Dispersion representations for hard exclusive processes

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Dispersion relations

- relate real and imaginary parts of the amplitude
- required to derive OPE for Compton scattering in Bjorken kinematics $Q^2 \sim W^2 \gg \Lambda_{QCD}$
- practical way to include two-loop corrections in DVCS [Muller, Pasek, Kumericki'07]
- was used to simplify NLO calculations for VM production [D. I., Schafer, Szymanowski, Krasnikov'04]

- our study was stimulated by recent work: [Teryaev'05], [Anikin, Teryaev'07]
  - at LO factorization formula for DVCS, VM prod. looks similar to DR

$$A(\xi) \sim \int\limits_{-1}^{1} dx \frac{H(x,\xi)}{x-\xi}, \quad \xi \sim Q^2/s$$

  - using polynamiality property of GPDs, it was shown

$$A(\xi) \sim \int\limits_{-1}^{1} dx \frac{H(x,\xi)}{x-\xi} = \int\limits_{-1}^{1} dx \frac{H(x,x)}{x-\xi} + const$$

  - up to energy independent const., amplitude can be expressed in terms of $H(x,x)$ - function of one variable instead of $H(x,\xi)$ - function of the two!
- we generalized this to all orders. The output:
  - interesting relations between polynomiality and DR
  - up to a const., amplitude can be expressed in terms of GPD in DGLAP region, $x > \xi$.
  - polarized quark and gluon GPDs

DVCS and VM production

$$\gamma^*(q) + p(p) \to \gamma(q') + p(p')\,, \quad \gamma^*(q) + p(p) \to M(q') + p(p')$$

Mandelstam variables: $s = (p+q)^2$**,** $t = (p-p')^2$**,** $u = (p-q')^2$
"crossing-symmetric" $\nu = (s-u)/2$

Invariant amplitudes (signature factor, $\sigma = \pm 1$):

$$\mathcal{F}^{[\sigma]}(-\nu, t) = \sigma \mathcal{F}^{[\sigma]}(\nu, t)$$

The fixed-t dispersion relation with no subtraction

$$Re\mathcal{F}^{[\sigma]}(\nu, t) = \frac{1}{\pi}\int_{\nu_{th}}^{\infty} d\nu'\, Im\mathcal{F}^{[\sigma]}(\nu', t)\left[\frac{1}{\nu'-\nu} + \sigma\frac{1}{\nu'+\nu}\right]$$

$$\mathcal{F}^{[+]}(\nu, t)_{|\nu|\to\infty} \to 0\,, \quad \nu^{-1}\mathcal{F}^{[-]}(\nu, t)_{|\nu|\to\infty} \to 0\,.$$

A dispersion relation with one subtraction,

$$Re\mathcal{F}^{[\sigma]}(\nu,t) - Re\mathcal{F}^{[\sigma]}(\nu_0,t)$$

$$= \frac{1}{\pi}\int_{\nu_{th}}^{\infty} d\nu'\, Im\mathcal{F}^{[\sigma]}(\nu',t)\left[\frac{1}{\nu'-\nu} + \sigma\frac{1}{\nu'+\nu} - \frac{1}{\nu'-\nu_0} - \sigma\frac{1}{\nu'+\nu_0}\right],$$

is valid if

$$\nu^{-2}\mathcal{F}^{[+]}(\nu,t)_{|\nu|\to\infty}\to 0$$

whereas for $\sigma=-1$ we have the same condition as with no subtraction.

note that

$$\xi = -\frac{(q+q')^2}{2\,(p+p')\cdot(q+q')} = \frac{Q^2}{s-u} = \frac{Q^2}{2\nu}$$

Our (the only) assumption: physical amplitudes grow at large energies less faster than $\sim s^2$, so one subtraction is enough.

the factorization formula can be written

$$\mathcal{F}^{q[\sigma]}(\xi) = \int_0^1 dx \, \frac{1}{\xi} \, C^{q[\sigma]}\Big(\frac{x}{\xi}\Big) \, F^{q[\sigma]}(x,\xi)$$

in terms of the combinations

$$F^{q[\sigma]}(x,\xi) = F^q(x,\xi) - \sigma F^q(-x,\xi)$$

for quark exchange of definite signature. $F^{q[+]}$ **corresponds to positive and** $F^{q[-]}$ **to negative charge conjugation parity in the** t**-channel.**

