# Integrable models of B.H. branes and cosmologies

A.T. Filippov (JINR, Dubna)

High dimensional theory (SUGRA, gravity, etc.) M-theory

↓

Reductions (K.M.K.F. = K.K.; compactif. on tori, spher. symm.)

e.g. 4d

↓

1 + 1 dimensional model of gravity + dilaton + matter

(generally, not integrable)

↓

Reduction to 0+1 dim. theory (static solution, e.g. B.H., or

⟶ homogeneous cosmologies)

(often integrable, even explicitly soluble; non-integrable may be analytically solved near horizons, etc.

- Old story: Euclidean Liouville e.a — 'another song'

- Our story starts in '91-'92
  (E. Witten, H. Verlinde, CGHS, ... )
  T. Banks and M. O'Loughlin -'91

- My own work '94-'96 (with V. de Alfaro and M. Cavaglia)
  Quantum B.H
  Quantum cosmologies } 0+1 dim theor.
  DISCUSSIONS!

- More general integrable models in 0+1 and 1+1 dim (A.T.F. '96-'97)
  N-Liouville in 1+1 and 0+1 (A.T.F. '01-'03)

- Dimensional reduction in gravity:
  '88 P. Breitenlohner, D. Maison, G. Gibbons
  '96 B. Julia, H. Nicolai e.a.

- '98 Reviews: P. Fre, K. Stelle
  '2000 T. Mohaupt
  a recent review: ('99) J. Lidsey e.a
  Superstring cosmology

* Symmetry approach v.s.
  Dynamical approach
  (or, Groups vs. Lagrangians)

2D metric: $-ds^2 = -A_1 dr^2 + A_2 dt^2 + 2A_3 dr\,dt$

$$\Delta \equiv A_3^2 + A_1 A_2 \,; \quad \sqrt{\Delta}\, R = \left(\frac{\dot{A}_1}{\sqrt{\Delta}}\right)^{\bullet} - \left(\frac{A_2'}{\sqrt{\Delta}}\right)' + \left(\frac{A_3'}{\sqrt{\Delta}}\right)^{\bullet} + \left(\frac{\dot{A}_3}{\sqrt{\Delta}}\right)' - \frac{1}{2\Delta^{3/2}}[\dots] \quad (\text{Gauß})$$

$$[\dots] = \varepsilon_{ijk} A_i A_j' \dot{A}_k \,; \quad \phi \equiv \frac{[\dots]}{2\Delta^{3/2}}\, dr \wedge dr$$

$$\hat{A}_i \equiv \frac{A_i}{\sqrt{\Delta}} \quad \begin{cases} \phi = \frac{1}{2}\left[\tilde{A}_1 d\tilde{A}_2 \wedge d\tilde{A}_3 + \dots\right] \\ \tilde{A}_3^2 + \tilde{A}_1 \tilde{A}_2 = 1 \end{cases}$$

* Prove that: $\phi = d\left(\ln \tilde{A}_1 d\tilde{A}_3\right)$ $\quad \boxed{\int_V d\phi = \int_{\partial V} \phi}$

Diagonal: $ds^2 = e^{2\alpha} dr^2 - e^{2\gamma} dt^2$

$$\overset{(2)}{R} = 2e^{-2\gamma}\left(\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma}\right) - 2e^{-2\alpha}\left(\gamma'' + \gamma'^2 - \gamma'\alpha'\right)$$

- $V_2$ is conformal flat: $ds^2 \Rightarrow e^{2\alpha}(dr^2 - dt^2)$
(prove this)

- Spherically symm. metric

$$ds_D^2 = \underbrace{g_{ij}\, dx^i dx^j}_{} + e^{2\beta(r,t)}\, \underbrace{d\Omega_{D-2}^2}_{S^{D-2}\text{-metric}}$$

$x^i = (r,t)\,; \quad g_{ij}(r,t)\,; \quad d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

$D=4$: $\quad ds^2 = e^{2\alpha}dr^2 + e^{2\beta}d\Omega_2^2 - e^{2\gamma}dt^2$

$$R = R^{(2)} + \underbrace{2e^{-2\beta}}_{R(S^2)} + 2(\nabla\beta)^2 -$$

$$\cdot\quad \underline{-2e^{-(\alpha+2\beta+\gamma)}\left((e^{2\beta})' e^{\gamma-\alpha}\right)'} \quad +$$

$$\cdot\quad \underline{+2e^{-(\alpha+2\beta+\gamma)}\left((e^{2\beta})^{\cdot} e^{\alpha-\gamma}\right)^{\cdot}}$$ (prove!)

$$\int d^4x\,\sqrt{-g^{(4)}}\,R = \int dt\,dr\,\sin\theta\,d\theta\,d\varphi\,\underline{e^{(\alpha+2\beta+\gamma)}}\,R$$

Thus, the last two terms give total derivatives in dim. reduction to 1+1. However, they should be <u>not</u> neglected!

$$\left( \nabla_i\beta \equiv \partial_i\beta\,;\quad \nabla^2\varphi \equiv \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}\,g^{ik}\partial_k\varphi\right) \right)$$

- Write explicit expression for the diag. metric.

- More general metric:

$$\begin{cases} ds^2 = g_{ij}dx^i dx^j + e^{2\beta}d\Omega_p^2 + e^{2\bar\beta}d\Omega^2_{\bar p;\bar\kappa} \\ i,j = 0,1\,;\quad p+\bar p+2 = D\,;\quad d\Omega^2_{p,\kappa} = d\theta_p^2 + S_\kappa^2(\theta_p)\,d\Omega^2_{p-1} \\ \underline{S_\kappa(\theta) \equiv \frac{\sin\kappa\theta}{\kappa}}\,,\qquad \kappa = 0, \pm 1 \end{cases}$$

- Geom. in $\underline{V_2}$ : $R^{(2)}$. In the light-cone coordinates $ds^2 = -4f(u,v)\,du\,dv$

$$R^{(2)} = \frac{1}{f}(\ln|f|)_{,uv} \qquad \text{(prove!)}$$

- Geom. in $\underline{V_3}$ : $R_{\alpha\beta}$ ($R_{\alpha\beta\gamma\delta}$ is linear in $R_{\alpha\beta}$) (find!)

