Nonlinear Waves in Disordered Media: Localization and Delocalization

S. Flach, MPIPKS Dresden

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Two lectures:

- **Obtaining Anderson localization**
- **Destruction of Anderson localization**

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Lecture II:

- Obtaining Anderson localization
- Destruction of Anderson localization

Defining the problem

- a disordered medium
- linear equations of motion: all eigenstates are Anderson localized
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

Will it delocalize?

Yes because of nonintegrability and ergodicity

No because of energy conservation –
spreading leads to small energy density,
nonlinearity can be neglected,
dynamics becomes integrable, and
Anderson localization is restored

Model 1: The discrete nonlinear Schrödinger Equation

$$\mathcal{H}_D = \sum_l \epsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - (\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l)$$

ϵ_l **uniformly from** $[-\frac{W}{2}, \frac{W}{2}]$ $\qquad \dot{\psi}_l = \partial \mathcal{H}_D / \partial (i\psi_l^\star)$

$$i\dot{\psi}_l = \epsilon_l \psi_l + \beta |\psi_l|^2 \psi_l - \psi_{l+1} - \psi_{l-1}$$

Conserved quantities: energy and norm $S = \sum_l |\psi_l|^2$

Varying the norm is strictly equivalent to varying β

Equations model light propagation and cold atom dynamics in structured media

Model 2: The Klein-Gordon chain

$$\mathcal{H}_K = \sum_l \frac{p_l^2}{2} + \frac{\tilde{\epsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}(u_{l+1} - u_l)^2$$

$$\ddot{u}_l = -\partial \mathcal{H}_K / \partial u_l$$

$\tilde{\epsilon}_l$ **uniformly from** $\left[\frac{1}{2}, \frac{3}{2}\right]$

$$\ddot{u}_l = -\tilde{\epsilon}_l u_l - u_l^3 + \frac{1}{W}(u_{l+1} + u_{l-1} - 2u_l)$$

Conserved quantity: energy only

Equations can be approximately mapped on model 1 for small amplitudes

Advantage:

- **test sensitivity to norm conservation**
- **numerical integration 10 times faster at same precision**

Back to model 1: $i\dot{\psi}_l = \epsilon_l \psi_l + \beta |\psi_l|^2 \psi_l - \psi_{l+1} - \psi_{l-1}$

The linear case: $\beta = 0 \qquad \psi_l = A_l \exp(-i\lambda t)$

Stationary states: $\lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1}$

Normal mode (NM) eigenvectors: $A_{\nu,l} \; (\sum_l A_{\nu,l}^2 = 1)$

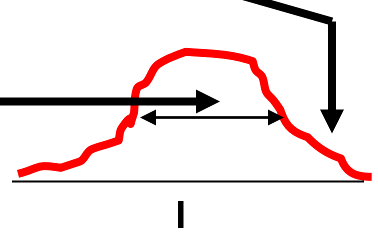
Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2}\right]$

Width of EV spectrum: $\Delta_D = W + 4$

Asymptotic decay: $A_{\nu,l} \sim \mathrm{e}^{-l/\xi(\lambda_\nu)}$

Localization length: $\xi(\lambda_\nu) \le \xi(0) \approx 100/W^2$

Localization volume of NM: V
V(W < 4) ≈ 3ξ   V(W > 10) ≈ 1



Equations in normal mode space:

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1} \phi_{\nu_2} \phi_{\nu_3}$$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

NM ordering in real space: $X_\nu = \sum_l l A^2_{\nu,l}$

Characterization of wavepackets in normal mode space:

$$z_\nu \equiv |\phi_\nu|^2 / \sum_\mu |\phi_\mu|^2 \qquad \bar{\nu} = \sum_\nu \nu z_\nu$$

Second moment: $m_2 = \sum_\nu (\nu - \bar{\nu})^2 z_\nu$ → location of tails

Participation number: $P = 1/\sum_\nu z_\nu^2$ → number of strongly excited modes

Compactness index: $\zeta = \dfrac{P^2}{m_2}$

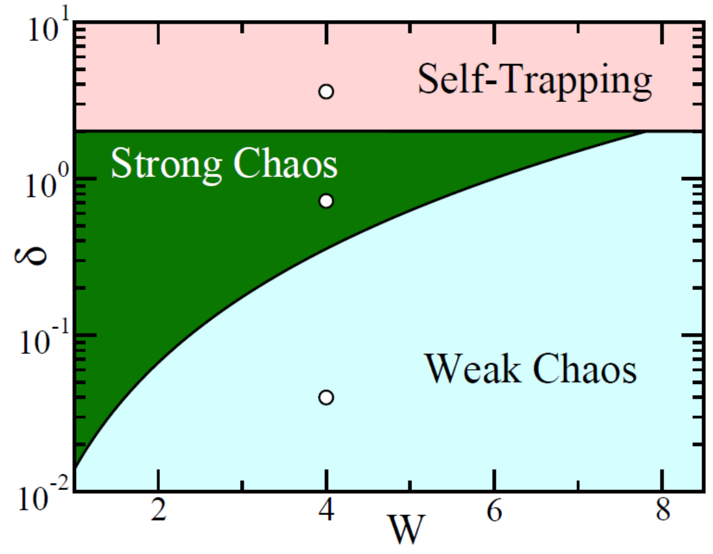
→ K adjacent sites equally excited: $\zeta = 12$

