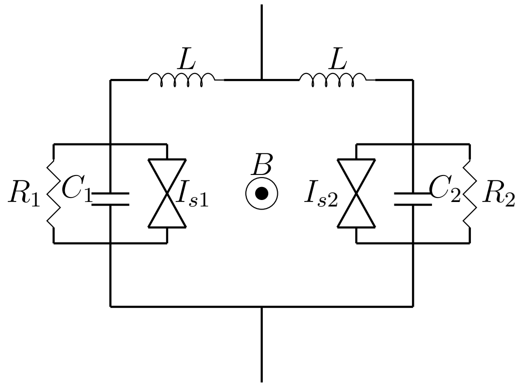

Figure 2: dc-SQUID equivalent circuit

Assuming a symmetrical dc-SQUID, and modeling the junctions by the resistively and capacitively shunted junction (RCSJ) model, as represented in fig. 2. The currents through the junctions can be written as:

$$I_1 = \frac{C\hbar}{2e}\frac{d^2\varphi_1}{dt^2} + \frac{\hbar}{2eR}\frac{d\varphi_1}{dt} + I_c\sin(\varphi_1) \tag{1}$$

$$I_2 = \frac{C\hbar}{2e}\frac{d^2\varphi_2}{dt^2} + \frac{\hbar}{2eR}\frac{d\varphi_2}{dt} + I_c\sin(\varphi_2) \tag{2}$$

Normalizing current to $i = \frac{I}{I_c}$ and time to $\tau = \frac{2\pi I_c R}{\Phi_0}t$, equations1 ,2 can be written as:

$$i_1 = \beta_c\frac{d^2\varphi_1}{dt^2} + \frac{d\varphi_1}{dt} + \sin(\varphi_1) \tag{3}$$

$$i_2 = \beta_c\frac{d^2\varphi_2}{dt^2} + \frac{d\varphi_2}{dt} + \sin(\varphi_2) \tag{4}$$

where $\beta_c = \frac{2\pi I_c R^2 C}{\Phi_0}$ is the McCumber parameter [6]

when magnetic field threads the superconducting loop, the magnetic flux is quantized according to:

$$\frac{1}{2\pi}(\varphi_1 - \varphi_2) + \frac{\Phi_t}{\Phi_0} = n \tag{5}$$

where the total flux Φ_t is:

$$\Phi_t = \Phi_e + LI_c\sin(\varphi_1) - LI_c\sin(\varphi_2) \tag{6}$$

$$\frac{2\pi\Phi_t}{\Phi_0} = \frac{2\pi\Phi_e}{\Phi_0} + \frac{2\pi L I_c}{\Phi_t}(\sin(\varphi_1) - \sin(\varphi_2)) \tag{7}$$

in dimensionless notation, with $\beta_L = \frac{2\pi L I_c}{\Phi_t}$:

$$2\pi\frac{\Phi_t}{\Phi_0} = 2\pi\frac{\Phi_e}{\Phi_0} + \beta_L(\sin(\varphi_1) - \sin(\varphi_2)) \tag{8}$$

β_L is the screening parameter, this parameter represents the ratio of the magnetic flux generated by the maximum possible circulating current I_c and $\Phi_0/2\pi$.

$$2\pi n - (\varphi_1 - \varphi_2) - 2\pi\frac{\Phi_e}{\Phi_0} = \beta_L[\sin(\varphi_1) - \sin(\varphi_2)] \tag{9}$$

the system of equations is used:

If $\beta_L = 0$

$$\dot{V_1} = \frac{1}{\beta_c}\big[\frac{1}{2}\{I - \sin(\varphi_1) - \sin(\varphi_2)\} - V_1\big]; \tag{10}$$

$$\dot{\varphi_1} = V_1 \tag{11}$$

$$\varphi_2 = 2\pi(\frac{\Phi_e}{\Phi_0} - n) - \varphi_1 \tag{12}$$

If $\beta_L \neq 0$

$$\dot{V_1} = \frac{1}{\beta_c}\big[\frac{1}{2}I - \sin(\varphi_1) + \frac{1}{\beta_L}(2\pi n - \varphi_1 + \varphi_2 - 2\pi\Phi_e) - V_1\big] \tag{13}$$