In $V_3$ there exist an orthogonal coord. syst

$$ds^2 = \sum_\alpha \varepsilon_\alpha e^{2F_\alpha}(dx^\alpha)^2$$

In this system: $R = \sum_{\alpha\beta} \frac{R_{\alpha\beta\alpha\beta}}{g_{\alpha\alpha}\, g_{\beta\beta}}$ (prove!)

- Any $V_D$ that has constant curvature is conformal to flat one. (prove for $S_k^{(D)}$)

- Prove that any sph. symm. space is conformal flat.

More general: $ds^2 = g_{\alpha\beta}\overbrace{dx^\alpha dx^\beta}^{m} + h_{ij}\underbrace{dy^i dy^j}_{n}$

$$R = R^{(m)} + R^{(n)} - \frac{2}{\sqrt{h}}\boxed{g}\sqrt{h} - \frac{2}{\sqrt{g}}\boxed{h}\sqrt{g}$$
$$+\frac{1}{4}\Big[g^{\alpha\beta}(\partial_\alpha h_{ij}\,\partial_\beta h^{ij}) + g^{\alpha\beta}(h^{ij}\partial_\alpha h_{ij})(h^{kl}\partial_\beta h_{kl})$$
$$+\,(g \leftrightarrow h;\ (\alpha,\beta,\gamma,\delta) \leftrightarrow (ijkl))\Big]$$

One more useful formula (L.L. p 345 exercise)

$$ds^2 = \sum \varepsilon_i e^{2F_i} (dx^i)^2, \quad F_i(x_1, \dots, x_D), \quad \varepsilon_i = \pm 1$$

(find $R_{ijkl}$, $R_{ij}$, prove the following for $R$)

$$e^{\sum_1^D F_m} R = \sum_i \varepsilon_i e^{\sum_m^{(i)} F_m - F_i} [\dots] =$$

$$\boxed{= \sum \varepsilon_i \Big\{ -2\left( (e^{\Sigma})_{,i} e^{-F_i} \right)_{,i} + e^{\Sigma - F_i} \Big[ (\Sigma_{,i})^2 - \sum_{m \neq i} F_{m,i}^2 \Big] \Big\}}$$

$$\Sigma \equiv \sum_{m \neq i}^{D} F_m$$

Very useful!

The Weyl transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}; \quad \tilde{\Box}\sigma = \Omega^{-2}\left[ \Box\sigma + (D-2) g^{\mu\nu} \frac{\Omega_{,\mu}\sigma_{,\nu}}{\Omega} \right]$$

$$R = \Omega^2 \left[ \tilde{R} + 2(D-1)\tilde{\Box} \ln\Omega - (D-2)(D-1)\tilde{g}^{\mu\nu} \frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2} \right]$$

(used for transforming the string frame $\leftrightarrow$ Einstein frame)

$\sqrt{-g}\, e^{\lambda\phi} R$ - term $\qquad$ $\sqrt{-g}\, R$ - term

This is not possible for $D=2$

But then W. can be used to remove $(\nabla\phi)^2$ term in the D.G.

# K-M-K-F - reduction $(D+p \mapsto D)$ torus

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta + h_{ij}(dy^i + A^i_\alpha dx^\alpha)(dy^j + A^j_\beta dx^\beta)$$

$$G_{\alpha\beta} = g_{\alpha\beta} + h_{ij}A^i_\alpha A^j_\beta; \quad G_{\alpha i} = h_{ij}A^j_\alpha, \quad G_{ij} = h_{ij}$$

$$G^{\alpha\beta} = g^{\alpha\beta}; \quad G^{\alpha i} = -g^{\alpha\beta}A^i_\beta, \quad G^{ij} = h^{ij} + A^{i\alpha}A^j_\alpha$$

$$R[G] = R[g] \underbrace{- \frac{2}{\sqrt{h}}\boxed{g}\sqrt{h}} + \frac{1}{4}\partial_\alpha h^{ij}\partial^\alpha h_{ij} +$$

$$+ \frac{1}{4}(h^{ij}\partial_\alpha h_{ij})(h^{kl}\partial^\alpha h_{kl}) - \frac{h_{ij}}{4}F^i_{\alpha\beta}F^{j\alpha\beta}$$

$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha$$ Abelian g. fields

$$\sqrt{G}\,R[G] = \sqrt{-g}\,\underbrace{\sqrt{h}}_{\text{dilaton factor}}\left\{R[g] - \frac{2}{\sqrt{h}}\boxed{g}\sqrt{h} + \dots\right\}$$

dilaton factor $\quad h_{ij}$ - scalar matter f.

