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Disclination vortices in elastic media

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Abstract. The vortex-like solutions are studied within the framework of the gauge model of disclinations in an elastic continuum. A complete set of model equations with disclination-driven dislocations taken into account is considered. Within the linear approximation an exact solution for a low-angle wedge disclination is found to be independent of the coupling constants of the theory. As a result, no additional dimensional characteristics (such as the core radius of the defect) are involved. The situation changes drastically for 2π vortices, where two characteristic lengths, l_ϕ and l_W, become of importance. The asymptotic behaviour of the solutions for both singular and non-singular 2π vortices is studied. Forces between pairs of vortices are calculated.

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1. Introduction

Topologically non-trivial objects arising in various physically interesting systems are the subject of considerable current interest. It will suffice to mention the ’t Hooft–Polyakov monopole in the non-Abelian Higgs model, instantons in quantum chromodynamics, solitons in the Skyrme model, Nielsen–Olesen magnetic vortices in the Abelian Higgs model, etc (see, e.g., the book [1]). Note that similar objects are known in condensed matter physics as well. For instance, vortices in liquids and liquid crystals, solitons in low-dimensional systems (e.g. in magnetics, linear polymers and organic molecules) as well as the famous Abrikosov magnetic vortices in superconductors are the matter of common knowledge. Mathematically, all these objects appear within the framework of nonlinear models as partial solutions of strongly nonlinear equations. As is well known, there are still no general methods to study such equations. This makes the derivation of the solutions difficult. An important point is also that all the solutions are topologically stable and belong to non-trivial homotopic sectors.

It should be noted that elastic media also leave room for topological defects known as dislocations and disclinations. However, these defects are usually treated within the linear theory of elasticity. In this case, all the information about defects is incorporated via source terms in the equilibrium condition. The sources are assumed to have a δ-function form multiplied by either the Burgers or the Frank vector, respectively (see, e.g., [2]). There were some attempts to invoke the nonlinear theory of elasticity for a description of dislocations (see details in [3]) which account for the nonlinear relation between stresses and strains, thus giving a possibility to determine stress and strain fields near a dislocation at large deformations. Recently [4], nonlinear problems in dislocation theory were studied within the framework of

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the gauge model of dislocations and disclinations proposed in [5, 6]. Unfortunately, all of these approaches [3, 4] are based on the perturbation scheme. It is known, however, that topologically non-trivial solutions cannot be found within the framework of any perturbation scheme.

As has been shown in [7–10], the gauge model [5] admits exact vortex-like solutions for wedge disclinations. This finding confirms the view of a disclination as a vortex of the elastic medium. Note that though the correspondence between dislocations and vortices has been known for many years (see, e.g., the review [11]) the explicit vortex-like solutions for disclinations were only obtained within the gauge-theory approach. It is interesting to note that the elastic flux due to rotational defects was found to be completely determined by the gauge vector fields associated with disclinations. In the continuum there are no restrictions on the value of the flux which is in fact the Frank vector, $\mathbf{\Omega}$. Here we will consider two cases: low-angle vortices with $|\mathbf{\Omega}| \ll 1$ and 2π vortices with $|\mathbf{\Omega}| = 2\pi$.

Before proceeding, let us mention that disclinations are of importance in various crystalline and non-crystalline materials (see, e.g., [12]). Among the current applications one can name the new class of carbon materials: fullerenes and nanotubes (see, e.g., [13]). According to Euler's theorem, these microcrystals can only be formed by having a total disclination of 4π. According to the geometry of the hexagonal network this means the presence of 12 60° disclinations on the closed hexatic elastic surface. Note that the disclinations in liquid crystals are one of the best-studied cases. In particular, the known exact 'hedgehog' solution has been obtained within the continuum model of nematics [2].

It is interesting that a hedgehog-like solution was also found for a point 4π disclination within the framework of the gauge model [14, 15]. An important advantage of the gauge model follows from the fact that it is similar to the known field theory models, first of all to the non-Abelian and Abelian Higgs models, where topological objects are studied well. Taking into account this similarity we have found two exact static solutions for linear disclinations [7, 9, 10] which will be discussed in this paper. In particular, within the linear scheme our model recovers the equilibrium conditions of the standard elasticity theory with a disclination-induced source being generated by gauge fields. It should be noted, however, that these solutions were obtained within the restricted model where the dislocation-induced contribution to the Lagrangian was neglected. As a matter of fact, this contribution always exists (so-called disclination-driven dislocations [5]). The goal of the present paper is to consider the most general model for linear disclinations in an elastic continuum involving all the main contributions.