→ K adjacent sites, every second empty or equipartition: $\zeta = 3$

Frequency scales

W=4 :

- **Eigenvalue (frequency) spectrum width:** $\Delta = W + 4$ **8**
- **Localization volume of eigenstate:** $V \approx 360/W^2$ **~18 (sites)**
- **Average frequency spacing inside localization volume:** $d = \Delta/V$ **0.43**
- **Nonlinearity induced frequency shift:** $\delta_l = \beta|\psi_l|^2$



Three expected evolution regimes:

Weak chaos : $\delta < d$

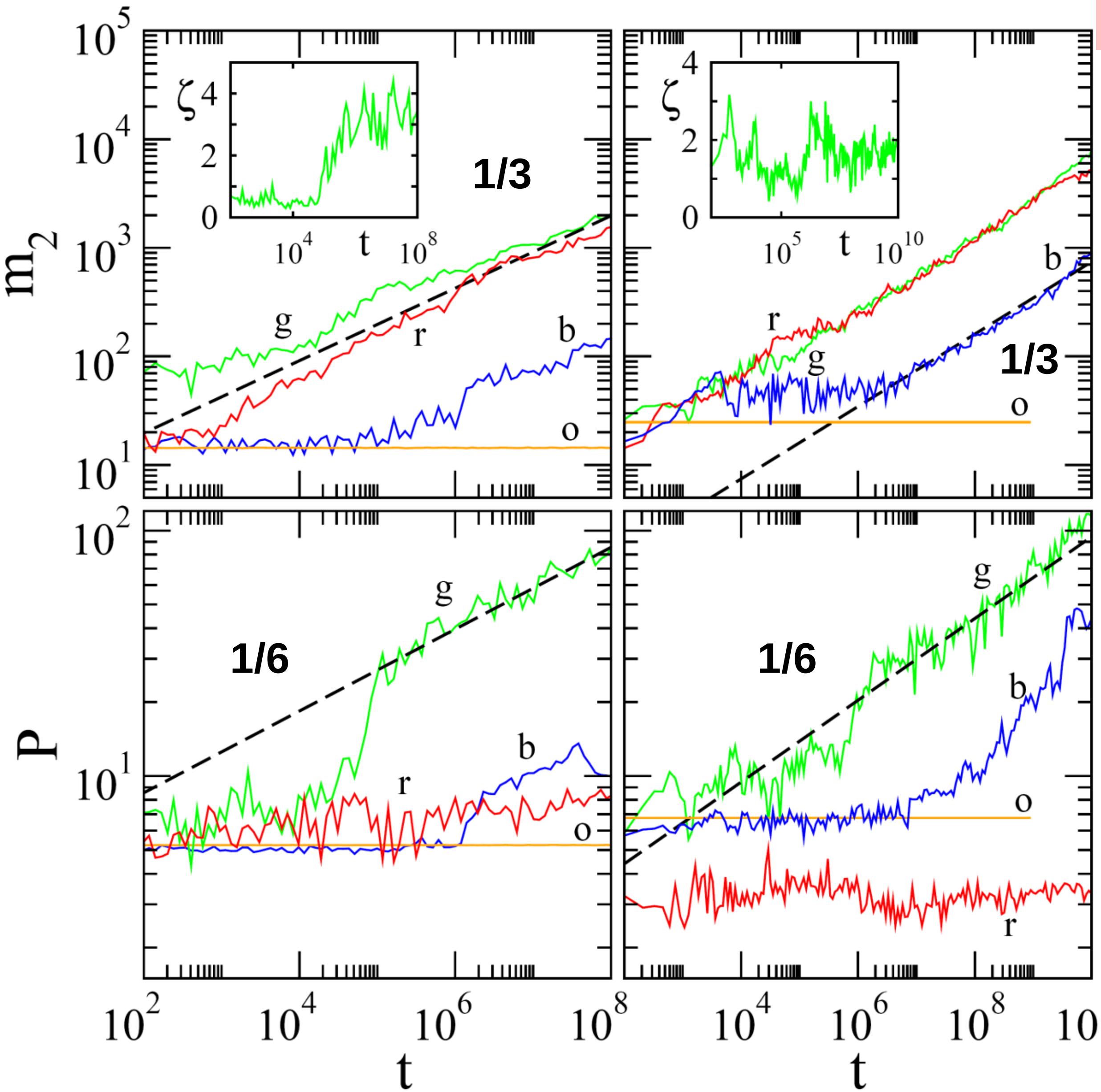
Strong chaos : $d < \delta < 2$

(partial) self trapping : $2 < \delta$

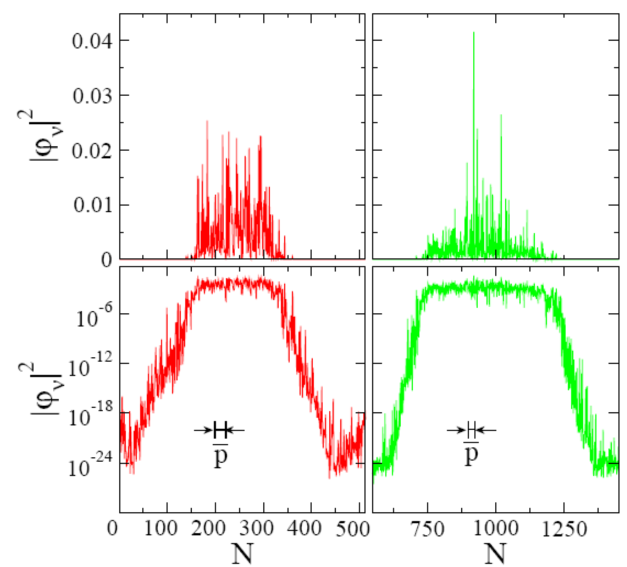
Results for single site excitations $\psi_l = \delta_{l,l_0} \quad \epsilon_{l_0} = 0$