$C^{q[\sigma]}(\omega)$ **– perturbatively calculable amplitude for scattering of** γ^* **on a quark. The Mandelstam variables for parton subprocess**

$$\hat{s} = xs - \tfrac{1}{2}(1-x)\,Q^2 \,, \quad \hat{u} = xu - \tfrac{1}{2}(1-x)\,Q^2 \,, \quad \frac{x}{\xi} = \frac{\hat{s}-\hat{u}}{Q^2} \,.$$

To LO in DVCS and VM production

$$C^{q[\sigma]}(\omega) \, \propto \, \frac{1}{1-\omega-i\epsilon} - \sigma \frac{1}{1+\omega-i\epsilon} \,, \quad Im C^{q[\sigma]}(\omega) \, \propto \, \pi \left[\delta(\omega-1) - \sigma\delta(\omega+1) \right]$$

Basic Idea

- **subprocess amplitude obeys its own fixed t- DR**

$$ReC^{q[\sigma]}\left(\frac{x}{\xi}\right) = \frac{1}{\pi}\int_1^\infty d\omega\, ImC^{q[\sigma]}(\omega)\left[\frac{1}{\omega - x/\xi} - \sigma\frac{1}{\omega + x/\xi}\right]$$

need no subtraction, since $C^{q[\sigma]}(-\omega) = -\sigma C^{q[\sigma]}(-\omega)$

- **two alternative dispersion representations**
  - **1) DR for the process amplitude, 2) insert there ampl. factorization formulae**
  - **1) amplitude factorization formulae, 2) insert there DR for subprocess ampl.**

- **consistency of these two representations**
  leads to nontrivial constrain on GPDs!

- representation based on DR for process amplitude

$$Re\mathcal{F}(\xi,t) = \frac{1}{\pi}\int_1^\infty d\omega\, ImC(\omega)\left\{\int_{-1}^1 dx\, F\Big(x,\frac{x}{\omega},t\Big)\left[\frac{1}{\omega\xi - x} - \sigma\frac{1}{\omega\xi + x}\right]\right.$$

$$\left. + \mathcal{I}(\omega,t)\right\},$$

includes GPD in DGLAP region only, $\omega \geq 1$ and $x \geq x/\omega$.
$\mathcal{I}(\omega,t)$ (energy, ξ, independent – subtr. const.) is nonzero only for unpolarized positive signature, $\sigma = +1$, quark and gluon GPDs.

- representation based on DR for subprocess amplitude

$$Re\mathcal{F}(\xi,t) = \frac{1}{\pi}\int_1^\infty d\omega\, ImC(\omega)\int_{-1}^1 dx\, F(x,\xi,t)\left[\frac{1}{\omega\xi - x} - \sigma\frac{1}{\omega\xi + x}\right]$$

includes GPD in the whole support domain, both DGLAP and ERBL regions.

- consistency – two dx integrals are equal at any $\omega \geq 1$!

At LO, $ImC(\omega) \sim \delta(1-\omega)$. We reproduce [Teryaev, Anikin] result at $\omega = 1$. Beyond LO one still needs not $F(x,x)$, but function of two variables $F\Big(x,\frac{x}{\omega}\Big)$.

Polynomiality property of GPDs Mellin moments

$$\int_{-1}^{1} dx\, x^{n-1}\, H^q(x,\xi,t) = \sum_{k=0,\text{even}}^{n-1} (2\xi)^k\, A^q_{n,k}(t) + (2\xi)^n C^q_n(t)$$

follows from the Lorentz covariance of the operator matrix elements which are parameterized by GPDs. $A^q_{n,k}(t), C^q_n(t)$ **– generalized form factors.**
C^q_n **is nonzero only for even** n**, in our terms** $\sigma = +1$ **case.**