This paper is structured as follows. In section 2 the gauge model of disclinations is presented. The Lagrangian describes elastic deformations and the self-energy of disclinations as well as a contribution from the rotational dislocations. A complete set of field equations is formulated for the static case. In section 3 the gauge model is studied within the linear approximation. An exact vortex-like solution for a low-angle straight wedge disclination is found to be independent of the coupling constants of the theory. Forces between vortices as well as dipole configurations are studied. Topologically stable 2π vortices are considered in section 4 as the solutions of strongly nonlinear equations of the theory. First, we consider an exact singular solution and show that the dislocation-induced contribution becomes important. Second, we analyse briefly a problem of the non-singular vortex. In particular, appropriate asymptotic solutions are found. Section 5 is devoted to concluding comments.

2. Gauge model

In accordance with the basic assumption of the gauge approach [5], topological defects can be introduced into the Lagrangian of the elasticity theory through gauge fields. Namely, the defect-free Lagrangian of elasticity theory is invariant under the homogeneous action of the space group $G = SO(3) \triangleright T(3)$. Let us consider this group as the gauge one and assume that the inhomogeneous action of G leaves the initial Lagrangian invariant. In this case, we arrive at a Yang–Mills-type theory which contains two different gauge fields associated with the rotational $SO(3)$ and translational $T(3)$ group, respectively. The model is based on the Yang–Mills minimal coupling arguments. While the initial Lagrangian is chosen to be quadratic in distortion fields, the presence of gauge fields makes it strongly nonlinear†. That is why the nonlinear relation between stresses and strains occurs from the very beginning.

For disclinations and rotational dislocations, only the $SO(3)$ group should be taken into account. The problem becomes simpler for rectilinear defects, which are of interest here. Indeed, in this case the rotational symmetry becomes broken only in the plane normal to the defect line, and, correspondingly, the gauge group reduces to $SO(2)$. Notice that such a gauge group is typical for models containing vortex-like objects.

The Lagrangian that is invariant under the inhomogeneous action of the $SO(2)$ gauge group takes the following form [9]:

$$\mathcal{L} = \mathcal{L}_\chi + \mathcal{L}_\phi + \mathcal{L}_W \tag{1}$$

where

$$\mathcal{L}_\chi = \tfrac{1}{2}\rho_0 B_3^i B_3^i - \tfrac{1}{8}[\lambda(\mathrm{tr}\, E)^2 + 2\mu\, \mathrm{tr}\, E^2] \tag{2}$$

describes the elastic properties of the material, while

$$\mathcal{L}_\phi = -\tfrac{1}{2}s_1 D^i_{ab} k^{ac} k^{bd} D^i_{cd} \tag{3}$$

and

$$\mathcal{L}_W = -\tfrac{1}{2}s_2 F_{ab} g^{ac} g^{bd} F_{cd} \tag{4}$$

describe defect-induced contributions. Here $E_{AB} = B^i_A B^i_B - \delta_{AB}$ is the strain tensor, $D^i_{ab} = \epsilon^i_j F_{ab}\chi^j$, $F_{ab} = \partial_a W_b - \partial_b W_a$, s_1 and s_2 are the coupling constants, tr $E = E_{AA}$ and summation over repeated indices is assumed. In accordance with the minimal replacement arguments, the distortion tensor is written as

$$B^i_a = \partial_a \chi^i + \epsilon^i_j \chi^j W_a \tag{5}$$

where $\chi^i(x^a) = \chi^i(x^A, T)$ characterizes the configuration at time T in terms of the coordinate cover (x^A) of a reference configuration, W_a is the compensating gauge field associated with the disclination field. In (3) the tensor k^{ab} is given by $k^{AB} = -\delta^{AB}$, $k^{33} = 1/y$ and $k^{ab} = 0$ for $a \neq b$, whereas in (4) $g^{AB} = -\delta^{AB}$, $g^{33} = 1/\xi$ and $g^{ab} = 0$ for $a \neq b$. The parameters y and ξ are the two positive 'propagation parameters', ϵ^i_j is a completely antisymmetric tensor, $\epsilon^1_2 = 1$, and λ and μ are the Lamé constants. (Indices $a, b, c, \ldots = 1, 2, 3$ and $A, B, C, \ldots = 1, 2$ are the space labels, whereas $i, j, k, \ldots = 1, 2$ belong to $SO(2)$.)

Notice that $\mathcal{L}_W$ describes a self-energy of pure rotational defects (disclinations). In accordance with the general approach it acquires a standard form of the $SO(2)$ Yang–Mills action. $\mathcal{L}_\phi$ is an additional invariant term associated with translational defects (rotational dislocations). This is reflected in the fact that the tensor D^i_{ab} in (3) is directly related to the density of dislocations defined as $\alpha^{Ai} = \epsilon^{ABC} D^i_{BC}$ (see [5]). Thus the appearance of this term in the Lagrangian reflects the known fact in the defect theory that the presence of disclinations implies the presence of dislocations.

† The cubic terms can be included into the Lagrangian as well (see [6]), but in so doing the model becomes too cumbersome for analysis.