DNLS W=4, β= 0, 0.1, 1, 4.5 **KG W=4, E= 0, 0.05, 0.4, 1.5**

SF,Krimer,Skokos (2009)
Skokos,Krimer,Komineas,SF (2009)



Wavepacket spreads way beyond localization volume.
DNLS at $t = 10^8$



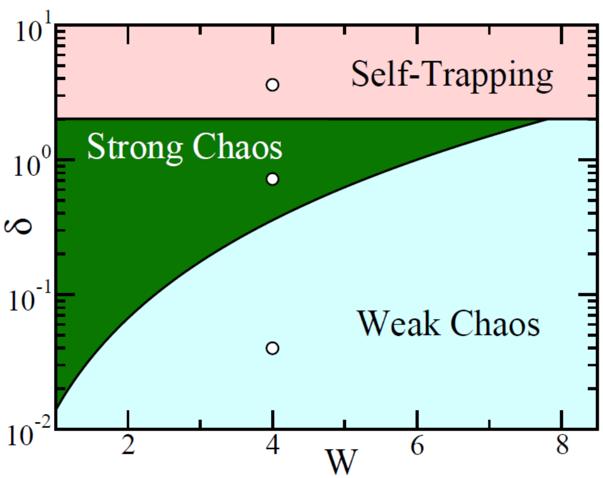
W=4

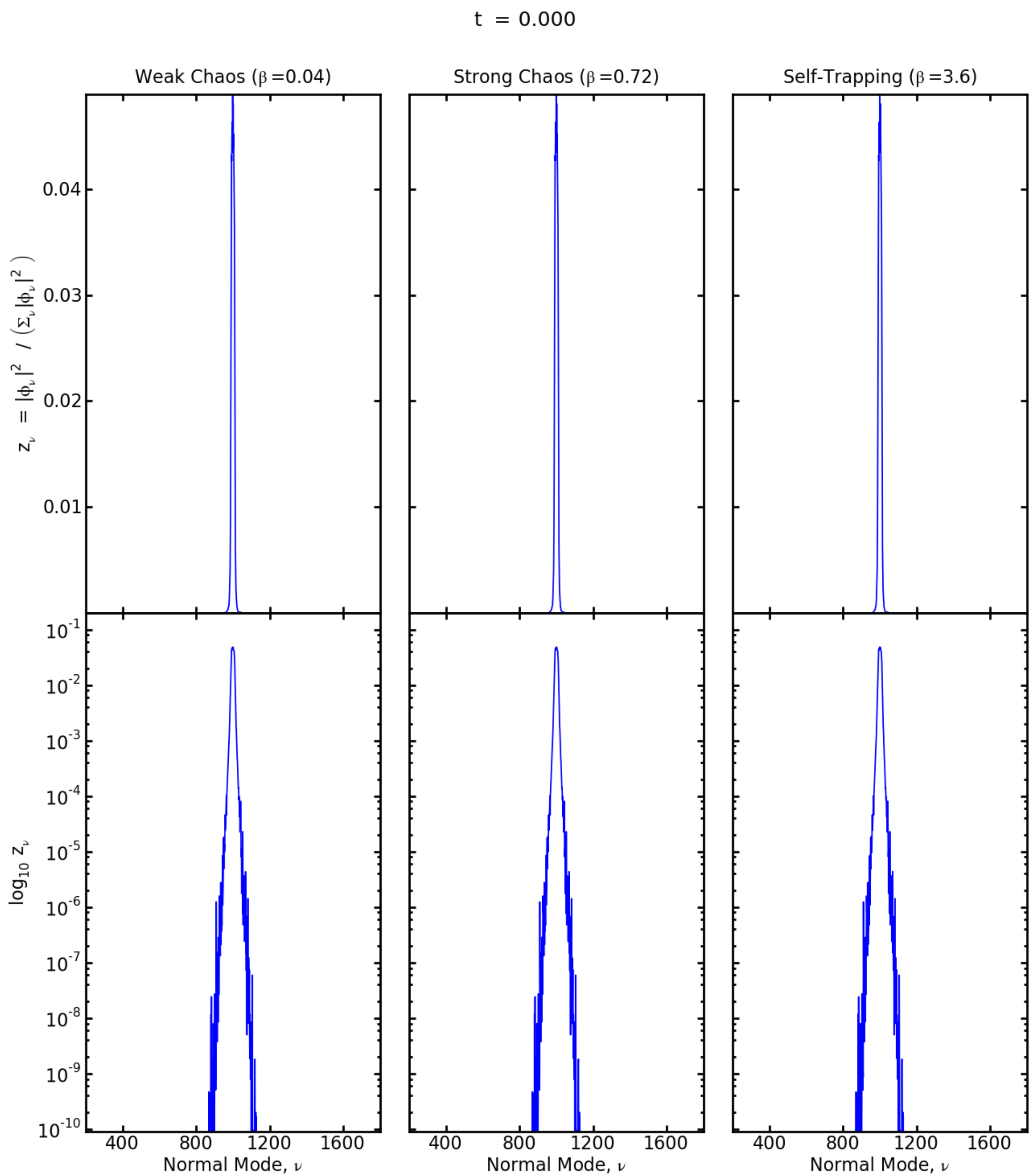
Wave packet with 20 sites

