Effective actions and functional determinants.

We wish to keep as much of the symmetries, we can. We have to work with the spectrum of the operators, i.e. to use the methods of spectral geometry.

Let D be a positive operator, and let λ be its' eigenvalue $(\lambda > 0)$. Then

$$\ln \lambda = -\int_0^\infty \frac{dt}{t} e^{-t\lambda}.$$

(to check this "identity" differentiate w.r.t λ)

Let us use $\ln \det D = \operatorname{Tr} \ln D$ and extend the eq. above to the whole operator:

$$W = \frac{1}{2} \ln \det D = -\frac{1}{2} \int_0^\infty \frac{dt}{t} K(t, D)$$

where

$$K(t, D) = \operatorname{Tr}_{L^2}\left(e^{-tD}\right).$$

We shall also need a more general ("smeared") heat kerner

$$K(t, f, D) = \operatorname{Tr}_{L^2}\left(f e^{-tD}\right)$$

where f is a smooth function.

The regularization

$$W_s = -\frac{1}{2}\mu^{2s}\int_0^\infty \frac{dt}{t^{1-s}} K(t,D)$$

s is a regularization parameter ($s \to 0$ ← removing the regularization). μ – a parameter of the mass dim. 1. One can perform the t-integration to obtain:

$$W_s = -\frac{1}{2}\mu^{2s}\Gamma(s)\zeta(s,D)$$

where $\zeta(s,D) := \zeta(s,1,D)$, and

$$\zeta(s,f,D) = \mathrm{Tr}_{L^2}(f D^{-s}) \quad \text{or}$$

$$\zeta(s,f,D) = \Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1} K(t,f,D)$$

Let $\zeta(s)$ be regular at $s=0$. Then, near $s=0$,

$$W_s = -\frac{1}{2}\left(\frac{1}{s} - \gamma_E + \ln\mu^2\right)\zeta(0,D) - \frac{1}{2}\zeta'(0,D)$$

If the pole can be removed by a renormalization

$$W^{\mathrm{ren}} = \underbrace{-\frac{1}{2}\zeta'(0,D)}_{\text{The Ray-Singer definition of the determinant}} - \frac{1}{2}\ln(\tilde{\mu}^2)\zeta(0,D)$$

"Usually" there is an asymptotic expansion as $t \to +0$:

$$K(t, f, D) \cong \sum_{k=0} t^{(k-n)/2} a_k(f, D)$$

and

$$a_k(f, D) = \operatorname*{Res}_{s=\frac{n-k}{2}} \left(\Gamma(s) \zeta(s, f, D) \right)$$

in particular

$$a_n(f, D) = \zeta(0, f, D)$$

Let D be a Laplacian, $D = -(\nabla^2 + E)$, on a compact manifold without boundaries, then $\quad a_{2l+1} = 0$,

$$a_0 = \operatorname{tr} \int d^n x \sqrt{g}\, f \cdot \frac{1}{(4\pi)^{n/2}}$$

$$a_2 = \operatorname{tr} \int d^n x \sqrt{g} \left(E + \frac{R}{6} \right) \frac{1}{(4\pi)^{n/2}}$$

etc

Return to noncommutativity

For a complex scalar field $W_m = \ln \det \Delta$.

Let us introduce

$$\Delta_{[\alpha]} = -e^{-\alpha\rho} \star \partial^2 e^{-\alpha\rho}$$

Then

$$\frac{d}{d\alpha} \zeta(s, \Delta_{[\alpha]}) = -s \,\mathrm{Tr}\left(\frac{d}{d\alpha} \Delta_{[\alpha]} \star \Delta_{[\alpha]}^{-s-1} \right)$$

$$= 2s \,\mathrm{Tr}\left(\rho \star \Delta_{[\alpha]}^{-s} \right) = 2s\, \zeta(\rho, s, \Delta_{[\alpha]})$$

or

$$\frac{d}{d\alpha} W_m(\alpha) = \frac{d}{d\alpha} \left(-\ln \mu^2 \zeta(0, \Delta_{[\alpha]}) - \zeta'(0, \Delta_{[\alpha]}) \right)$$

$$= -2 a_2 (\rho, \Delta_{[\alpha]})$$

Initial condition :

$$W_m(\alpha = 0) = 0$$

We have to calculate the $t \to +0$ asympt. expansion of

$$K(f,t,\Delta) = \int d^2x \int \frac{d^2k}{(2\pi)^2} e^{-ikx} \star f(x) \star e^{-t\Delta} \star e^{ikx}$$

orthonormal basis

Next: push e^{ikx} to the left, $\partial \to \partial + ik$ everywhere except for "$\star$" (!).

$$K(f,t,\Delta) = \int d^2x \int \frac{d^2k}{(2\pi)^2} f(x) \star$$

$$\star \exp\big(\underbrace{-tk^2 e^{-2\rho}}_{\text{keep ...}} \underbrace{+ t e^{-\rho} \star \partial^2 e^{-\rho} + 2i e^{-\rho} \star (k\partial) e^{-\rho}}_{\text{expand ...}}\big)$$

... and integrate over k with the help of

$$\int \frac{d^2k}{(2\pi)^2} e^{-ak^2} k^{2n} = \frac{n!}{4\pi\, a^{n+1}}$$

$$\int \frac{d^2k}{(2\pi)^2} e^{-ak^2} k_\mu k_\nu k^{2n} = \frac{1}{2} \delta_{\mu\nu} \frac{(n+1)!}{4\pi\, a^{n+2}}$$

(valid also in the NC case)

Results:

$$a_0(f,\Delta) = \frac{1}{4\pi}\int d^2x \; f \star e^{2\rho}$$

$$a_2(f,\Delta) = \frac{1}{4\pi}\int d^2x \; f(x) \star \Big(-\frac{1}{3}\partial^2\rho + \frac{1}{30}[[\rho,\partial_\mu\rho],\partial_\mu\rho]$$

$$+ \frac{7}{90}\partial_\mu[[\partial_\mu\rho,\rho],\rho] + O(\rho^4)\Big)$$

Now we can calculate the effective action

$$W_M = -\frac{1}{4\pi}\int d^2x \Big(\underbrace{-\frac{1}{3}\rho\star\partial^2\rho}_{\text{Polyakov}} + \underbrace{\frac{1}{45}[\rho,\partial_\mu\rho]\star[\rho,\partial_\mu\rho] + O(\rho^5)}_{\text{NC corrections}\;\dots} \Big)$$