The Euler–Lagrange equations for (1) take the following form in the static case:

$$\partial_A \sigma_k^A = \epsilon_k^j W_C \sigma_j^C + s_1 \chi^l \delta_{lk} F_{AB} F^{AB} \tag{6}$$

$$\sigma_A^j \delta_{ij} \epsilon_l^i \chi^l = 2\partial_B \left[(s_1 (\chi)^2 + s_2) F^{BA} \right] \tag{7}$$

where $(\chi)^2 = \chi^l \chi_l$, $l = 1, 2$. To avoid cumbersome expressions we will sometimes omit the right order of the top and bottom indices which can be easily restored by using the appropriate δ-symbols. The stress tensor σ_C^j is determined to be

$$\sigma_C^j = \tfrac{1}{2}[\lambda(\mathrm{tr}\, E) B_C^j + 2\mu(E_{CB} B_B^j)]. \tag{8}$$

It is convenient to introduce the dimensionless variables via $x^i = \sqrt{s_2/s_1}\, \tilde{x}^i$ and $W_A = \sqrt{s_1/s_2}\, \tilde{W}_A$. The Euler–Lagrange equations (6) and (7) become

$$\tilde{\partial}_A \tilde{\sigma}_k^A = \epsilon_k^j \tilde{W}_C \tilde{\sigma}_j^C + \frac{s_1^2}{\mu s_2} \tilde{\chi}^l \delta_{lk} \tilde{F}_{AB} \tilde{F}_{AB} \tag{9}$$

$$\tilde{\sigma}_A^j \delta_{ij} \epsilon_l^i \tilde{\chi}^l = \frac{2 s_1^2}{\mu s_2} \tilde{\partial}_B [((\tilde{\chi})^2 + 1) \tilde{F}^{BA}] \tag{10}$$

where $\chi^l(x^B) = \sqrt{s_2/s_1}\, \tilde{\chi}^l(\tilde{x}^B)$, $\partial_B = \sqrt{s_1/s_2}\, \tilde{\partial}_B$, $\tilde{F}_{AB} = \tilde{\partial}_A \tilde{W}_B - \tilde{\partial}_B \tilde{W}_A$ and the stress tensor is found to be

$$\tilde{\sigma}_j^A = \tfrac{1}{2} L(\mathrm{tr}\, \tilde{E}) \tilde{B}_A^j + \tilde{E}_{AC} \tilde{B}_C^j. \tag{11}$$

Here $L = \lambda/\mu$, and the strain tensor takes the form

$$\tilde{E}_{AB} = \tilde{B}_A^i \tilde{B}_B^i - \delta_{AB} \tag{12}$$

with $\tilde{B}_A^i = \tilde{\partial}_A \tilde{\chi}^i + \epsilon_j^i \tilde{\chi}^j \tilde{W}_A$. To simplify the notation, below we will omit the 'tilde' symbol. As is seen, the self-consistent system of field equations (9) and (10) is strongly nonlinear. In some respects (see the discussion in [7, 9]) it is similar to that in the Abelian Higgs model. An analogous system of equations appears in the study of type-II superconductors in a magnetic field directed along the z-axis. This observation was helpful in finding the exact vortex-like solution to (9) and (10) for $s_1 = 0$ [7]. A possible way to study these equations is the linearization procedure which is valid for low-angle defects. It was found [10] that there is an exact vortex-like solution at $s_1 = 0$ which reproduces the known strain and stress fields of straight wedge disclinations. Note that both solutions are found to be singular at the disclination line. While this singularity is well known in dislocation theory, the question arises whether there exist non-singular solutions. Besides, it turns out that for $s_1 = 0$ the coupling constant s_2 drops out of the problem as well. Let us re-examine the vortex-like solutions within the most general model.

3. Low-angle disclination vortices

Let us consider first the linearized equations of the model. The linearization procedure is based on a homogeneous scaling of the gauge group generators (for details see [5]). It is clear that in employing the linear approximation we restrict our consideration to small deformations or, correspondingly, to the low-angle defects. The displacement vector u^i can be introduced as follows:

$$\chi^i(x^B) = \delta_A^i x^A + u^i(x^B). \tag{13}$$

Then, with the scaling parameter ϵ all the fields are expanded in series of ϵ

$$u^i = \epsilon u_1^i + \epsilon^2 u_2^i + \cdots \qquad W_A = \epsilon W_{1A} + \epsilon^2 W_{2A} + \cdots. \tag{14}$$

Taking into account only defect-induced displacements u_1^i that are of interest here, we obtain the first-order equations in the following form:

$$\partial_B[(x^2+y^2+1)F^{BA}] = 0 \tag{15}$$

$$\Delta \boldsymbol{u} + (L+1)\nabla\,\mathrm{div}\,\boldsymbol{u} = \boldsymbol{j} + \boldsymbol{x}\frac{s_1^2}{\mu s_2}F_{AB}F^{AB} \tag{16}$$

where the density of the elastic flow due to a disclination, $\boldsymbol{j}$, is completely determined by gauge fields

$$j_C = -(L-1)\epsilon_{AC}W_A - L\epsilon_{AB}x_B\partial_C W_A - \epsilon_{AB}x_B\partial_A W_C - \epsilon_{CB}x_B\partial_A W_A. \tag{17}$$