Norm density = 1

Random initial phases

Averaging over
1000 realizations





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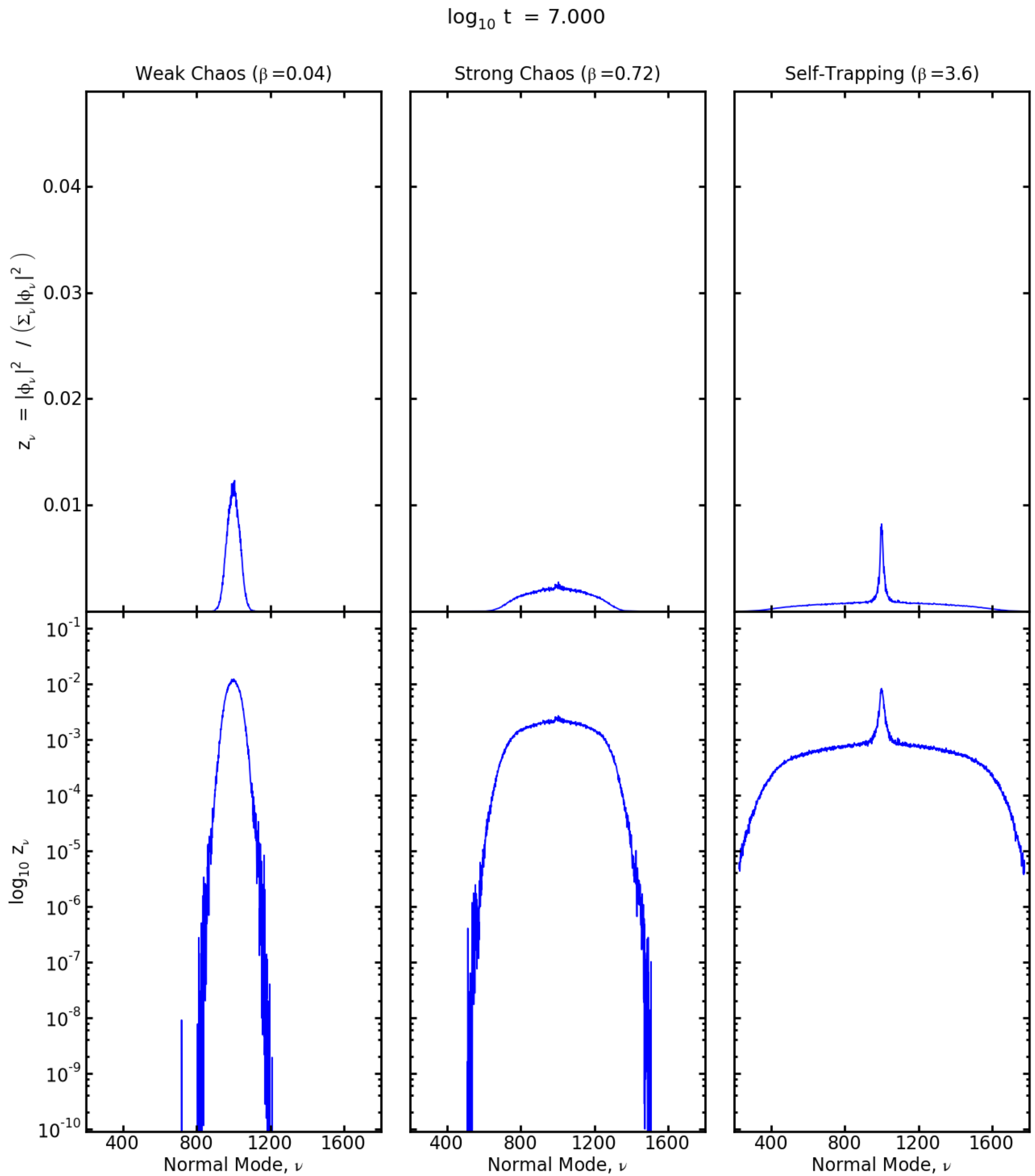
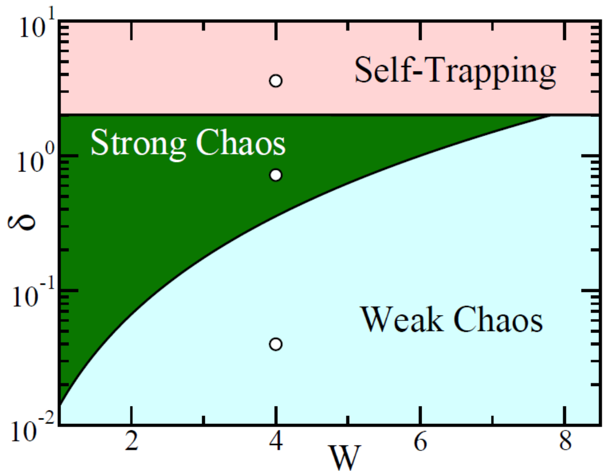
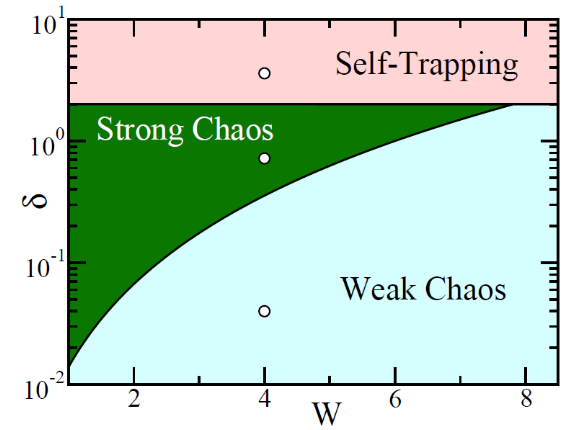
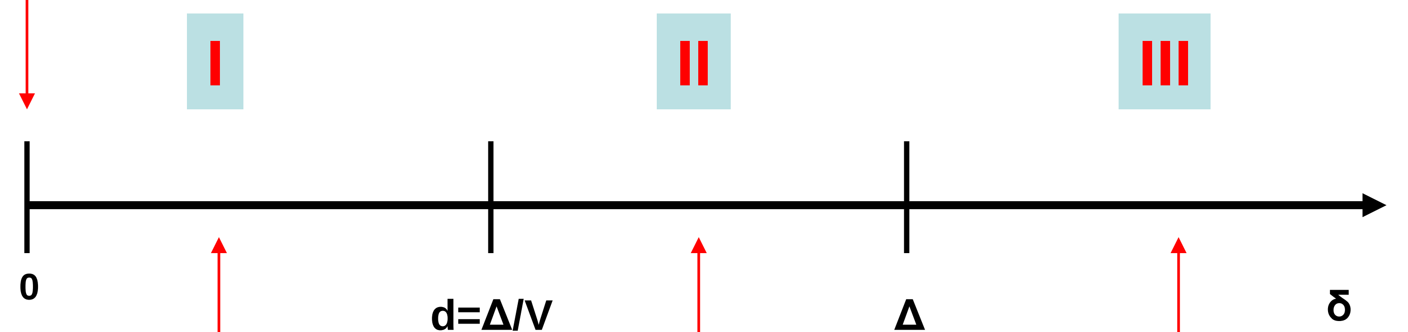