$$\dot{\varphi_1} = V_1 \tag{14}$$

$$\dot{V_2} = \frac{1}{\beta_c}\big[\frac{1}{2}I - \sin(\varphi_2) + \frac{1}{\beta_L}(2\pi n - \varphi_1 + \varphi_2 - 2\pi\Phi_e) - V_2\big] \tag{15}$$

$$\dot{\varphi_2} = V_2 \tag{16}$$

3 dc-SQUID characteristics

3.1 S-state

Fig 3.(a) shows the critical current of the dc-SQUID vs. applied flux $I_c(\Phi_e)$ for 3 values of the screening parameter β_L. .Fig 3.(b); the SQUID handbook [7] results for the same values, to be compared. As the screening parameter β_L increases the squid sensitivity to external magnetic flux decreases.

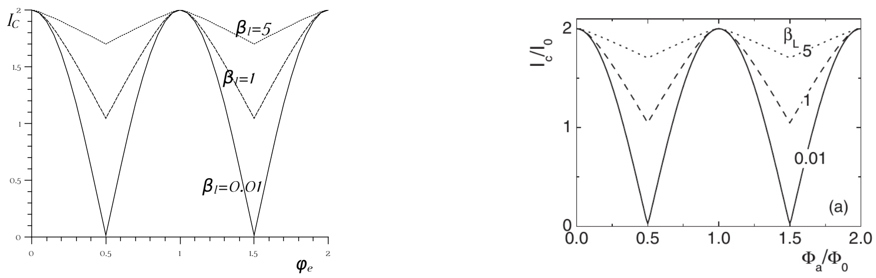

Figure 3: (a) Critical current modulation of dc-SQUID for screening parameter β_L= 0.01, 1 and 5,
(b) A (Screen-shot) of Fig. 2.7(a)- SQUID Handbook [7]

If $\beta_L = 0$, it's straight forward to get a mathematical expression for **the critical current** dependence on the external flux $I_c(\Phi_e)$.
Starting with quantization:

$$\varphi_2 - \varphi_1 = 2\pi \frac{\Phi_e}{\Phi_0} \tag{17}$$

In that case the super-current is

$$I_s = I_c\big(\sin(\varphi_1) + \sin(\varphi_2)\big) \tag{18}$$

$$I_s = 2I_c \sin\big(\frac{\varphi_1 + \varphi_2}{2}\big)\cos\big(\frac{\varphi_1 - \varphi_2}{2}\big) \tag{19}$$

using $\frac{\varphi_1+\varphi_2}{2} = \varphi_1 - \frac{\varphi_1-\varphi_2}{2}$ and $\frac{\varphi_1-\varphi_2}{2} = -\pi\frac{\Phi_t}{\Phi_0}$

$$I_s = 2I_c \sin(\varphi_1 + \pi\frac{\Phi_t}{\Phi_0})\cos(\pi\frac{\Phi_t}{\Phi_0}) \tag{20}$$

then the dc-SQUID critical current is

$$I_{c,SQUID} = I_s^m = 2I_c|\cos(\pi\frac{\Phi_t}{\Phi_0})| \tag{21}$$

3.2 Voltage state

The CVCs of a ds-SQUID for ($\beta_c = 1, \beta_L = 0.01$) is shown in fig **??**, and for ($\beta_c = 1, \beta_L = 1$) is shown in fig **??**. The external magnetic flux changes from 0 to $\frac{\Phi_0}{2}$ by a $0.1\Phi_0$ step.