Reduction of other fields:

if $p=1$

$h_{D+1,D+1} \equiv e^{2\sigma}$

$x^{D+1} \equiv \theta$

as $\partial_\theta \psi \equiv 0$, we have for scalar fields $G^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi =$

$= g^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi$. For a 3-form $H^{(D+1)}_{\mu\nu\lambda}$

$$H^{(D+1)}_{\mu\nu\lambda}H^{(D+1)\mu\nu\lambda} = \tilde{H}^{(D)\alpha\beta\gamma}\tilde{H}^{(D)}_{\alpha\beta\gamma} + 3e^{-2\psi}H^{(D)}_{\alpha\beta\theta}H^{(D)\alpha\beta\theta}$$

$$\tilde{H}^{(D)}_{\alpha\beta\gamma} = H^{(D)}_{\alpha\beta\gamma} - (A_\alpha H^{(D)}_{\beta\gamma\theta} + \underset{\text{cyclic}}{[\alpha\beta\gamma]})$$

## From 11d → (1+1)d (SUGRA → DGM)

Bosonic sector:

$$S^{(11)} = \int dx^{11} \left[ \sqrt{-g} \left( R - \frac{1}{48} F_{[4]}^2 \right) + \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right]$$

$F_{[4]} = d A_{[3]}$ $\quad S^1 \to$ compactif $\quad$ IIA $d=10$ $\updownarrow$ T-duality IIB $d=10$

$$\Rightarrow S^{(10)} = \int dx^{10} \sqrt{-g} \underbrace{\left[ R(g) + 4 (\nabla \varphi^{(10)})^2 - \frac{1}{12} H_{[3]}^2 \right] e^{-2\varphi^{(10)}}}_{\text{NS-NS sector of IIA th.}}$$

$\varphi$ – dilaton (the same for IIB)

Further reductions: e.g. K-K, compact. on tori, ...

Brane*) + K.K $\quad ds^2 = g^{(6)}_{\mu\nu} dx^\mu dx^\nu + e^{2\psi} dx^m dx_m$

$$g^{(6)}_{\mu\nu} = \begin{pmatrix} g^{(5)}_{\alpha\beta} + e^{\psi_1} A_\alpha A_\beta & e^{\psi_1} A_\alpha \\ e^{\psi_1} A_\beta & e^{\psi_1} \end{pmatrix} \quad m, n = 6,7,8,9$$

$\psi, \psi_1, F_{\mu\nu} = dA, \bar{F}_{\mu\nu} = H_{\mu\nu x^5}$

$\Rightarrow$ 5-dim gravity + dilaton + matter

$$\mathcal{L}^{(5)} = \sqrt{-g^{(5)}}\, e^{2\varphi_1} \left\{ R^{(5)} + 4(\nabla\varphi)^2 - 4(\nabla\psi)^2 - \frac{1}{4}(\nabla\psi_1)^2 - \frac{1}{12} H'^2 - \frac{1}{4}\left( e^{-\psi_1} \bar{F}^2 + e^{\psi_1} F^2 \right) \right\}$$

$\varphi_1 = \varphi^{(10)} + \psi + \frac{1}{4}\psi_1$

Sph. symm.: $\quad ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + e^{-2\psi_2} d\Omega^2_{(3)}$

$$\Rightarrow \text{DGM}: \mathcal{L}^{(2)} = \sqrt{-g} \left\{ R^{(2)} + (\alpha e^{2\psi_2} + \beta e^{6\psi_2}) + 4(\nabla\varphi)^2 - 4(\nabla\psi)^2 - \frac{1}{4}(\nabla\psi_1)^2 - 3(\nabla\psi_2)^2 - \frac{1}{4}\left( e^{-\psi_1} F^2 + e^{\psi_1} \bar{F}^2 \right) \right\} e^{-2\varphi}$$

*) Rem. $A_{[n-1]} : A_{\mu_1 \dots \mu_{n-1}} = \varepsilon_{\mu_1 \dots \mu_{n-1}} e^{A(r,t)}$ (electric) for spherical symm. reduction on a p-brane (similarly – magnetic field)

- From $\underline{10 \text{ dim}}$ to $\underline{1+1}$

$$\mathcal{L}^{(10)} = \mathcal{L}^{(10)}_{NS} + \mathcal{L}^{(10)}_{RR} + \text{C.S.}$$

$$\mathcal{L}^{(10)}_{NS} = \sqrt{-G}\, \underbrace{e^{-2\varphi_s}}_{\text{dilaton}} \left[ R + 4(\nabla\varphi_s)^2 - \frac{1}{12} H_3^2 \right]$$

$$H_3 = dB_2 \,, \qquad G = \det G_{\mu\nu}$$

II A: $$\mathcal{L}^{(10)}_{RR} = \sqrt{-G} \left[ -\frac{1}{4} F_2^2 - \frac{1}{48} F_4^2 \right]$$

$$F_2 = dA \,, \quad F_4 = dB_3 \,, \quad F_1^2 = (\nabla f)^2 \,, \ldots$$

II B: $$\mathcal{L}^{(10)}_{RR} = \sqrt{-G} \left[ -\frac{1}{2} F_1^2 - \frac{1}{240} F_3^2 \right]$$

Reductions generate scalars, lower-rank forms..

Consider a generic $d$-dim. $\mathcal{L}$.

$$\mathcal{L}^{(d)} = \sqrt{-G} \Big\{ e^{-2\varphi_d} \left[ R + 4(\nabla\varphi_d)^2 - \frac{1}{2}(\nabla\psi)^2 \right] -$$

Note signs!

$$- X_0(\varphi_d, \psi) - X_1(\varphi_d,\psi)(\nabla\sigma)^2 - X_2(\varphi_d,\psi) F_2^2 \Big\}$$

(there may be several $\psi$-fields, $\sigma$-fields, higher-rank forms ($F_3, F_4$) but in 1+1 dim they give similar terms)

Weyl transform.: $$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}$$

$$\downarrow$$

In Einnstein frame:



$$\mathcal{L}_E^{(d)} = \sqrt{-G}\Big[\breve{R}^{(d)} - 4\nu(\nabla\varphi_d)^2 - \frac{1}{2}(\nabla\psi)^2 - $$
$$- e^{2d\varphi_d\nu} X_0 - X_1 e^{2\varphi_d}(\nabla\sigma)^2 - X_2 e^{2\varphi_d\,\nu(d-4)} F_2^2\Big]$$

$\underline{\nu \equiv 1/n}$, $\underline{n \equiv d-2}$ $(\Omega = e^{-2\nu\varphi_d})$