The emerging picture



Anderson Localization



Weak Chaos

Strong Chaos

Selftrapping, part of packet IS spreading

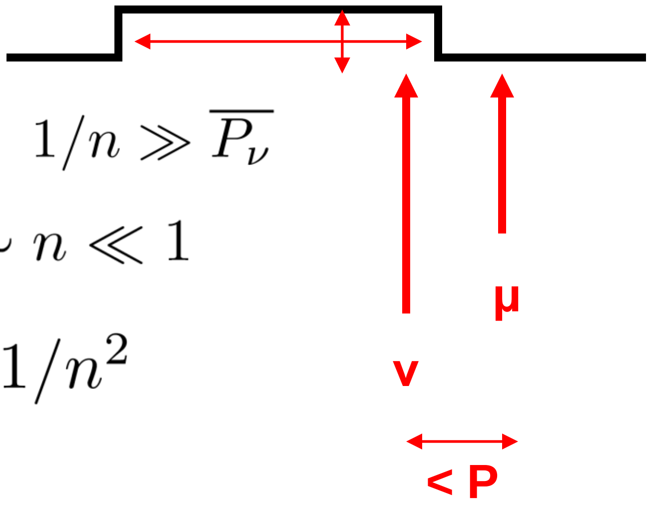
SF,Krimer,Skokos (2009)
Shepelyansky and Pikovsky (2008)
Molina (1998)

SF (2010)
Bodyfelt,Lapteva,Krimer,
Skokos,SF (2010)

Kopidakis, Komineas,
SF, Aubry (2008)

In all cases subdiffusive spreading

Explaining subdiffusion?