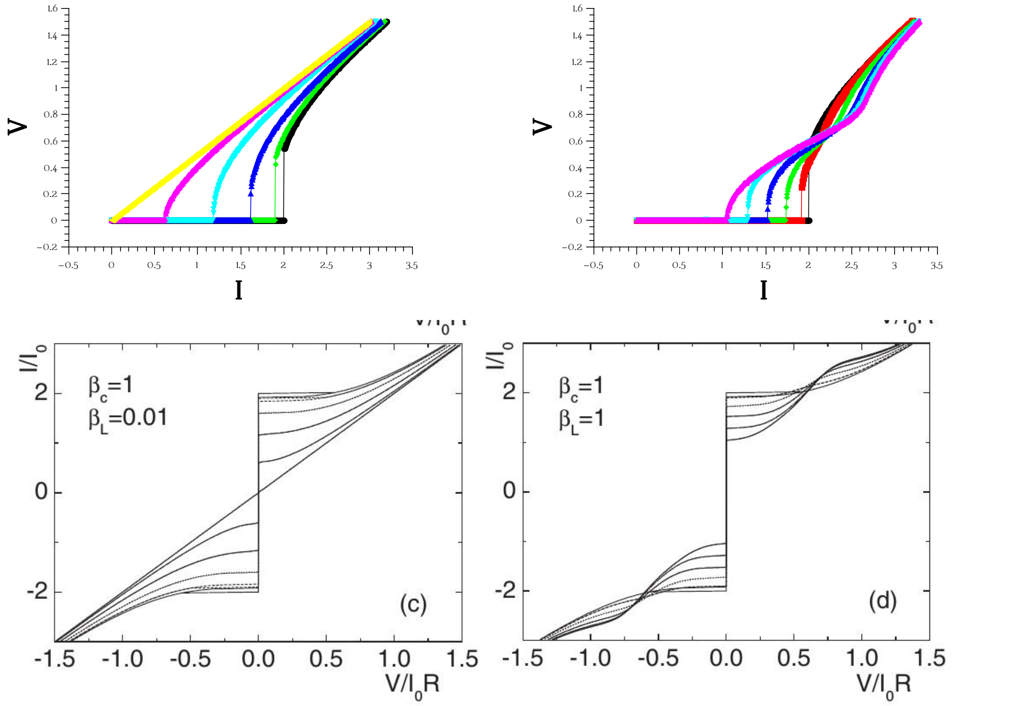

Figure 4: **a** $\beta_L = 0.01, \beta_c = 1$, **b** $\beta_L = 1, \beta_c = 1$, **c** (Screen-shot)Fig. 2.8(c-d)-SQUID Handbook [7]

The dc-SQUID can be considered as a **flux-to-voltage transducer**, which produces an output voltage in response to small variations of the input flux. Figure 5 shows the modulation $V(\Phi_e)$ for several values of the bias current for $\beta_L = 1$and $\beta_c = 1$. We see that the modulation in I_c directly transfers into a modulation of V.

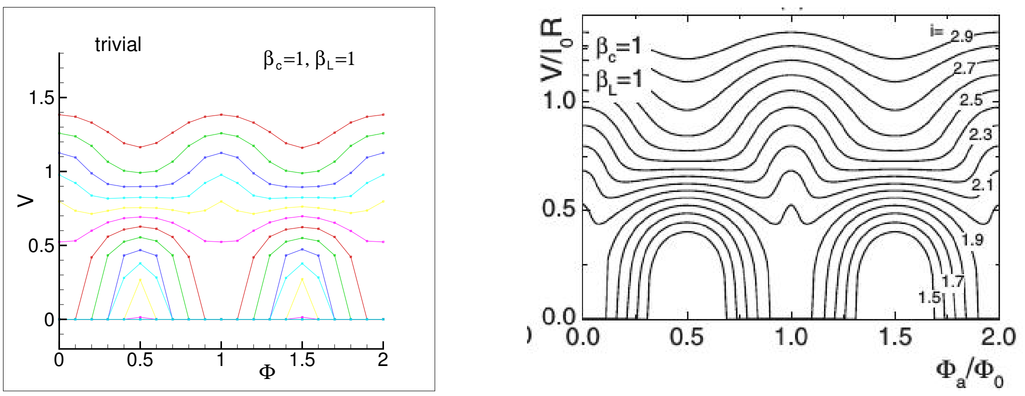

Figure 5: $V(\Phi_e)$ characteristics for trivial dc-SQUID

4 Quantum computing and Majorana fermions

Decreasing the components size of the microprocessors brings quantum mechanical effects to domination. which give rise unpredictable and unwanted operation in classical microprocessor designs.

