Numerical computation of physical observables at two-loop level

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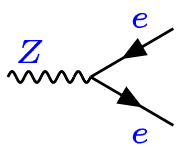
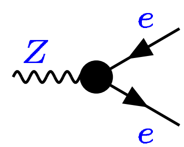
“CALC-2006”

The path to physical observables

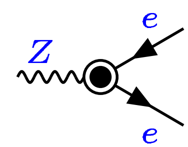
- Start from the renormalized Langrangian at 2-loop level



Normal vertices



1-loop counterterms



2-loop counterterms

- Generate the diagrams for the process
- Introduce projectors
- Compute the trace of Dirac matrices

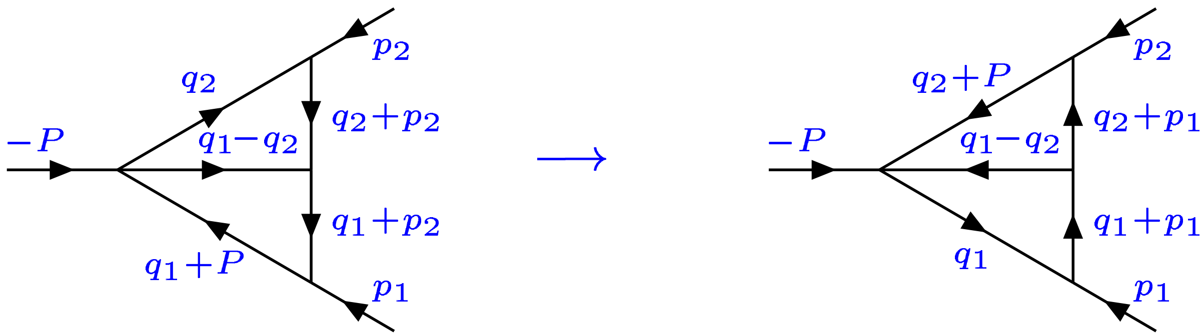
$$\Downarrow$$

All loop momenta are contracted with other momenta

- Recursive application of:
  - Obvious reduction:

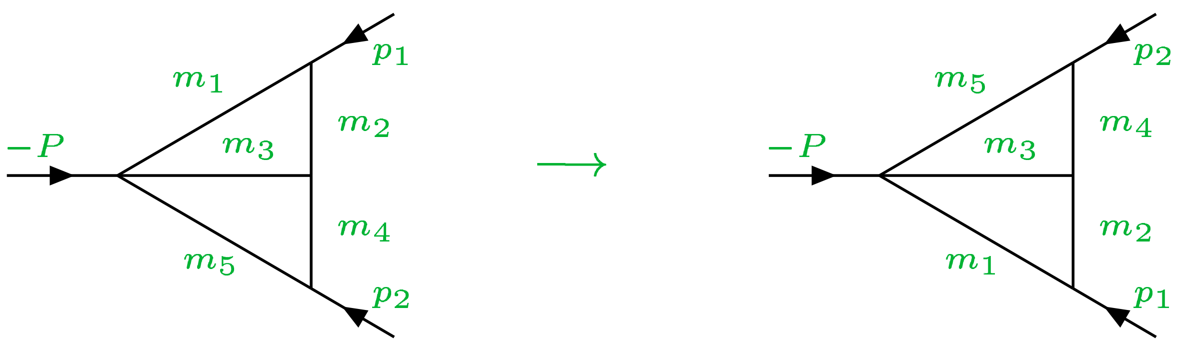
$$\frac{2\,q.p}{(q^2+m^2)\,[(q+p)^2+M^2]} = \frac{1}{q^2+m^2} - \frac{1}{(q+p)^2+M^2} - \frac{p^2-m^2+M^2}{(q^2+m^2)\,[(q+p)^2+M^2]}$$

  - Mapping on a fixed standard routing for loop momenta:



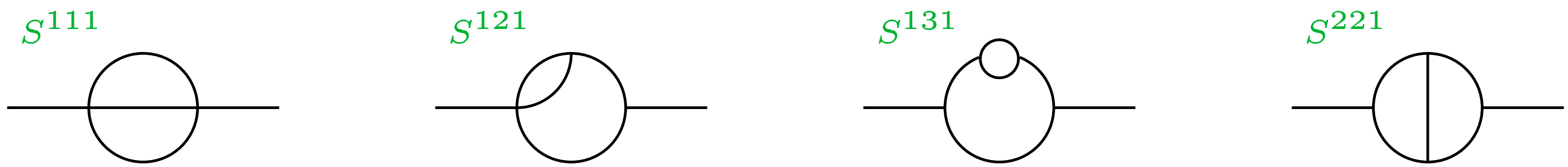
$$\begin{pmatrix} q_1 \to -q_1 - P \\ q_2 \to -q_2 - P \end{pmatrix}$$

  - Symmatrization:

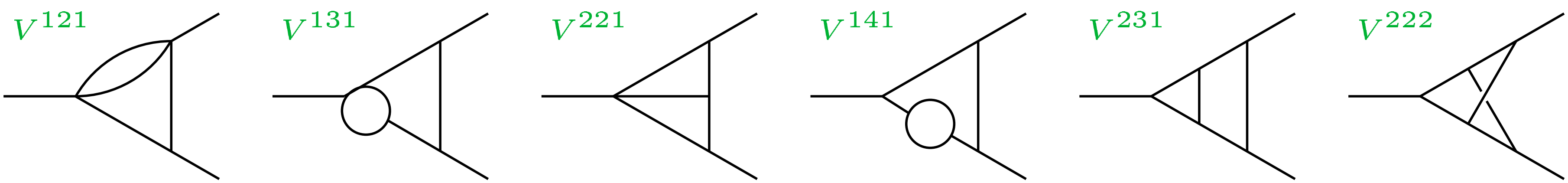


$$\begin{pmatrix} q_1 \to -q_2 - P \\ q_2 \to -q_1 - P \end{pmatrix}$$

- We end with integrals up to rank 3:
  - 1-loop functions
  - 2-loop self-energies (4 topologies)