The generic d-dim. $\mathcal{L}$ (after some redef.)

$$\mathcal{L}_E^{(d)} = \sqrt{-G}\Big[R^{(d)} - \frac{1}{2}(\nabla\tilde{\varphi}_d)^2 - \frac{1}{2}(\nabla\psi)^2 - X_0 e^{a_0\tilde{\varphi}_d} - (\nabla\sigma)^2 e^{a_1\tilde{\varphi}_d} X_1$$
$$- X_2 e^{a_2\tilde{\varphi}_d} F_2^2\Big]$$

$(\tilde{\varphi}_d = \sqrt{8\nu}\,\varphi_d)$

Spherical reduction to 1+1 dim:

$$ds_d^2 = g_{ij}\,dx^i dx^j + e^{-4\nu\varphi_2}\,d\Omega_n^2\,, \qquad i,j = 0,1$$

$$\mathcal{L}^{(2)} = \sqrt{-g}\,e^{-2\varphi_2}\Big[R^{(2)} + n(n-1)e^{4\nu\varphi_2} + 4(1-\nu)(\nabla\varphi_2)^2 -$$
$$- X_2 e^{a_2\tilde{\varphi}_d} F_2^2 - \frac{1}{2}(\nabla\tilde{\varphi}_d)^2 - \frac{1}{2}(\nabla\psi)^2 - X_1 e^{a_1\tilde{\varphi}_d} -$$
$$- X_0 e^{a_0\tilde{\varphi}_d}\Big]$$

Weyl: to remove $(\nabla\varphi_2)^2$, $\tilde{\varphi}_d \equiv x$, $e^{-2\varphi_2} \equiv \varphi$

1+1 dim <u>dilaton-gravity-matter</u> (DGM)

$$\Bigg[\mathcal{L} = \sqrt{-g}\Big[\varphi R + n(n-1)\varphi^{-\nu} - X_2 e^{a_2 x}\varphi^{2-\nu} F^2$$
$$- X_0 e^{a_0 x}\varphi^{\nu} - \frac{1}{2}\varphi\big((\nabla x)^2 + (\nabla\psi)^2 + 2X_1 e^{a_1 x}(\nabla\sigma)^2\big)\Big]\Bigg]$$

$X_0, X_1, X_2 \geq 0$ $\qquad n(n-1) \geq 0$

- abstract DGM in 1+1 dim.

$$\mathcal{L} = \sqrt{-g}\,\Big[\overbrace{U(\varphi)R(g) + V(\varphi) + W(\varphi)(\nabla\varphi)^2}^{DG} +$$

$$+ \underbrace{\sum_a X_a(\varphi,\psi)F^2_{(a)}}_{X_a = X_a(\varphi)} + Y(\varphi,\psi) + \sum_n Z_n(\varphi,\psi)(\nabla\psi^{(n)})^2$$

$$\mathcal{F}(\varphi,\psi;F^2_{(a)}) \qquad (\nabla\varphi)_i = \partial_i\varphi = \varphi_i\,;\dots$$

We may solve the eqs for $(F_{(a)})_{ij}$ $\Rightarrow$

$$Y \to Y_{eff} \doteq Y(\varphi,\psi) - \sum X^{(a)}(\varphi,\psi)\,F^{(a)}_{ij}F^{(a)ij} \doteq$$

$$\doteq Y(\varphi,\psi) + \sum 2Q_a^2 / X^{(a)}(\varphi,\psi)$$

Using a special Weyl transform:

$$\mathcal{L} = \sqrt{-g}\,\Big[U(\varphi)R + Y_{eff}(\varphi,\psi) + \sum_n Z^{(n)}(\varphi,\psi)\,g^{ij}\psi_i^{(n)}\psi_j^{(n)}\Big] \quad (*)$$

(including $V(\varphi)$)

1) D.G. is explicitly integrable with arbitrary $U(\varphi), V(\varphi), W(\varphi), X_a(\varphi)$ (or $\mathcal{F}(\underline{\underline{\varphi}}, F_a^2)$)
Reduced to 0+1 dim. D.G.

2) D.G.M. (*) usually not integrable

Exceptions: 1. $Z^{(n)} = -\gamma_n = \text{const}$

$Y_{eff}$ very special -

2. $Y_{eff} = 0$, $Z^{(n)}$ - very special functs. of $\varphi$

Integrable cases are related to <u>Liouville</u>, Toda, ?.

- $\mathcal{L}_{eff} = \sqrt{-g}\left(\varphi R + V(\varphi,\psi) + Z(\varphi,\psi)\cdot(\nabla\psi)^2\right)$

E.O.M. $\underline{\partial_u \partial_v \varphi + f V} = 0$ $\underline{(1)}$

$$\partial_u (Z \partial_v \psi) + \partial_v (Z \partial_u \psi) + f V_\psi = Z_\psi \partial_u \partial_v \quad \underline{(2)}$$

constr. $\left\{ f \cdot \partial_i \left(\frac{\partial_i \varphi}{f}\right) = Z \cdot (\partial_i \psi)^2 \right\}, \; i = u, v;$ $\underline{(3)}$

$\boxed{ds^2 = -4 f(u,v)\, du\, dv \;;}$ $\quad V_\psi \equiv \frac{\partial V}{\partial \psi}, \ldots$

If $V_\psi(\varphi, \psi_0) = 0 \Rightarrow \exists$ solution with

$f = \varphi_u b'(v) = \varphi_v a'(u)$, $\varphi_{,a} = \varphi_{,b}$ $\quad \psi \equiv \psi_0 = \text{const}$

Then $(3) \mapsto \begin{cases} \varphi(u,v) = \varphi(\tau), & \boxed{\tau = a(u) + b(v)} \\ f(u,v) = \varphi'(\tau)\, a'(u)\, b'(v) \end{cases}$