> The component size of individual transistors on modern microprocessors is becoming so small that quantum effects will soon begin to dominate. Unfortunately, quantum mechanical behaviour will tend to result in unpredictable and unwanted operation in classical microprocessor designs. We therefore have two choices: keep trying to suppress quantum effects in classically fabricated electronics, or move to the field of quantum information processing (QIP) where we exploit them. This leads to a paradigm shift in the way we view and process information and has commanded considerable interest from physicists, engineers, computer scientists and mathematicians. The counter-intuitive and strange rules of quantum physics offer enormous possibilities for information processing and the development of a large-scale quantum computer is the holy grail for many researchers worldwide. While the advent of Shor?s algorithm [Sho97] certainly spawned great interest in QIP, demonstrating that quantum algorithms could be far more efficient than those used in classical computing, there was a great deal of debate surrounding the practicality of building a large scale, controllable, quantum system. It was well known even before the introduction of quantum information that coherent quantum states were extremely fragile and many believed that to maintain large, multi-qubit, coherent quantum states for a long enough time to complete any quantum algorithm was unrealistic [8]
>
> On the most basic level, a set of Majorana-carrying vortices or domain walls non-locally encodes quantum information in the degenerate ground-state space, enabling immediate applications for long-lived ?topological quantum memory?. In the longer term the prospect of manipulating that information in a manner that avoids decoherence would constitute an important breakthrough for quantum computation. This is made possible by the most coveted manifestation of Majorana fermions: non-Abelian statistics.
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Together these properties give rise to non-Abelian statistics of vortices: if one performs sequential exchanges, the final state depends on the order in which they are carried out. [9]

5 Topologicaly nontrivial DC-SQUID

Topological nontrivial SQUID, is a SQUID with topologically nontrivial element (junctions or the superconducting loop). In such devices, Majorana Fermios (MF) are predicted to exist. this existence of MF changes the supercurrent to be $I_s = I_c \sin(\phi/2)$.
Also the flux quantization eqn. 5 will change to:

$$\frac{\varphi_1}{\gamma_{j1}} - \frac{\varphi_2}{\gamma_{j2}} + \frac{2\pi}{\gamma_l}\frac{\Phi_t}{\Phi_0} = 2\pi n \tag{22}$$

where γ_{j1},γ_{j2} and γ_l are related to the charge carrier $q = \frac{2e}{\gamma}$ in the junctions and the loop respectively.
The total flux of the system is given by:

$$\Phi_t = \Phi_e + I_c L(\chi_1 - \chi_2) \tag{23}$$

where

$$\chi_i = \alpha_i \sin(\varphi_i) + (1 - \alpha_i)\sin(\alpha_i/2) \tag{24}$$

The current using the RCSJ model, in normalized form:

$$i_1 = \beta_c \frac{d^2\varphi_1}{dt^2} + \frac{d\varphi_1}{dt} + \chi_1 \tag{25}$$

$$i_2 = \beta_c \frac{d^2\varphi_2}{dt^2} + \frac{d\varphi_2}{dt} + \chi_2 \tag{26}$$

the total current pass through the dc-SQUID:

$$i = 2\beta_c \frac{d^2\varphi_2}{dt^2} + 2\frac{d\varphi_2}{dt} + \chi_1 + \chi_2 \tag{27}$$

the flux quantization eqn. 22 can be written as:

$$\frac{\varphi_1}{\gamma_{j1}} - \frac{\varphi_2}{\gamma_{j2}} + \frac{1}{\gamma_l}(2\pi\Phi_e + \beta_L(\chi_1 - \chi_2)) = 2\pi n \tag{28}$$