Hereafter we omit the index 1 denoting the order of the approximation. Let us emphasize that here we assume $s_1^2/\mu s_2 \sim 1/\epsilon$. For the other two possibilities, $s_1^2/\mu s_2 \sim \epsilon$ and $s_1^2/\mu s_2 \sim 1$, terms with s_1 in (15) and (16) are of little importance, thus the standard theory [10] is recovered. Note that for $s_1 = 0$ a solution of these equations was found in [10]. An interesting property of this solution is its independence of the parameter s_2 as well. This can be seen directly from (15) and (16) where s_2 is completely absent. Let us choose the following vortex-like ansatz:

$$W_A = -W(r)\epsilon_{AB}\partial_B \log r \qquad u^i = x^i G(r) \tag{18}$$

where $r^2 = x^2 + y^2$. With (18) taken into account one can rewrite (15) and (16) as follows:

$$\partial_r\left[(r^2+1)\frac{W'(r)}{r}\right] = 0 \tag{19}$$

$$(L+2)\left(G''(r) + \frac{3G'(r)}{r}\right) = L\frac{W'(r)}{r} - \frac{2W(r)}{r^2} + \frac{2s_1^2}{\mu s_2}\left(\frac{W'(r)}{r}\right)^2 \tag{20}$$

where G' stands for $\mathrm{d}G/\mathrm{d}r$ and W' for $\mathrm{d}W/\mathrm{d}r$. A solution to (19) takes the form

$$W(r) = C_1 \ln(r^2+1) + C_0 \tag{21}$$

with C_0 and C_1 being the integration constants. Note that a disclination flow through the plane xy is given by a circular integral

$$\frac{1}{2\pi}\oint \boldsymbol{W}\,\mathrm{d}\boldsymbol{r} = \nu. \tag{22}$$

Taking this into account, we immediately obtain that $C_1 = 0$. Thus the constant C_0 turns out to be, in fact, a topological characteristic of the defect, that is the Frank index ν. For $W(r) = \nu$ (20) becomes remarkably simpler and has a solution

$$G(r) = -\frac{\nu}{L+2}\ln r - \tfrac{1}{2}C_2 r^{-2} + C_3. \tag{23}$$

Since the boundary condition requires $u^i(0) = 0$ we must put $C_2 = 0$. Returning to the dimensional variables, we finally obtain

$$u^i = -x^i\left(\frac{\nu}{L+2}\ln\sqrt{\frac{s_1}{s_2}}r + C_3\right) \tag{24}$$

where C_3 is still an arbitrary constant. As is seen, the term with s_1 and s_2 only renormalizes the constant C_3. As an example, for the straight wedge disclination on a disc of radius R with a boundary condition in the form $u^i(R) = 0$ one obtains

$$u^i = -x^i\frac{\nu}{L+2}\ln\frac{r}{R}. \tag{25}$$

Similarly, for the most-used boundary condition, $\sigma_{kl}n_l = 0$ at the free surfaces, one can reproduce the well known stress fields for a wedge disclination on a disc (see details in [10]). It is seen that parameters s_1 and s_2 actually drop out from (24) and (25). Thus one can conclude that the information carried by the coupling constants s_1 and s_2 is lost within the linear approximation. What does this mean? As is known, the classical theory of elasticity introduces a characteristic velocity $\sqrt{\mu/\rho_0}$, but does not lead to a characteristic length. For this reason there is no room within the linear theory of elasticity for a description of the core region. It is interesting that the gauge theory of defects [5, 6] introduces appropriate length scales. These are the dislocation length scale, $l_\phi^2 = s_1/\mu$, and the disclination length scale, $l_W^4 = s_2/\mu$. Nevertheless, as we have just seen, in the linear approximation the gauge theory loses these parameters thus making the description of the core region impossible. One can expect†, however, that these parameters would be of importance in a study of the basic model equations (6) and (7). We will consider this problem in section 4.

3.1. Forces between vortices, dipole configurations

Let us consider two low-angle vortices with parallel Frank vectors oriented along the z-axis. This corresponds to a pair of straight wedge disclinations. For simplicity, we suppose that disclination lines coincide with their axes of rotations. The stress field due to the disclination results in the force acting on the second defect (in perfect analogy to the known Peach–Koehler force in dislocation theory). Generally, it can be written as [16] (per unit length of the disclination line)

$$\mathcal{F}_c = \epsilon_{bkc}\epsilon_{amn}\Omega_m X_n \sigma_{ab}\xi_k \tag{26}$$

where $\vec{\xi}$ is a tangent vector at the disclination line, Ω_m are the components of the Frank vector, $X_n = x_n - x_n^0$, and x_n^0 is a point on the axis of the disclination. In our case, $x_n^0 = 0$ and one has to put in (26) $k = 3$ and $m = 3$. Note that the same expression follows from the general equations of the gauge model [5, 6] in the linear approximation. As was shown above, the stress fields take the well known form in the linear theory. Evidently, the force between two parallel low-angle wedge disclinations also has the known form. For example, when the first vortex is situated at the point $(0, 0)$ while the second one is at point $(d, 0)$ on the xy-plane we obtain from (26)