$\varphi''(\tau) + \varphi'(\tau) V(\varphi) = 0 \quad \rightarrow \quad (\varphi'(\tau) + N(\varphi))' = 0 \qquad ' = \frac{d}{d\tau}$

From $(1) \mapsto \underline{\varphi' + N(\varphi) \equiv M}$ (integral of M.)

$$\left[ N(\varphi) \stackrel{\text{def}}{=} \int d\varphi\, V(\varphi, \psi_0) \right]$$

Thus : $\left\{ \begin{aligned} f(u,v) &\equiv h(\tau)\, a'(u)\, b'(v) \equiv \\ &= [M - N(\varphi)]\, a'(u)\, b'(v) \end{aligned} \right.$

$\int \frac{d\varphi}{M - N(\varphi)} = \tau - \tau_0$ defines $\varphi(\tau)$ ∎

$h(\tau) = 0$ or $M - N(\varphi) = 0 \mapsto$ horizon

* Any solution of (1)-(3) with $\psi \equiv \psi_0$ has at least one horizon (for some M)

- DGM in 0+1 dimension ($\psi$ arb.) or $V_\psi(\varphi,\psi_0) \neq 0$

Let $ds^2 = -4f(u,v)\,du\,dv$ (LC coord.)

! Suppose that $\varphi = \varphi(\tau)$, $\psi^{(n)} = \psi^{(n)}(\tau)$

$\boxed{\tau = a(u) + b(v)}$ : $f = h(\tau)\,a'(u)\,b'(v)$

0+1 dim Lagrangian: ($\ell(\tau)$ – Lagrange multiplier)

$$\mathcal{L} = -\frac{1}{\ell}\Big(\dot\varphi\,\frac{\dot h}{h} + \sum_n Z^{(n)}(\varphi,\psi)\,\dot\psi_n^2\Big) + \ell h\, Y_{eff}(\varphi,\psi).$$

* Important note: if $Z^{(n)}$, $Y_{eff}$ do not depend on, say, $\psi_1 \mapsto$ the eq. for $\psi_1$ can be solved and give the eff. potential $\ell C_1 / Z^{(1)}(\varphi,\psi)$, $C_1 = \text{const.}$

(We will have $\ell h Y_{eff}$ and $\ell Y'_{eff}$)

(Pure) dilaton gravity

$$\mathcal{L} = -\frac{1}{\ell}\dot\varphi\,\frac{\dot h}{h} + \ell h V(\varphi) :$$

Define $\begin{cases} \bar\ell = \ell h N'(\varphi) \\ N' = \dfrac{dN}{d\varphi} \\ V(\varphi) = N'(\varphi) \end{cases}$

$$\Rightarrow \mathcal{L} = -\frac{1}{\bar\ell}\dot N \dot h - \bar\ell$$

can be explicitly solved (main trick)

$h = p_N T$, $N = p_h T$, $T = \int \bar\ell(\tau)\,d\tau$

$\boxed{h \sim (M - N(\varphi))}$ Always exists a horizon!

- B.H. and cosmology in a simple example

$D=4$

$Z = \varphi \equiv e^{-2\phi} \equiv r^2 \qquad U = e^{-2\phi}\ ; \ \nu = 2$

$W = 2e^{-2\phi} \qquad \frac{w'}{w} \equiv \frac{W}{U'} = -1, \quad w = e^{-\phi} = r = \sqrt{\varphi}$

In 0+1 dim $\qquad ds^2 = -4h(\underbrace{a+b}_{\tau})\, da\, db$

Integral of m.: $\underbrace{Z\psi = C_0} = \text{const}$

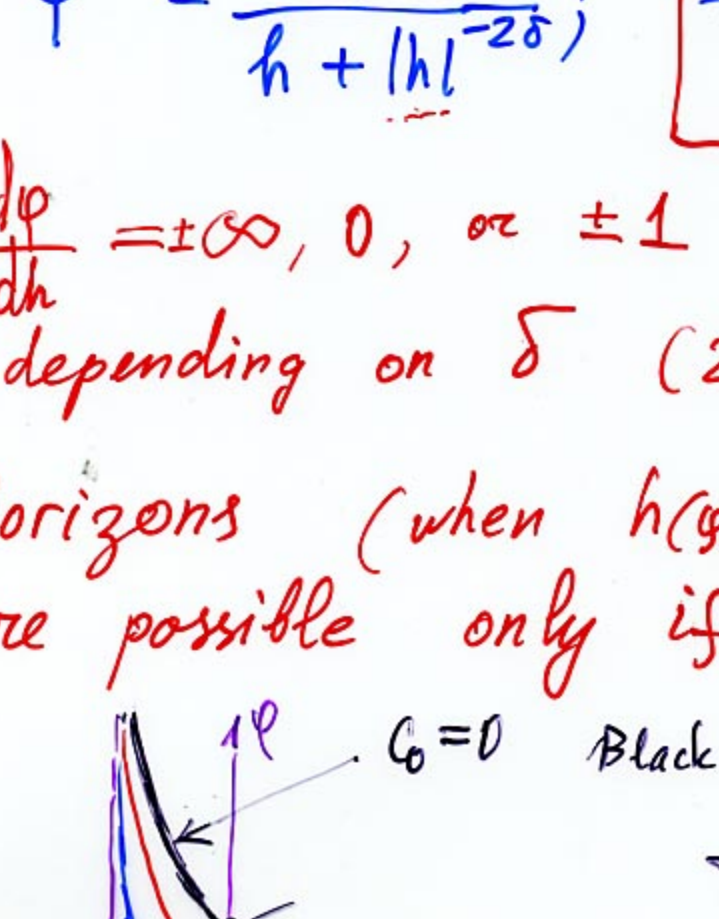
The solution (excluding $\tau$-dependence)
(normalized)