we solve the following system of equations

$$
\begin{aligned}
&\text{If } \beta_L = 0\\
&\dot{V_1} = \frac{1}{\beta_c}(\frac{1}{2}(I - \alpha_1 \sin(\varphi_1) - (1-\alpha_1)\sin(\frac{\varphi_1}{2}) - \alpha_2 \sin(\varphi_2) - (1-\alpha_2)\sin(\frac{\varphi_2}{2})) - V_1)\\
&\qquad \dot{\varphi_1} = V_1\\
&\qquad \varphi_2 = \gamma_{j2}[\frac{\varphi_1}{\gamma_{j1}} + 2\pi(\frac{\Phi_e}{\gamma_l} - n)]\\
&\text{If } \beta_L \neq 0\\
&\dot{V_1} = \frac{1}{\beta_c}(\frac{1}{2}I - \alpha_1 \sin(\varphi_1) - (1-\alpha_1)\sin(\frac{\varphi_1}{2}) + \frac{\gamma_l}{\beta_L}(2\pi n - \frac{\varphi_1 - \varphi_2}{\gamma_j} - 2\pi\frac{\Phi_e}{\gamma_l}) - V_1)\\
&\qquad \dot{\varphi_1} = V_1\\
&\dot{V_2} = \frac{1}{\beta_c}(\frac{1}{2}I - \alpha_2 \sin(\varphi_2) - (1-\alpha_2)\sin(\frac{\varphi_2}{2}) + \frac{\gamma_l}{\beta_L}(2\pi n - \frac{\varphi_1 - \varphi_2}{\gamma_j} - 2\pi\frac{\Phi_e}{\gamma_l}) - V_2)\\
&\qquad \dot{\varphi_2} = V_2
\end{aligned}
\tag{29}
$$

5.1 S-state

Critical current If $\beta_L = 0$, the flux quantization can be written as ($n = 0$): $\alpha_1 = \alpha_2$

$$\frac{\varphi_1}{\gamma_{j1}} - \frac{\varphi_2}{\gamma_{j2}} = \frac{2\pi}{\gamma_l}\frac{\Phi_e}{\Phi_0} \tag{30}$$

The super-current is

$$
\begin{aligned}
I_s = {} & I_c(\alpha_1\ sin(\varphi_1) + (1-\alpha_1)\sin(\varphi_1/2))\\
& + I_c(\alpha_2\ sin(\varphi_2) + (1-\alpha_2)\sin(\varphi_2/2))
\end{aligned}
\tag{31}
$$

Assuming that the loop is trivial and the junctions are topologically nontrivial ($\gamma_l = 1$, $\alpha_1 = \alpha_2 = 0$ and $\gamma_{j1} = \gamma_{j2} = 2$) ,the super-current is:

$$I_s = I_c(\sin(\varphi_1/2)) + \sin(\varphi_2/2)) \tag{32}$$

$$I_s = 2I_c \sin(\frac{\varphi_1 + \varphi_2}{4})\cos(\frac{\varphi_1 - \varphi_2}{4}) \tag{33}$$

using $\frac{\varphi_1+\varphi_2}{4} = \frac{\varphi_1}{2} - \frac{\varphi_1-\varphi_2}{4}$ and $\frac{\varphi_1-\varphi_2}{2} = 2\pi\frac{\Phi_t}{\Phi_0}$

$$I_s = 2I_c \sin(\frac{\varphi_1}{2} - \pi\frac{\Phi_t}{\Phi_0})\cos(\pi\frac{\Phi_t}{\Phi_0}) \tag{34}$$

then the dc-SQUID critical current is

$$I_{c,NT-SQUID} = I_s^m = 2I_c|\cos(\pi\frac{\Phi_t}{\Phi_0})| \tag{35}$$

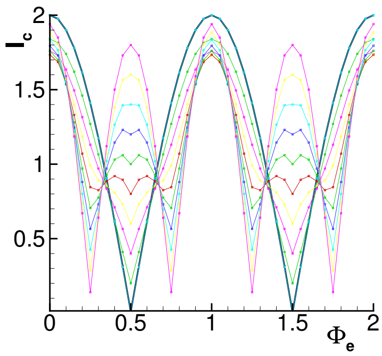


Figure 6: Critical current dependence on the external magnetic flux, $\beta_L = 0$, by changing the portion of Majorana fermions to cooper-pairs α from 0to1 by increment of $\delta\alpha = 0.1$

Otherwise, the maximum of I_s can be found by differentiating eqn. 31 with respect to φ_1, then find the zeros numerically..