A way to ensure polynomiality – double distribution representation [Radyushkin]

$$H^q(x,\xi,t) = H^q_f(x,\xi,t) + sign(\xi)\, D^q\Big(\frac{x}{\xi}, t\Big)$$

with

$$H^q_f(x,\xi,t) = \int d\beta\, d\alpha\ \delta(x-\alpha\xi-\beta)\, f^q(\beta,\alpha,t)$$

where f^q **is commonly referred to as double distribution and** D^q **as the** D**-term**

$$\int_{-1}^{1} d\alpha\, \alpha^{n-1} D^q(\alpha,t) = 2^n C^q_n(t)$$

Inserting DD representation for GPDs into our integral relation, one obtains:

- **consistency relation is fulfilled (for any "polynomial" GPD)**
- **subtraction constant in DR is related to D- term**

$$\mathcal{I}(\omega, t) = 2 \sum_{n=2\text{even}}^{\infty} \left(\frac{2}{\omega}\right)^n C_n^q(t) = 2 \int_{-1}^{1} dx \, \frac{D^q(x,t)}{\omega - x}$$

Subtraction constant – energy independent contribution, which is related with spin zero exchange in the t- channel.

Which GPDs allow spin zero exchange?

- $\sigma = +1$ **unpolarized quark/gluon GPDs,** $H^{[+]}, E^{[+]}$
- **proton helicity flip,** $\sigma = +1$**, polarized quark/gluon GPD,** $\tilde{E}^{[+]}$

Table 1: Quantum numbers of t-channel exchanges for combinations of generalized quark distributions of definite charge conjugation parity. The entries with $C = +1$ also hold for the corresponding gluon distributions.

distribution	J^{PC}
$H^q(x,\xi,t) - H^q(-x,\xi,t)$	$0^{++}, 2^{++}, \ldots$
$E^q(x,\xi,t) - E^q(-x,\xi,t)$	$0^{++}, 2^{++}, \ldots$
$\widetilde{H}^q(x,\xi,t) + \widetilde{H}^q(-x,\xi,t)$	$1^{++}, 3^{++}, \ldots$
$\widetilde{E}^q(x,\xi,t) + \widetilde{E}^q(-x,\xi,t)$	$0^{-+}, 1^{++}, 2^{-+}, 3^{++}, \ldots$
$H^q(x,\xi,t) + H^q(-x,\xi,t)$	$1^{--}, 3^{--}, \ldots$
$E^q(x,\xi,t) + E^q(-x,\xi,t)$	$1^{--}, 3^{--}, \ldots$
$\widetilde{H}^q(x,\xi,t) - \widetilde{H}^q(-x,\xi,t)$	$2^{--}, 4^{--}, \ldots$
$\widetilde{E}^q(x,\xi,t) - \widetilde{E}^q(-x,\xi,t)$	$1^{+-}, 2^{--}, 3^{+-}, 4^{--}, \ldots$

No D- term and no subtraction constant $\mathcal{I}(\omega, t)$ for polarized, proton helicity flip quark/gluon GPD, $\tilde{E}^{[+]}$!
Where is spin zero (pion exchange) t-channel exchange contribution?

Answer
is somewhat "kinematical". Within standard definition $\tilde{E}^{[+]}$ has opposite (negative) signature in comparison to unpolarized GPDs $E^{[+]}, E^{[+]}$. Besides, in physical amplitudes it always contribute as $\xi \tilde{E}^{[+]}$. Therefore, if write DR for $\xi \tilde{\mathcal{E}}^{[+]}$ – the situation become similar to unpolarized case:

- **the spin-zero exchange contribution appears directly as a subtraction constant, with**
$$\sum_{n=1\mathrm{odd}}^{\infty} \left(\frac{2}{\omega}\right)^n \widetilde{B}^q_{n,n-1}(t)$$
playing the same role as $\mathcal{I}(\omega, t)$ in unpolarized case.