$\varphi = \frac{1}{h + |h|^{-2\delta}}; \quad \left[ \begin{array}{l} -2\delta = 1 - \sqrt{1+2C_0^2} \approx \\ \approx -C_0^2 \text{ for small } \psi \end{array} \right.$

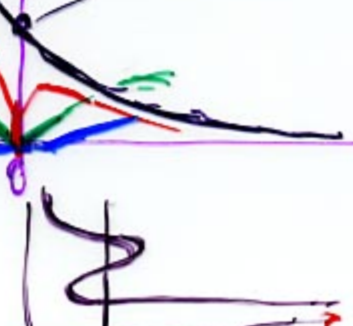
$2\delta > 0$

$\frac{d\varphi}{dh} = \pm\infty,\ 0,$ or $\pm 1$
depending on $\delta$ $(2\delta < 1,\ 2\delta > 1,\ 2\delta = 1)$

Horizons (when $h(\varphi_0) = 0$)
are possible only if $C_0 = 0$




The bifurcation picture of the solutions



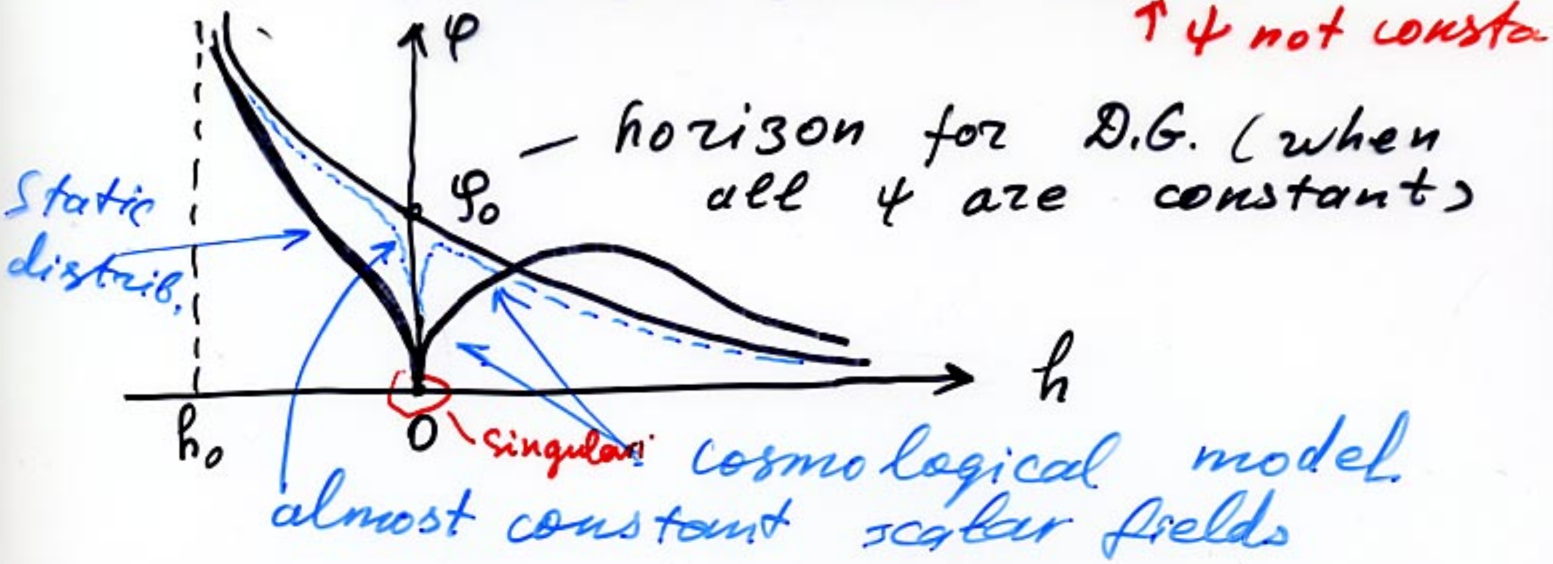
other examples

several horizons

• No horizon theorem

If $Z^{(n)}$, $Y_{eff}$ do not depend on $\psi$, there is no 'horizon' (i.e. no zero of $h$ for finite $\varphi, \psi$)

↑ $\psi$ not consta




When $Z^{(n)}$, $\underline{Y_{eff}}$ depend on $\psi$

the horizons may exist (but have a somewhat different structure)

• In a fairly general (not integrable model) we have the following structure

(A.T.F. + D. Maison)

$$\begin{cases} Y_{eff}(\varphi_0,\psi_0) \neq 0 \\ \text{non degenerate} \end{cases} \begin{cases} h = \underline{h_0}\, x\,(1 + h_1 x + \dots) \\ \varphi = \underline{\varphi_0} + \varphi_1 x + \dots \\ \psi = \underline{\psi_0} + \psi_1 x + \dots \end{cases}$$

$x \equiv \dot{\varphi}(\tau)$

$h(0) = 0$ hor.

• $h_1, h_2, \dots \varphi_1, \varphi_2 \dots$ are derive recursively

• The expansions are (locally) convergent

• Global solutions may be found in integr. mod.