- at some time t packet contains 1/n modes: $1/n \gg \overline{P_\nu}$
- each mode on average has norm $|\phi_\nu|^2 \sim n \ll 1$
- the second moment amounts to $m_2 \sim 1/n^2$

Two mechanisms of exciting a cold exterior mode:

- heated up by the packet (nonresonant process)
- directly excited by a packet mode (resonant process)
- in both cases the relevant modes are in a layer of the width of the localization volume at the edge of the packet

Heating

Simplest assumption:

- some modes in packet interact resonantly and therefore evolve chaotic
- Probability of resonance: P(βn)
- all phases decohere after some time scale
- spreading = heating of cold exterior

exterior mode:

$$i\dot{\phi}_\mu \approx \lambda_\mu \phi_\mu + \beta n^{3/2} \mathcal{P}(\beta n) f(t)$$

$$\langle f(t) f(t') \rangle = \delta(t - t')$$

$$|\phi_\mu|^2 \sim \beta^2 n^3 (\mathcal{P}(\beta n))^2 t$$

The momentary diffusion rate of packet equals the inverse time the exterior mode needs to heat up to the packet level:

$$D = 1/T \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2$$

Counting resonances inside the packet:

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1} \phi_{\nu_2} \phi_{\nu_3}$$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

substitute $\phi_\nu = \mathrm{e}^{-i\lambda_\nu t} \chi_\nu \qquad |\chi_\nu|^2 = n_\nu$

$$i\dot{\chi}_\nu = \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \chi^*_{\nu_1} \chi_{\nu_2} \chi_{\nu_3} \mathrm{e}^{i(\lambda_\nu + \lambda_{\nu_1} - \lambda_{\nu_2} - \lambda_{\nu_3})t}$$

Ignore secular terms, eg $\nu = \nu_2 \ , \ \nu_1 = \nu_3$

Resonance if no averaging is possible: $\longrightarrow$

$$n_\nu^{1/2} \leq \frac{\beta I_{\nu,\nu_1,\nu_2,\nu_3} (n_{\nu_1} n_{\nu_2} n_{\nu_3})^{1/2}}{|\lambda_\nu + \lambda_{\nu_1} - \lambda_{\nu_2} - \lambda_{\nu_3}|}$$

Probability of packet mode satisfying $$n_\nu^{1/2} \le \frac{\beta I_{\nu,\nu_1,\nu_2,\nu_3}(n_{\nu_1} n_{\nu_2} n_{\nu_3})^{1/2}}{|\lambda_\nu + \lambda_{\nu_1} - \lambda_{\nu_2} - \lambda_{\nu_3}|}$$

$$|\phi_\nu^{(1)}| = \beta\sqrt{n_{\mu_1} n_{\mu_2} n_{\mu_3}} R_{\nu,\vec{\mu}}^{-1}, \quad R_{\nu,\vec{\mu}} \sim \left| \frac{\vec{d\lambda}}{I_{\nu,\mu_1,\mu_2,\mu_3}} \right| \qquad \vec{d\lambda} = \lambda_\nu + \lambda_{\mu_1} - \lambda_{\mu_2} - \lambda_{\mu_3}$$

We perform a statistical numerical analysis for the quadruplet case. For a given NM ν we obtain $R_{\nu,\vec{\mu}_0} = \min_{\vec{\mu}} R_{\nu,\vec{\mu}}$. Collecting $R_{\nu,\vec{\mu}_0}$ for many ν and many disorder realizations, we find the probability density distribution $\mathcal{W}(R_{\nu,\vec{\mu}_0})$ (Fig. 9).

$$\mathcal{P} = \int_0^{\beta n} \mathcal{W}(x)\,dx$$

C ≠ 0!

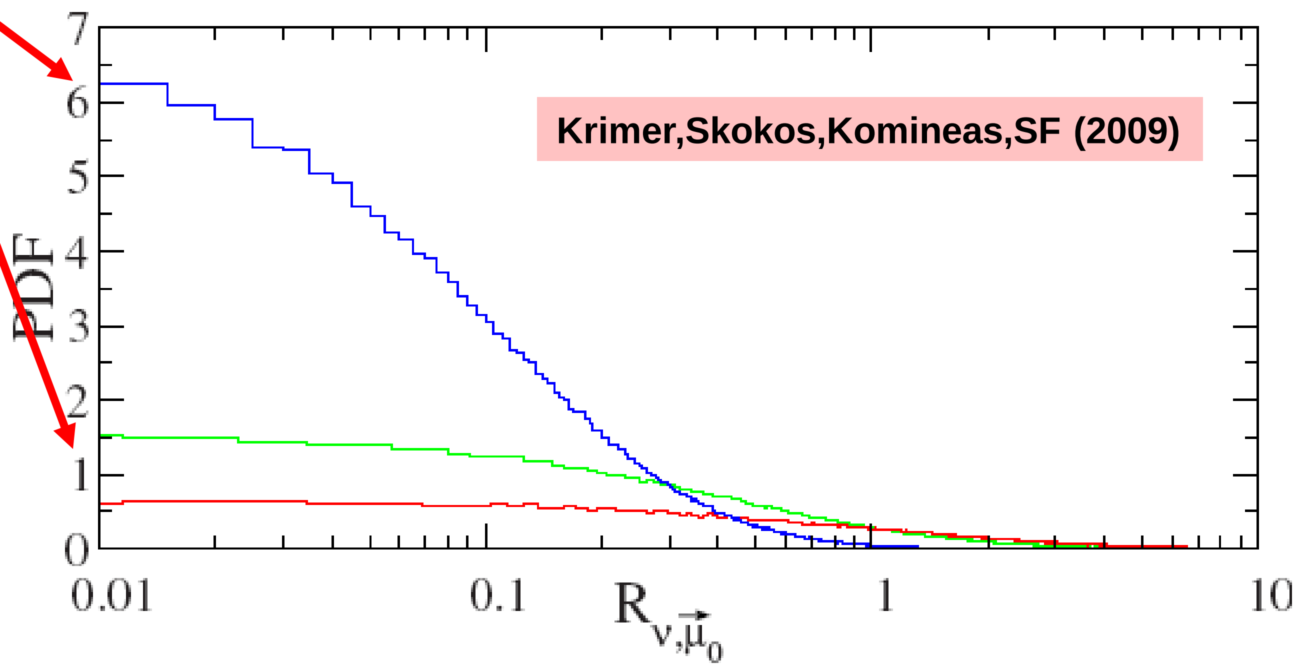


FIG. 9. (Color online) Statistical properties of NMs of the DNLS model. Probability densities $\mathcal{W}(R_{\nu,\vec{\mu}_0})$ of NMs being resonant (see Sec. IV B for details). Disorder strength $W=4,7,10$ (from top to bottom).