- **Constraint on GPDs we derived from DR is equivalent to polynomiality condition. It is one of the manifestations of general deep relation between analyticity of the amplitudes and such property of a theory as Lorentz invariance.**

- **Our dispersion representation of the amplitude provides a practical consistency check for GPD models of in which Lorentz invariance (polynomiality) is not exactly satisfied.**

Popular Forward Model [Freund, McDermott, Strikman'03]

here focus on the quark singlet distribution:

$$\Sigma(x) = \sum_q \left[\, q(x) + \bar{q}(x) \,\right]$$

and its generalized counterpart:

$$H(x,\xi) = \sum_q H^{q[+]}(x,\xi)$$

in DGLAP region $x \geq \xi$ – coincides with forward singlet

$$H(x,\xi) = \Sigma(x)$$

in ERBL region $x \leq \xi$ – simple polynomial

$$H(x,\xi) = \Sigma(\xi)\,\frac{x}{\xi}\left[1 + \frac{15}{2}a(\xi)\left(1 - \frac{x^2}{\xi^2}\right)\right]$$

parameter $a(\xi)$ chosen to satisfy the polynomiality condition

$$\int_0^1 dx\, x\, H(x,\xi) = \sum_q \int_{-1}^1 dx\, x\, H^q(x,\xi) = \int_0^1 dx\, x\, \Sigma(x) + 4\xi^2 C_2$$

for the lowest nontrivial Mellin moment, where $C_2 = \sum_q C_2^q(t=0)$.

Shortage of this procedure:
Higher Mellin moments of are not polynomials in ξ.

At small $\xi \sim 10^{-3}$, typical for HERA experiments, one may expect that this is not important.

However, relation between scattering amplitudes and GPDs Mellin moments is notrivial (nonlinear)!

Question: **Can such small inconsistency in Moments lead to big effects in scattering amplitudes?**

Two representations for the real part calculated:
directly from factorization formula:

$$Re\mathcal{H}_{dir}(\xi) = \int_0^1 dx\, H(x,\xi)\left[\frac{1}{\xi - x} - \frac{1}{\xi + x}\right] \,,$$

derived (from polynomiality) from consistency of two disp. relations:

$$Re\mathcal{H}_{\xi_0}(\xi) = \int_0^1 dx \left\{ H(x,x)\ \left[\frac{1}{\xi - x} - \frac{1}{\xi + x}\right] \right.$$

$$\left. + \Big[H(x,\xi_0) - H(x,x)\Big]\ \left[\frac{1}{\xi_0 - x} - \frac{1}{\xi_0 + x}\right]\right\} .$$

should give the same result for any subtraction point ξ_0.

By construction, the two representations coincide of course for $\xi = \xi_0$.

How much the two alternative representations for real part differ for the model?

For a numerical study we use:

$$x\Sigma(x) = 0.34\,x^{-0.25}(1-x)^4(1+25.4\,x)$$

it gives good approximation of the CTEQ6M distributions at scale $\mu = 2$ GeV.

We take two options for subtraction point:

the s-channel threshold $\xi_0 = 1$, where $H(x,1) \propto x\,(1-x^2)$.

As an alternative choice: $\xi_0 = 0.01$ in the small-ξ region.

In numerics we put $C_2 = 0$. This term gives equal contribution, $20\,C_2$, to both integrals and can not restore the discrepancy between them.

Results:

ξ	$\frac{Re\mathcal{H}_{1.0}}{Re\mathcal{H}_{dir}}$	$\frac{Re\mathcal{H}_{0.01}}{Re\mathcal{H}_{dir}}$
10^{-4}	0.37	0.37
10^{-3}	0.35	0.39
10^{-2}	0.23	1
0.1	0.37	10
0.3	0.70	16
0.5	0.76	26

The discrepancy is severe and does not improve with decreasing ξ.

We must conclude:

even for small ξ, amplitude calculated with the model violates strongly dispersion relations.

Summary

- **Consistency condition of DR for the physical amplitude and DR amplitude of parton-level subprocess turned to be nontrivial for exclusive reactions. (In forward DIS case they trivially fulfilled.)**

- **The integral relation for GPDs we derived is equivalent to polynomiality property.**

- **Amplitude can be expressed in terms of GPD in DGLAP region, $x > \xi$, up to energy independent constant (related with spin zero exchange).**

- **Our results may serve as a practical check for the GPD models, in which polynomiality property does not "build in". We found that popular forward model leads to serious conflicts with dispersion relations when used for calculating the real part of scattering amplitudes.**