# • Integrable theories in 0+1 dim

## I. N-Liouville models

consider a class of models

$$\mathcal{L} = \frac{1}{l}\left(-\dot{F}\dot{\varphi} - \sum_{n=3}^{N} z_n \dot{\psi}_n^2\right) + l\varepsilon \sum_{n=1}^{N} \frac{1}{2} g_n e^{q_n}$$

$$\boxed{q_n = \sum_{m=1}^{n} \psi_m a_{mn}}, \quad \psi_1 = \frac{1}{2}(F+\varphi), \quad \psi_2 \equiv \frac{1}{2}(F-\varphi)$$

Suppose that $z_n = -1$ $\quad \left[\begin{array}{l} a_{1n} = 1 + a_n \\ a_{2n} = 1 - a_n \end{array}\right.$

Suppose that $a_{mn}$ satisfy the relations $(\varepsilon_1 = -1,\ \varepsilon_m = 1,\ m \geq 2)$

$$\boxed{\sum_{m=1}^{N} \varepsilon_m a_{ml} a_{mn} = \frac{1}{\gamma_n}\delta_{ln}} \quad \text{p-orth.}$$

Then the e.o.m. are reduced to N Liouville equations

$$\boxed{\ddot{q}_n + \bar{g}_n e^{q_n} = 0}, \quad \bar{g}_n = -\frac{g_n}{\gamma_n}\varepsilon$$

$(\bar{g}_n \gtrless 0$ or $\bar{g}_n = 0)$ $\qquad \varepsilon \equiv h/|h|.$

$$\boxed{e^{-q_n} = \frac{|\bar{g}_n|}{2\mu_n^2}\left(e^{\mu_n(\tau-\tau_n)} + e^{-\mu_n(\tau-\tau_n)} + 2\varepsilon_n\right)}$$

$\mu_n^2 \in \mathrm{Re}$ $\qquad \varepsilon_n \equiv \bar{g}_n / |\bar{g}_n|$

The constraint: $\boxed{\sum_{n=1}^{N} \gamma_n \mu_n^2 = 0}$

Moduli: $\mu_1, \ldots, \mu_{N-1};\ \tau_1, \ldots, \tau_{N-1}$

• One and only one $\gamma_n$ is $< 0$ $(\gamma_1)$

$$\sum \gamma_n = 0$$

## • Horizons in N-Liouville.

I If $\underline{|\tau| \to \infty}$, $q_n \to -\infty$ ($\mu_n \in \mathrm{Re}$)
but scalar fields may be finite
for special relations between
moduli $\mu_n$. If $\mu_n = \mu$; [redacted] : $R \to R_0$

($h = e^F$) A) $F \to -\infty$ $(h \to 0)$, $\underline{\phi \to \phi_0}$ – horizon

other possibility: B) $F \to F_0$, $\underline{\phi \to \pm\infty}$ { flat space $R \to$

II. If $\underline{|\tau - \tau_n| \to 0}$, $\varepsilon_n < 0$, $\forall n$
$q_n \to +\infty$ singularity!

However, for $\tau_n \equiv \tau_0$ $\forall \psi_n$ are finite!

$F \to +\infty$, $\phi \to \phi_0$, $R \to R_0$ in a special case $a_n > 0$ reg. even

$\boxed{R^{(2)} = \frac{1}{f}(\log f)_{uv} \underset{\text{'st.'}}{\to} \frac{1}{h}(\log h)''}$ (no singularity and $R$ is finite)

[There exist nonsingular solutions! (Regular Cosmology?)]

It is not yet clear whether
such models may be found
among dim. reduced high-dim.
theories. Also interesting cosmological models!

• There exist also models not reducible to N-Liouville but integrable

Consider the above example

• $$\mathcal{L}^{(1)} = \frac{\varphi}{\ell}\left(-\frac{\dot{\varphi}}{\varphi}\dot{F} + \frac{1}{2}(\dot{x}^2 + \dot{\varphi}^2) + X_1 e^{a_1 x}\dot{\sigma}^2\right)$$
$$+ \underbrace{\frac{\ell}{\varphi}}_{=\bar{\ell}}\varepsilon e^{F}\left(n(n-1)\varphi^{1-\nu} - X_3 e^{a_3 x}\varphi^{1+\nu} - \frac{2Q^2}{X_2}e^{-a_2 x}\varphi^{-(1-\nu)}\right)$$

$\varphi \equiv e^{\phi}$ and integrate out $X_1$-term ($\sigma$-term)

• $$\mathcal{L}^{(1)}_{eff} = \frac{1}{\bar{\ell}}\left(-\dot{\phi}\dot{F} + \frac{1}{2}(\dot{x}^2 + \dot{\varphi}^2)\right) + \bar{\ell}\varepsilon e^{F}\left(n(n-1)e^{(1-\nu)\phi} - X_3 e^{a_3 x + (1+\nu)\phi} - \frac{2Q^2}{X_2}e^{-a_2 x - (1-\nu)\phi}\right)$$
$$+ \bar{\ell}\left(-\frac{c_1^2}{X_1}e^{-a_1 x}\right)$$

Note a different $F$-dependence

Such theories may be integrable if equiv. to N-Liouville or Toda (or to some degenerate theories)

Otherwise, they are (most probably) NOT INTEGRABLE.

23

Integrability is a miracle (explained by some symm)

# 1+1 dimensional integrable $\mathcal{N}$ - Liouville theories

Let $Z_n = \text{const}$ $(Z_n = -1)$

$$\mathcal{L} = \varphi\, \partial_u \partial_v \ln|f| - Z\, \partial_u \psi^{(n)} \partial_v \psi^{(n)} + f\, V(\varphi, \psi^{(n)}) \qquad V = \sum \varepsilon g_n e^{q_n - \ln|f|}$$

$$\ln|f| \equiv F\,, \quad \varepsilon = \operatorname{sgn} f\,, \quad q_n = F + a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn}$$

$$(*) \quad C_i \equiv f\, \partial_i (\partial_i \varphi / f) + \sum_{n=3}^{N} (\partial_i \psi_n)^2 = 0\,, \quad i = u, v$$

$$\boxed{\partial_u \partial_v q_n - \tilde{g}_n e^{q_n} = 0} \quad \text{Liouville.}$$

$$\left[\tilde{g}_n = \varepsilon \lambda_n g_n\,; \quad \lambda_n = \sum \varepsilon_m a_{mn}^2\,; \quad \varepsilon_1 = -1\,, \ \varepsilon_m = +1,\ m \geq 2\right.$$