$$\mathcal{F}_x = d\Omega\sigma_{yy} \qquad \mathcal{F}_y = -d\Omega\sigma_{yx} \tag{27}$$

in accordance with [16]. Note that for vortices with equal but oppositely directed Frank vectors such a configuration corresponds to a wedge disclination dipole with non-skew axes of rotation. It is interesting to reproduce the solution for the dipole within the gauge model. Since the previous analysis shows that the constants s_1 and s_2 are inessential in the linear approximation, we will drop terms with s_1 and put $W(r) = \nu$ from the beginning. A dipole solution to (15) then reads

$$W_B = -\nu\epsilon_{BC}\partial_C(\log r_1 - \log r_2). \tag{28}$$

The simplest way to solve (16) is via the Airy stress function. Namely, let us differentiate (16). After straightforward calculations one can rewrite this equation as

$$\left(\lambda/4\mu + \tfrac{1}{2}\right)\Delta\,\mathrm{tr}\,E = \epsilon_{AB}\partial_A W_B. \tag{29}$$

† We would like to thank Professor A M Kosevich for attracting our attention to this possibility.

The last term in the right-hand side of (29) describes a source due to disclination fields. For solution (28) it takes the form

$$\epsilon_{AB}\partial_A W_B = \nu\Delta(\log r_1 - \log r_2) = 2\pi\nu(\delta(\vec{r}_1) - \delta(\vec{r}_2)).$$

Introducing the Airy stress function $\chi(\vec{r})$ by $\sigma_{BA} = \epsilon_{BM}\epsilon_{AN}\partial_M\partial_N\chi(\vec{r})$, one can finally rewrite (29) as

$$K_0^{-1}\Delta^2\chi = 2\pi\nu(\delta(\vec{r}_1) - \delta(\vec{r}_2)). \tag{30}$$

Here $K_0 = 4\mu(\lambda+\mu)/(\lambda+2\mu)$, and tr $E = (1/(\lambda+\mu))\Delta\chi(\vec{r})$. Evidently, a solution to (30) is the sum $\chi = \chi_1 + \chi_2$ with $\chi_i = Ar_i^2 \ln r_i$ $(i = 1, 2)$. One can easily find that $A = \pm\nu K_0/4$. Finally, turning back to σ_{BA} one can exactly reproduce the known stress fields for disclination dipoles (cf, e.g., [12, 17]).

4. 2π disclination vortices

Let us choose the following ansatz for (6) and (7) to meet the necessary symmetry requirements

$$\chi^1(x^A) = F(r)\cos\theta \qquad \chi^2(x^A) = F(r)\sin\theta \tag{31}$$

and

$$W_x(x^A) = -\frac{y}{r^2}W(r) \qquad W_y(x^A) = \frac{x}{r^2}W(r) \tag{32}$$

where $r^2 = x^A x_A$ $(r, \theta$ are the polar coordinates). All the variables here are again dimensionless. We restrict ourselves by the topological sector with $n = 1$. As is seen, (31) and (32) describe a 2π vortex, that is the circular integral in (22) is equal to $\nu = 1$. With (31) and (32) taken into account (9) and (10) reduce to

$$\begin{aligned} 4\frac{F}{r^2}W'^2 = \frac{W-1}{r} f\big[K(g^2+f^2-2) + 2P(f^2-1)\big] + K\left[\frac{\mathrm{d}}{\mathrm{d}r}(f^2 g) + \frac{f^2 g}{r}\right] \\ +\big[3(K+2P)g'g^2 - 2(K+P)g'\big] + \frac{1}{r}\big[(K+2P)g^3 - 2(K+P)g\big] \end{aligned} \tag{33}$$

$$4\frac{\mathrm{d}}{\mathrm{d}r}\left[(1+F^2)\frac{W'}{r}\right] = Ff[K(f^2+g^2-2) + 2P(f^2-1)] \tag{34}$$

where $K = \lambda s_2/s_1^2$, $P = \mu s_2/s_1^2$, $g = \mathrm{d}F(r)/\mathrm{d}r$, $f = F(r)(1-W(r))/r$, $W' = \mathrm{d}W(r)/\mathrm{d}r$. This system of equations is our interest in this section. We will consider two possible cases.