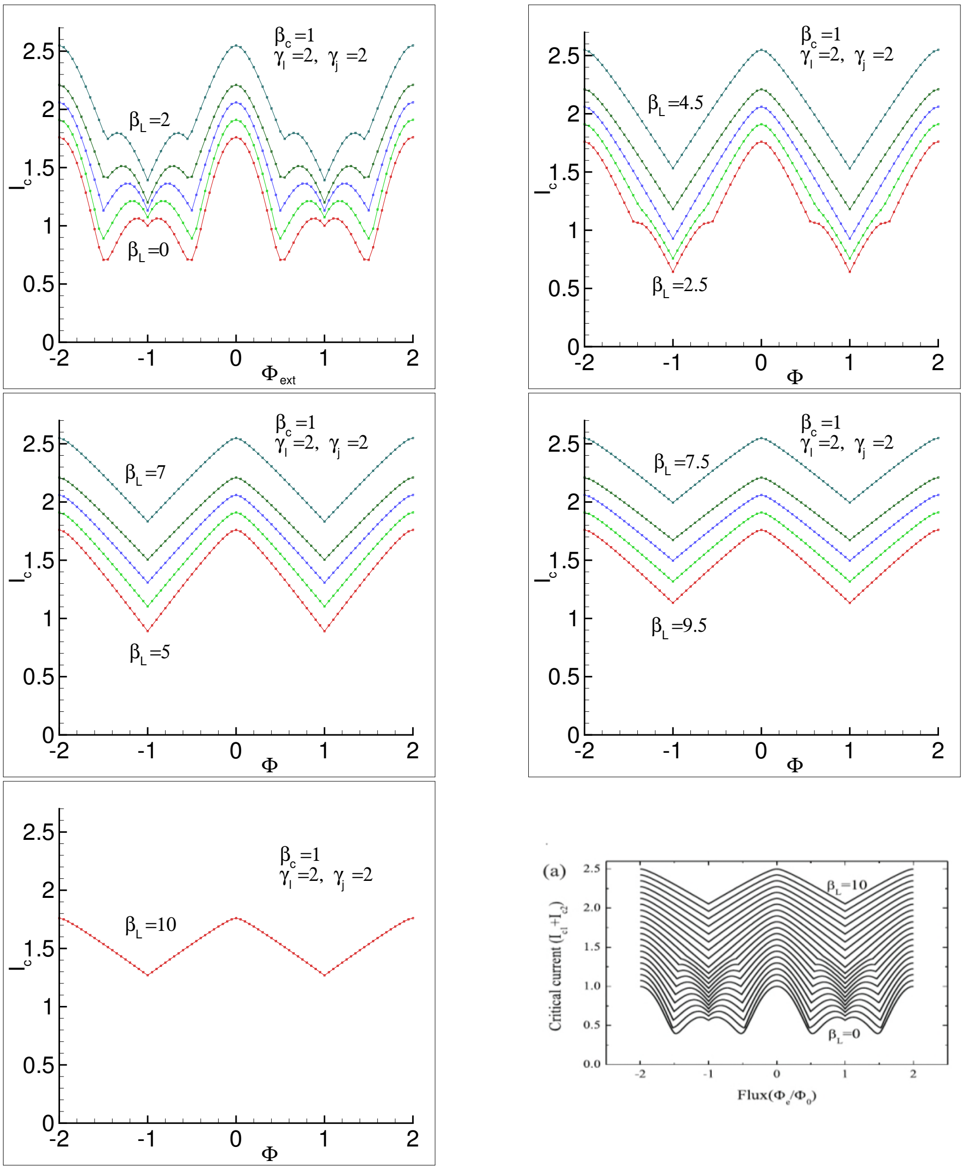
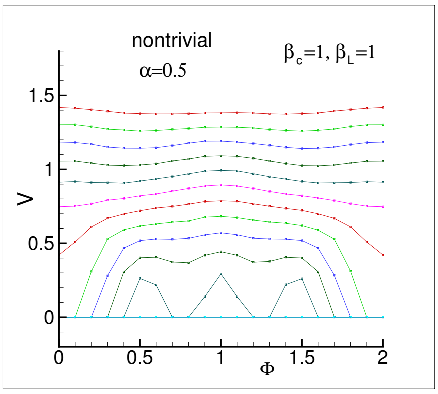