The conditions on $a_n, a_{mn}$ (orthogonality) are the same as in 0+1 dimensional case

Solving the constraints (*) is a non-trivial problem

They can be solved because the norms $\lambda_n \equiv \sum \varepsilon_m a_{mn}^2$ satisfy the constraint

$$\boxed{\sum_{n=1}^{N} \lambda_n^{-1} = 0. \qquad (!)}$$

Let $\underline{e^{-q_n/2} \equiv X_n}$ Liouville is equiv. to

$$\left[ -X_n \partial_u \partial_v X_n + \partial_u X_n \partial_v X_n = \lambda_n \frac{\varepsilon g_n}{2} \equiv \frac{\tilde{g}_n}{2} \right.$$

$X_n = a_n(u) b_n(v) + \bar{a}_n(u) \bar{b}_n(v)$ , $a_n, b_n$ - arbitr

$$\bar{a}_n = -a_n(u) \int \frac{w_a^{(n)}}{a_n^2(u)} du \qquad \bar{b}_n = -b_n(v) \int \frac{w_b^{(n)}}{b_n^2(v)} dv$$

Constraints: $C_i = \sum_{m=1}^{N} \gamma_m \left( q_{m,i}^2 - 2 q_{m,ii} \right)$ , $i = u, v$

$$\gamma_m = \frac{1}{\lambda_m}$$

$$C_u = 4 \sum_{m=1}^{N} \gamma_m \frac{a_m''(u)}{a_m(u)} = -4c^2(u) \quad (\text{or, } c(u) \equiv 0)$$

$$C_u = 4 \sum \gamma_m \left[ \left( \frac{a_m'}{a_m} \right)' + \left( \frac{a_m'}{a_m} \right)^2 \right]$$

Let $\dfrac{a_n'}{a_n} = \rho_n(u) - X(u)$ Then $C_i = -c^2(u)$

↑ arbitrary

if and o. if

$$\boxed{X = \frac{1}{2} \left[ \ln \left| \sum \gamma_n \rho_n \right| \right]' + \frac{1}{2} \frac{\sum \gamma_n \rho_n^2}{\sum \gamma_n \rho_n} + c^2}$$

$$\rightarrow \quad a_n = \exp \int [\rho_n(u) - X(u)] \, du$$

(actually, we have $N-1$ arbitrary functions, not $N$ !)

- e.g. for $N=2$, $a_1$ and $a_2$ are defined in terms of $\rho_1(u) - \rho_2(u)$

# Cosmological reduction from 1+1 to 1+0

$$ds_2^2 = e^{2\alpha} dr^2 - e^{2\gamma} dt^2$$

$$ds_4^2 = e^{2\alpha}(dr^2 + S_k^2(r)\, d\Omega_2^2) - e^{2\gamma} dt^2,$$

i.e. $e^{2\beta} = S_k^2(r)\, e^{2\alpha(t)}$ under cosm reduction

So, forget about 4 dim. and consider, instead of the naive reduction $\alpha = \alpha(t)$, $\gamma = \gamma(t)$, $\beta = \beta(t)$ the following: $\begin{cases} \beta = \alpha(t) + \ln \underline{S_k(r)} \\ \alpha = \alpha(t), \quad \gamma = \gamma(t) \end{cases}$

$S_k = \frac{\sin\sqrt{k}r}{\sqrt{k}}$

The gravit. part of the reduced Lag.:

$$\mathcal{L}_{g^{(2)}}^{(2)} = \sqrt{|g|}\left\{ e^{-2\phi}\left[R^{(2)} + 2e^{2\phi} + 2g^{ij}\phi_{,i}\phi_{,j}\right] - \right.$$
$$- 2e^{-(\alpha+2\beta+\gamma)}\left((e^{2\beta})' e^{\gamma-\alpha}\right)' + 2e^{-(\alpha+2\beta+\gamma)}\left((e^{2\beta})^{\bullet} e^{\alpha-\gamma}\right)^{\bullet}$$

where $2\phi = -2\beta \mapsto -2\alpha(t) - 2\ln S_k(r)$

If we neglect total derivatives ($\sqrt{|g|}e^{-2\phi} \equiv e^{\alpha+2\beta+\gamma}$ !) we may get something reasonable (but not completely correct) only for $k = +1$. The correct result:

$$\sqrt{g}\{\ldots\} \mapsto 6ke^{\alpha+\gamma} - 6\dot{\alpha}^2 e^{3\alpha-\gamma} + 2\left((e^{3\alpha})^{\bullet} e^{-\gamma}\right)^{\bullet}$$

# The simplest possible repr. of the solutions

Take arbitrary $\mu_n(u)$, $\nu_n(v)$ satisfying the constraints

$$\sum \gamma_n \mu_n^2(u) = 0 = \sum \gamma_n \nu_n^2(v)$$

Then the general solution is

$$\begin{cases} a_n(u) = \left(\sum \gamma_m \mu_m\right)^{-1/2} \exp\left(\int \mu_n(u)\, du\right) \\ b_n(v) = \left(\sum \gamma_m \nu_m\right)^{-1/2} \exp\left(\int \nu_n(v)\, dv\right) \end{cases}$$

$$e^{-q_n/2} \equiv X_n(u,v) = a_n(u) b_n(v) + \bar{a}_n(u) \bar{b}_n(v)$$

$$\text{where} \begin{cases} \bar{a}_n = c_n a_n(u) \int \frac{du}{a_n^2(u)} \\ \bar{b}_n = D_n b_n(v) \int \frac{dv}{b_n^2(v)} \end{cases} \quad c_n / D_n = \text{fixed otherwise arbitrary}$$

In terms of $a_n, b_n$ one may give an interesting classification of the Liouville solutions.

They are also convenient for quantizing having a group th. meaning.