4.1. Singular vortex

In the case of $s_1 = 0$ an exact solution to (33) and (34) for a static disclination vortex was found in [7, 8]. The solution is singular on the disclination line with $W(r) = 1$. It is interesting to note that the same solution is valid for the general case when $s_1 \neq 0$. Indeed, for $W(r) = 1$ one obtains $f = 0$. In such an event, both sides of (34) turn out to be zero, whereas (33) reduces to

$$\big(3g'g^2 - N_0^2 g'\big) + \frac{1}{r}\big(g^3 - N_0^2 g\big) = 0 \tag{35}$$

where $N_0^2 = 2(K+P)/(K+2P) = 2(\lambda+\mu)/(\lambda+2\mu)$. Carrying out an integration in (35) one obtains finally the algebraic equation

$$|g^3(r) - N_0^2 g(r)| = \frac{C_0}{r} \tag{36}$$

where C_0 is an arbitrary integration constant. The solution to (36) is written as $g(r) = (2/\sqrt{3})N_0\bar{g}(r)$ with

$$\bar{g}(r) = \begin{cases} \bar{g}_1(r) = \cosh[\frac{1}{3}\cosh^{-1}(r_0/r)] & r \leqslant r_0 \\ \bar{g}_2(r) = -\cos[\frac{1}{3}\cos^{-1}(r_0/r) + \frac{2}{3}\pi l] & r \geqslant r_0 \end{cases} \tag{37}$$

where $r_0 = 3\sqrt{3}C_0/2N_0^3$ is a characteristic parameter, and $l = 0, 1, 2$. Note that in the natural variables r_0 becomes dimensional. It was supposed in [7] that r_0 could be considered as a core radius of the defect. Indeed, according to (37) the point $r = r_0$ is prominent. It should be mentioned, however, that this attractive possibility for the description of the core radius requires an involvement of the additional phenomenological parameter, C_0, into the theory. At the same time, the model parameters s_1 and s_2 drop out of (35) and (37).

The main reason is that the chosen ansatz $W(r) = 1$ (the pure gauge for all r) is too restrictive. Obviously, any solutions with no constant $W(r)$ are of special interest. However, equations (33) and (34) look rather cumbersome, and a search for non-trivial exact solutions still remains an open problem. Let us try another way of looking at the problem. For this purpose, one can put $C_0 = 0$ in (37). The exact solution to (35) then takes the essentially simpler form

$$W(r) = 1 \qquad F_{1,2}(r) = \pm N_0 r \qquad F_3(r) = 0. \tag{38}$$

As a first step, let us consider small perturbations of the exact solution (38). It will be shown below that even this simplified consideration allows us to obtain important information about the role of the coupling constants s_1 and s_2 in the theory. Let us write

$$F(r) = N_0 r + \epsilon u_1(r) + \epsilon^2 u_2(r) + \cdots \tag{39}$$

$$W(r) = 1 - \epsilon w_1(r) + \cdots. \tag{40}$$

We consider here the case $F(r) = F_1(r)$. Substituting this expansion into (33) and (34) one obtains

$$\left(\frac{1}{r} + N_0^2 r\right) w_1''(r) + \left(N_0^2 - \frac{1}{r^2}\right) w_1'(r) - \frac{P}{2} N_0^4 r w_1(r) = 0 \tag{41}$$

$$u_2''(r) + \frac{1}{r}u_2'(r) = \frac{N_0}{(K+P)}\frac{(w_1')^2}{r} - N_0\frac{w_1^2}{2r} - \frac{KN_0}{K+2P}w_1 w_1'. \tag{42}$$

It should be noted that $u_1 = 0$. This follows from the requirement $F(r) \to 0$ at $r \to 0$. Let us analyse (41). As is known, the equation of the type $w'' + H(x)w' + Q(x)w = 0$ can be put into the form $z'' + I(x)z = 0$ by a substitution

$$w(x) = z(x)\exp\left(-\frac{1}{2}\int H(x)\,\mathrm{d}x\right)$$

where $I = -\frac{1}{2}H' - \frac{1}{4}H^2 + Q$. For (41)

$$H(r) = \frac{N_0^2 r^2 - 1}{r\left(N_0^2 r^2 + 1\right)} \qquad Q(r) = -\frac{P N_0^4 r^2}{2(N_0^2 r^2 + 1)}. \tag{43}$$

In this case, the general solution is not yet known. Instead, let us derive two limiting cases.

1. In the limit $N_0^2 r^2 \gg 1$ one obtains $H(r) = 1/r$, and $w_1(r) = z(r)/\sqrt{r}$. The equation for $z(r)$ takes the form

$$z''(r) = \left(\frac{PN_0^2}{2} - \frac{1+2P}{4r^2}\right)z(r). \tag{44}$$

This is a special form of the Whittaker equation with the solution

$$z(r) = C_l W_{0,m}(\beta r) \tag{45}$$

where C_l is a constant, $W_{0,m}$ is the Whittaker function, $m = \pm \mathrm{i}\sqrt{P/2}$, and $\beta = N_0\sqrt{2P}$. In this case, $W(r)$ is found to be

$$W(r) = 1 - C_l r^{-1/2} W_{0,m}(\beta r). \tag{46}$$

Note that C_l includes ϵ. Depending on β two asymptotics can be obtained.