Figure 7: $I\,\Phi$ characteristics of a dc SQUID composed of a topologically nontrivial superconducting ring and nontrivial junctions. (a) dc SQUID oscillations for two symmetric junctions with equal amplitude $\sin(\phi)$ and $\sin(\phi/2)$ components, $\alpha = 0.5$. Increasing β_L (in steps $\delta\beta_L = 0.5$, and shifted for clarity)

5.2 Voltage state

the time-averaged voltage versus the applied flux for different values of the bias current.



(a) $V(\Phi_e)$ characteristics for nontrivial dc-SQUID

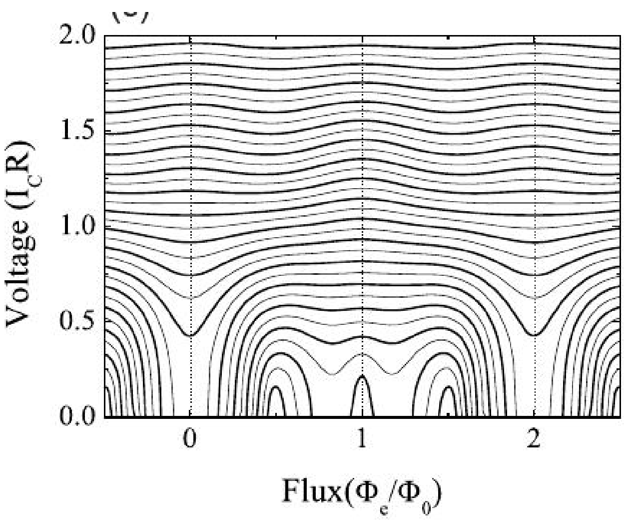
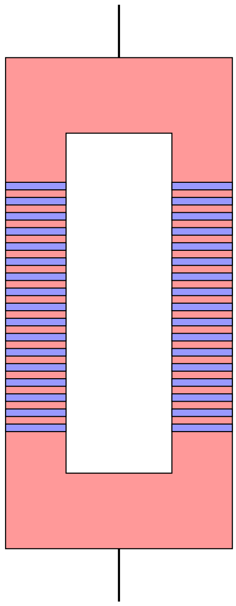


Figure 8: $V(\Phi_e)$ characteristics for nontrivial dc-SQUID

Figure 8 shows the voltage flux dependency for a nontrivial dc-SQUID , $\alpha = 0.5$, our results is compared to [5]

6 dc-SQUID with IJJ

The flux quantization is



$$\sum_{1}^{N_1} \frac{\psi_m}{\nu_j} - \sum_{1}^{N_2} \frac{\varphi_k}{\nu_j} + 2\pi(\frac{\Phi_e}{\gamma_l} - n) = \frac{\beta_L}{2}(I_2 - I_1) \tag{36}$$

$$I_1 = \beta_c \frac{\partial U_m}{\partial t} + \frac{\partial \psi_m}{\partial t} + \chi_m \tag{37}$$

$$I_2 = \beta_c \frac{\partial V_k}{\partial t} + \frac{\partial \varphi_k}{\partial t} + \rho_k \tag{38}$$

$$I = I_1 + I_2 \tag{39}$$

$$\frac{\partial \psi_m}{\partial t} = U_m + \alpha(U_{m+1} + U_{m-1} - 2U_m) \tag{40}$$

$$\frac{\partial \varphi_k}{\partial t} = V_k + \alpha(V_{k+1} + V_{k-1} - 2V_k) \tag{41}$$