$$\mathcal{W}(R) \approx C\mathrm{e}^{-CR}$$

$$\mathcal{P} = 1 - \mathrm{e}^{-C\beta n}$$

$$\frac{1}{C} \approx d$$

SF (2010)

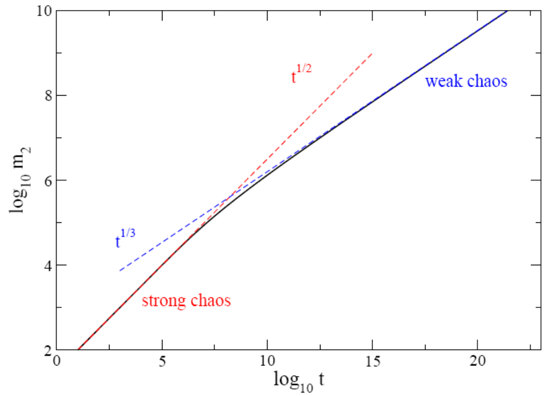
Do you remember? $D = 1/T \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2$

With $m_2 \sim 1/n^2$ the diffusion equation $m_2 \sim Dt$ yields

$$\frac{1}{n^2} \sim \beta(1 - e^{-C\beta n}) t^{1/2} \; .$$

$$m_2 \sim (\beta^2 t)^{1/2} \; , \; \text{strong chaos} \; , \; C\beta n > 1$$
$$m_2 \sim C^{2/3} \beta^{4/3} t^{1/3} \; , \; \text{weak chaos} \; , \; C\beta n < 1$$

Crossover from strong to weak chaos: $C\beta n_c \approx 1.86$



SF (2010)

Asymptotic regime of weak chaos

Krimer,Skokos,Komineas,SF (2009)

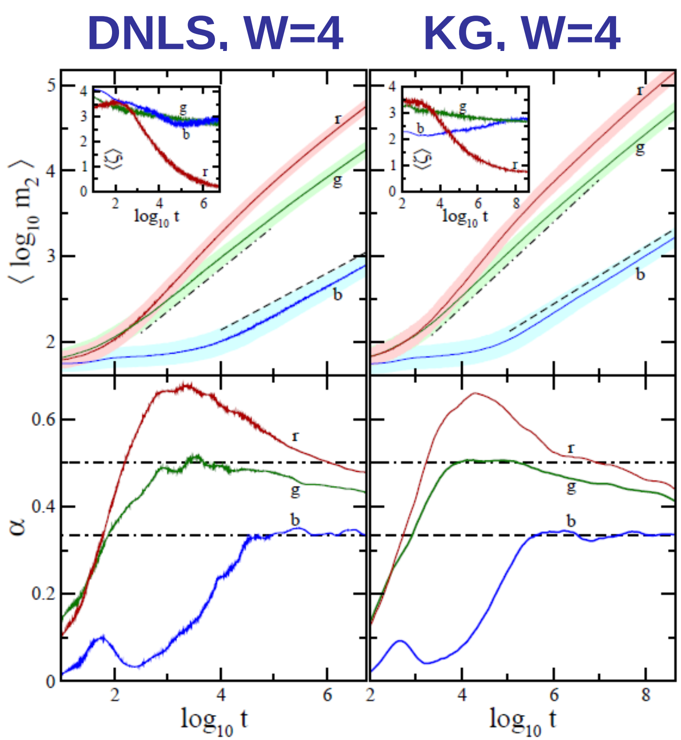
We averaged the measured exponent over 20 realizations:

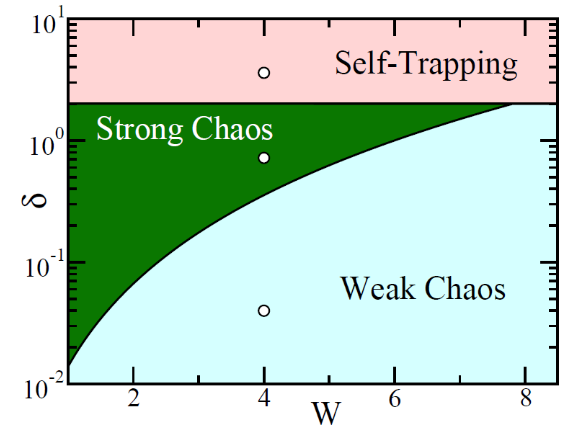
$\alpha = 0.33 \pm 0.02$ (DNLS)
$\alpha = 0.33 \pm 0.05$ (KG)