(i) For $\beta r \gg 1$ one obtains

$$W(r) = 1 - C_l r^{-1/2} \exp(-\tfrac{1}{2}\beta r) \tag{47}$$

and

$$F(r) = N_0 r + C_l^2 \frac{K}{4(K+2P)} \frac{1}{N_0 r} \exp(-\beta r). \tag{48}$$

(ii) In the limit $\beta r \ll 1$ one has

$$W(r) = 1 - C_l \cos\left(\sqrt{\tfrac{1}{2}P} \ln(\beta r)\right) \tag{49}$$

and

$$F(r) = \tfrac{1}{2} N_0 (2 - C_l^2) r. \tag{50}$$

2. Let us consider the limit $N_0^2 r^2 \ll 1$. In this case, $H(r) = -1/r$ and $w_1(r) = \sqrt{r} z(r)$. The equation for $z(r)$ reads

$$z''(r) = \left(\frac{3}{4r^2} + \frac{P}{2} N_0^4 r^2\right) z(r) \tag{51}$$

with a solution in the form

$$z(r) = 2C_s \frac{\sinh(\tfrac{1}{2}\gamma^2 r^2)}{\sqrt{\gamma r}} \tag{52}$$

where $\gamma^2 = N_0^2 \sqrt{P/8}$. Thus,

$$W(r) = 1 - \frac{2C_s}{\sqrt{\gamma}} \sinh(\tfrac{1}{2}\gamma^2 r^2). \tag{53}$$

Two limiting cases are of interest.

(i) For $\gamma^2 r^2 \gg 1$ one obtains

$$W(r) = 1 - \frac{C_s}{\sqrt{\gamma}} \exp(\tfrac{1}{2}\gamma^2 r^2) \tag{54}$$

and

$$F(r) = N_0 r + \Omega_1 C_s^2 r^3 \sum_{n=0}^{\infty} \frac{(\gamma^2 r^2)^n}{n!} \frac{1}{(2n+3)^2} \tag{55}$$

where $\Omega_1 = N_0 \gamma (\sqrt{2P} - K)/(K + 2P)$.

(ii) If $\gamma^2 r^2 \ll 1$ we obtain

$$W(r) = 1 - C_s \gamma^{3/2} r^2 \tag{56}$$

and

$$F(r) = N_0 r + \tfrac{4}{9} C_s^2 \gamma^3 \frac{N_0}{K+P} r^3. \tag{57}$$

For a better understanding of the obtained results let us return to the dimensional variables. It is interesting that both characteristic lengths of the theory turn out to be involved. Namely, the dimensional r reads $r = \sqrt{s_2/s_1}\,\tilde{r}$, and the important parameters β and γ are determined via $\sqrt{P} = l_W^2/l_\phi^2$. In [6] three physically interesting limits were discussed. In particular, in accordance with [6] the typical condition which is valid in crystals and polycrystals is $l_W \gg l_\phi$, in some polycrystals and amorphous bodies $l_W \sim l_\phi$, while the most exotic limit which can be expected in some special amorphous materials is $l_W \ll l_\phi$. Our consideration shows that vortex-like solutions have different asymptotics in each case. For $l_W \gg l_\phi, l_W \sim l_\phi$ and $l_W \ll l_\phi$ they follow (1(i), 2(i)), (1(i), 2(ii)) and (1(ii), 2(ii)), respectively (see above). In accordance with (11) and (12) this results in different asymptotic behaviour of strains and stresses due to disclinations thus giving a possibility for the experimental verification. On the other hand, the obtained results indicate that the proper information about the core region, if any, can be obtained only within the framework of the complete gauge model which should include rotational dislocations.

4.2. *Non-singular* 2π *vortex*

Let us discuss briefly a possibility for a non-singular solution in (31) and (32) which provides a finite energy of the vortex. This means that the condition $W(r) \to 0$ for $r \to 0$ should be satisfied. A simple analysis of (33) and (34) shows that there is an asymptotic solution at small r in the form

$$W(r) \sim r^\alpha \qquad F(r) \sim ar^\mu \tag{58}$$

where $\mu = 1$, $\alpha = 2$ and a is an arbitrary constant with the only restriction $a \neq 1$ following from (34). Note that this resembles the behaviour of the Abrikosov–Nielsen–Olesen vortex. For large r, the asymptotics found in the previous subsection are valid. To prove the existence of the solution for any r the numerical calculations of variations in the energy density which has the following form:

$$E(r) = \lambda\left[F'^2 + \frac{F^2}{r^2}(1-W)^2 - 2\right]^2 + 2\mu\left[F'^4 + \frac{F^4}{r^4}(1-W)^2 - 2F'^2 - 2\frac{F^2}{r^2}(1-W)^2 + 2\right]$$
$$+\frac{s_1^2}{s_2}\left[(F^2+1)\left(\frac{W'}{r}\right)^2\right] \tag{59}$$

where $F' = \mathrm{d}F/\mathrm{d}r$ and $W' = \mathrm{d}W/\mathrm{d}r$ must be used. This study will give a final answer about whether the above asymptotics come from the unique solution or not. The corresponding calculations are now in progress.