$$\frac{\partial\psi_m}{\partial t} = U_m + \alpha(U_{m+1} + U_{m-1} - 2U_m)$$

$$\frac{\partial\varphi_k}{\partial t} = V_k + \alpha(V_{k+1} + V_{k-1} - 2V_k)$$

$$\frac{\partial U_m}{\partial t} = \frac{1}{\beta_c}\big[\frac{I}{2} - \frac{1}{\beta_L}\{\frac{1}{\nu_j}(\sum_1^{N_1}\psi_m - \sum_1^{N_2}\varphi_k) + 2\pi(\frac{\Phi_e}{\gamma_l} - n)\} - \frac{\partial\psi_m}{\partial t} - \chi_m\big]$$

$$\frac{\partial V_k}{\partial t} = \frac{1}{\beta_c}\big[\frac{I}{2} + \frac{1}{\beta_L}\{\frac{1}{\nu_j}(\sum_1^{N_1}\psi_m - \sum_1^{N_2}\varphi_k) + 2\pi(\frac{\Phi_e}{\gamma_l} - n)\} - \frac{\partial\varphi_m}{\partial t} - \rho_m\big]$$

6.1 Preliminary results

$c = p, T_i = 50, T_f = 1000, J_{P0} = 0.005, T_P = 0.05, I_0 = 0.1, I_{more1} = 0.2, I_{more2} = 3, noismax = 10^{-8}$,
$\beta_c = 25, \beta_L = 1, \alpha = 0.1, \gamma = 1$
$\nu_j = 1, \nu_l = 1, \epsilon = 1, n = 0, \Phi_e = 0$,

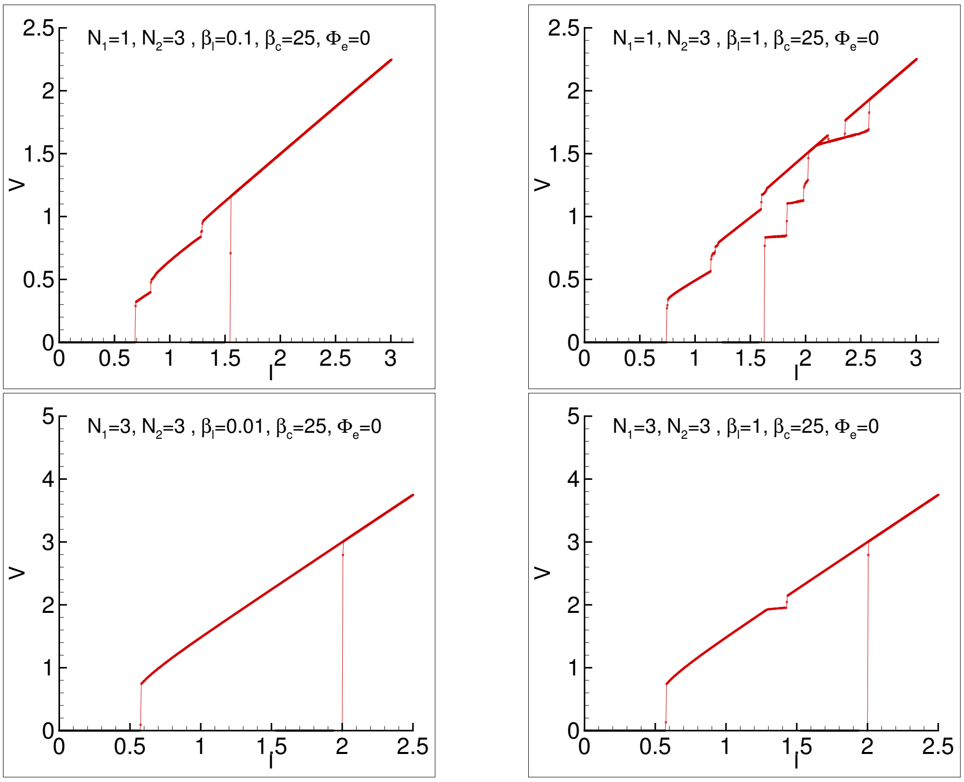


Figure 9: CVC of dc-SQUIDs with IJJ

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