Strong chaos and crossover to weak chaos

Bodyfelt,Lapteva,Krimer,Skokos,SF (2010)

Averaging over 1000 realizations, taking local derivatives in log-log scales:





Generalization to larger dimensions and different nonlinearity powers:

SF (2010)

$$i\dot{\psi}_{\boldsymbol{l}} = \epsilon_{\boldsymbol{l}}\psi_{\boldsymbol{l}} - \beta|\psi_{\boldsymbol{l}}|^{\sigma}\psi_{\boldsymbol{l}} - \sum_{\boldsymbol{m}\in D(\boldsymbol{l})} \psi_{\boldsymbol{m}}$$

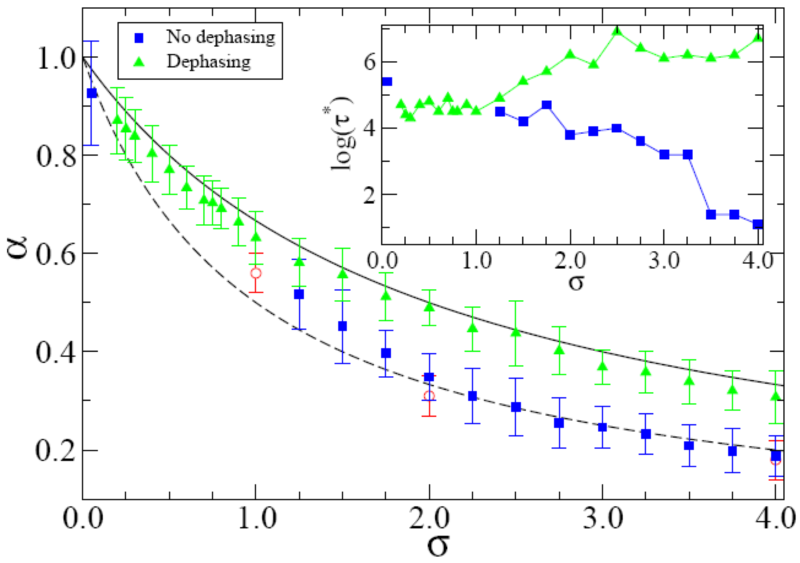
$$m_2 \sim (\beta^2 t)^{\frac{2}{2+\sigma \boldsymbol{D}}} \,,\ \text{strong chaos}$$
$$m_2 \sim (\beta^4 t)^{\frac{1}{1+\sigma \boldsymbol{D}}} \,,\ \text{weak chaos}$$

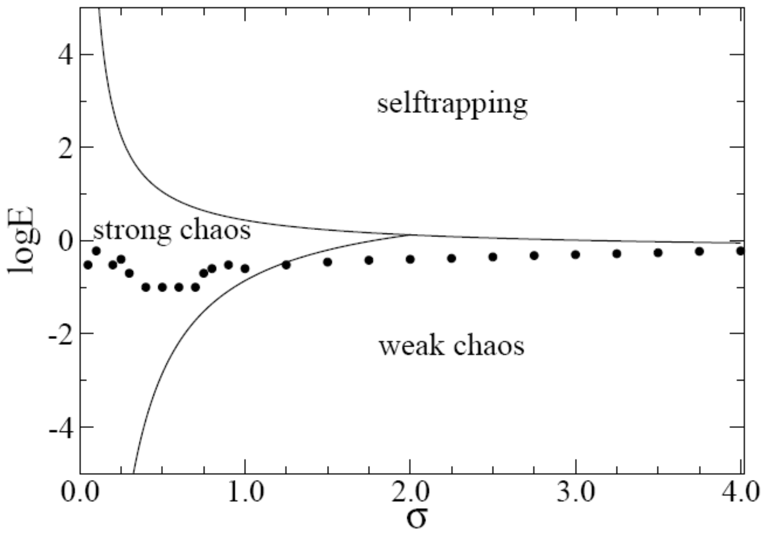
We also find that the number of surface resonances will grow

$$\boldsymbol{D} > \boldsymbol{D_c} = \frac{1}{1-\sigma/2} \,,\ \sigma < 2\,.$$

Numerical evaluation for KG, D=1, W=4, single site excitations:

Skokos,SF (2010)





Conclusions

Some corrections to the results of a famous writer …

“Well, in my country,” said Alice, still panting a little, “you would generally get to somewhere else, if you ran very fast for a long time, as we’ve been doing”. “A slow sort of country!”, said the queen. “Now *here*, it takes all the running you alone can do, to stay in the same place. But with friends, if running together, and more and more, the more time flows, you will finally get to somewhere else, no matter how far that place will be. We call this game SUBDIFFUSION.”

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- T.Lapteva, J.D.Bodyfelt,D.O.Krimer, Ch.Skokos, SF, EPL 91, 30001 (2010)
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Open questions, prob

- is there a KAM regi
- will a spreading pa
- is the spreading wave packet equilibrating inside, if yes, how?
- is the observed spreading Arnold diffusion, or not?
- will the spreading slow down into a kind of Arnold diffusion, or not?
- are the computational results affected by roundoff errors, or not?
- finite systems: how is the KAM threshold scaling with system size?
- characteristics of energy diffusion at finite norm/energy densities ?
- relation to quantum many body localization ?