4.3. *Forces*

Let us discuss briefly the force between two 2π vortices. In accordance with the classical formula (26) one obtains

$$\mathcal{F}_A = \Omega f\left[\lambda(g^2 + f^2 - 2) + 2\mu(f^2 - 1)\right]x_A. \tag{60}$$

Thus the force turns out to be exactly zero for the solution $W(r) = 1, F(r) = N_0 r$, that is in the case of 2π singular vortices. Assuming a small perturbation of this exact solution we obtain the following asymptotics.

1. For $N_0^2 r^2 s_1/s_2 \gg 1$

$$\mathcal{F}_A = -2\mu N_0^2 C_l \Omega\left(r\sqrt{\frac{s_1}{s_2}}\right)^{-1/2} W_{0,m}\left(\beta\sqrt{\frac{s_1}{s_2}}r\right)x_A \tag{61}$$

2. For $N_0^2 r^2 s_1/s_2 \ll 1$

$$\mathcal{F}_A = -4\mu N_0^2 C_s \Omega \frac{1}{\sqrt{\gamma}} \sinh\left(\gamma^2 \frac{s_1}{s_2} r^2\right) x_A. \tag{62}$$

It is important to note that in the strict sense we have to use the general expressions for the forces given by the gauge model (see [5, 6]). These calculations, however, are too cumbersome and will be omitted here. Note only that the main conclusions agree very closely with (60)–(62).

5. Conclusion

In this paper we have studied the vortex-like solutions for disclinations within the most general gauge model of rotational defects when both disclinations and rotational dislocations are taken into account. The model contains two additional parameters, coupling constants s_1 and s_2, which allow us to introduce two characteristic lengths l_ϕ and l_W. The appearance of these lengths is the unique property of the gauge model that distinguishes it from the classical elasticity theory as well as from other known models of the elastic continuum with topological defects.

There are two distinctive features of the vortices in elastic media. First, the elastic flux is 'classical' in its origin, i.e. there is no quantization as opposed to the magnetic vortex. This means that generally there are no restrictions on the value of ν in (22) apart from $\nu > -1$ for topological reasons. However, if we take into account the symmetry group of the underlying crystal lattice the available values of ν become 'quantized' in accordance with this group (e.g. $\nu = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots$ for a hexagonal lattice). Second, the singular character of the solution on the defect line is typical for the dislocation theory. As a result, all the known solutions for dislocations contain a logarithmic divergence in the energy. To avoid this difficulty, one introduces a cut-off from below by using r_0 as a core radius of the defect. The core region itself is assumed to be beyond the scope of the linear theory of elasticity. For this reason, any non-singular solution will be of essential interest.

Finally, let us note that a similar problem appears for point 4π disclinations. In this case, the gauge group $SO(3)$ should be considered. We expect that the inclusion of rotational dislocations will clarify the role of the characteristic lengths as well as the problem of the core region in this case. This study is now in progress.

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References

[1] Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North-Holland)
[2] Landau L D and Lifshitz E M 1960 *Theory of Elasticity* (Oxford: Pergamon)
[3] Gairola B K D 1980 *Dislocations in Solids* vol 1, ed F R N Nabarro (Amsterdam: North-Holland) p 223
[4] Osipov V A 1991 *J. Phys. A: Math. Gen.* **24** 3237
[5] Kadić A and Edelen D G B 1983 *A Gauge Theory of Dislocations and Disclinations (Lecture Notes in Physics vol 174)* ed H Araki, J Ehlers, K Hepp, R Rippenhahn, H A Weidenmüller and J Zittarz (Berlin: Springer)
[6] Edelen D G B and Lagoudas D C 1988 Gauge theory and defects in solids *Mechanics and Physics of Discrete Systems* vol 1 ed G C Sih (Amsterdam: North-Holland)
[7] Osipov V A 1992 *Phys. Lett.* A **164** 327
[8] Konoplich R V, Selivanov E V and Zhukov V P 1992 *Phys. Lett.* A **167** 75

[9] Osipov V A 1993 *J. Phys. A: Math. Gen.* **26** 1375
[10] Osipov V A 1994 *Phys. Lett.* A **193** 97
[11] Marcinkowski M J 1989 *Phys. Status Solidi* B **152** 9
[12] Vladimirov V I and Romanov A E 1986 *Disclinations in Crystals* (Leningrad: Nauka) (in Russian)
[13] Ebbesen T W 1996 *Phys. Today* June 26
[14] Osipov V A 1990 *Phys. Lett.* A **146** 67
[15] Osipov V A 1991 *Physica* A **175** 369
[16] Das E S P, Marcinkowski M J, Armstrong R W and de Wit R 1973 *Phil. Mag.* A **27** 369
[17] de Wit R 1973 *J. Res. Nat. Bur. Stand.* A **77** 607