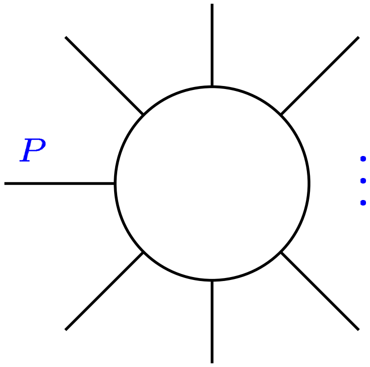
  - 2-loop vertices (6 topologies)



- Full scalarisation $\rightarrow$ possible for 1-loop diagrams and 2-loop self-energies
- For all, few scalar products are remaining

Numerical evaluation in parametric space

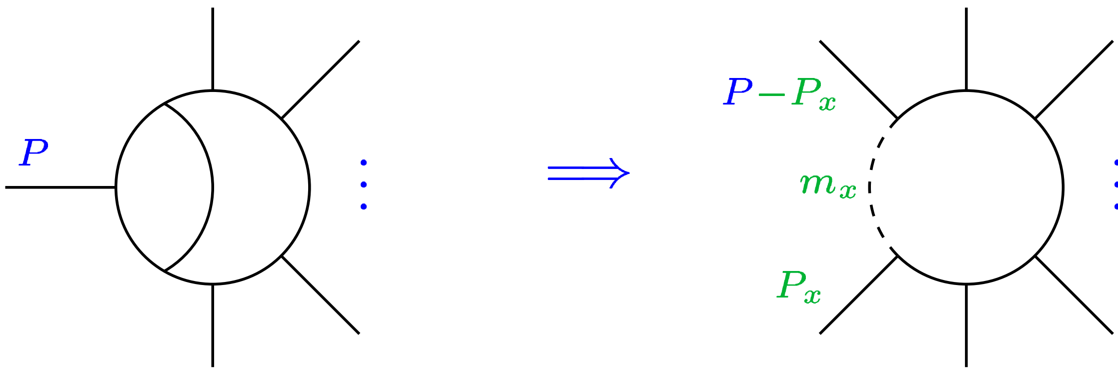
- One-loop diagrams



$$G_N = \int dS_{N-1}(x)\, Q(x)\, V(x)^{2-N-\epsilon/2}$$

$$V(x) = x^t H x + 2K^t x + L$$
$$\int dS_n(x) = \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n,$$

- Two-loop diagrams

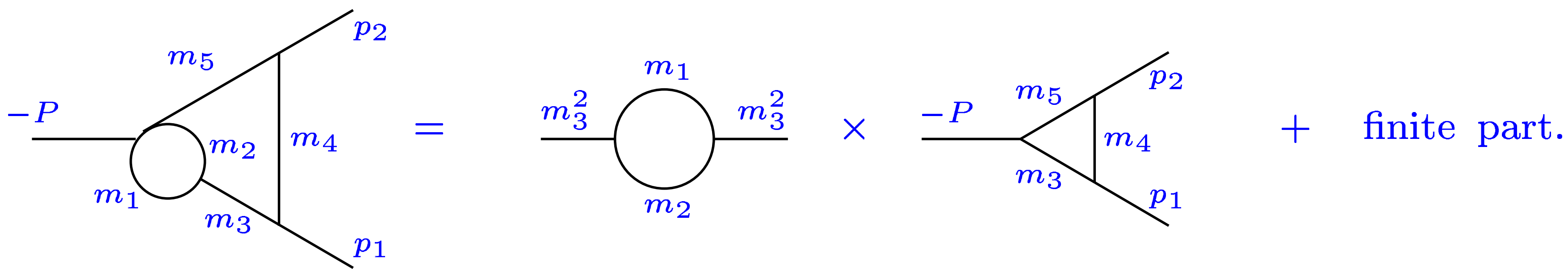


$$G_{acb} = \int dS_{a+b-1}(x)\, dS_c(y)\, [x_a(1-x_a)]^{2-a-b-\frac{\epsilon}{2}}\, Q(x,y)\, y_c^{a+b-3+\frac{\epsilon}{2}}\, V_x(y)^{4-a-b-c-\epsilon}$$

$$V_x(y) = y^t H_x y + 2K_x^t y + L_x$$

Extraction of the UV poles

- UV poles of 1-loop diagrams → trivial ($\Gamma(\epsilon/2)$)
- UV poles of 2-loop diagrams:
  - Overall divergency → trivial ($\Gamma(\epsilon)$)
  - Singularities coming from sub-loops → hidden in the integrand
  - The single pole can always be expressed in terms of 1-loop functions.



Write the **finite part** in one of the following forms:

1. $\int dx \, \frac{Q(x)}{V(x)}$ $V(x)$ polinomial positive definite
2. $\frac{1}{B} \int dx \, Q(x) \ln^n V(x)$ B constant $\neq 0$.
3. $\int dx \, \frac{Q(x)}{V(x)} \, f\left(\frac{V(x)}{P(x)}\right)$ $f(0) = 0,$ $f(x) = ln^n(1+x),\; Li_n(x),\; S_{n,p}(x)$

Bernstein-Sato-Tkachov (BST) theorem for quadratics

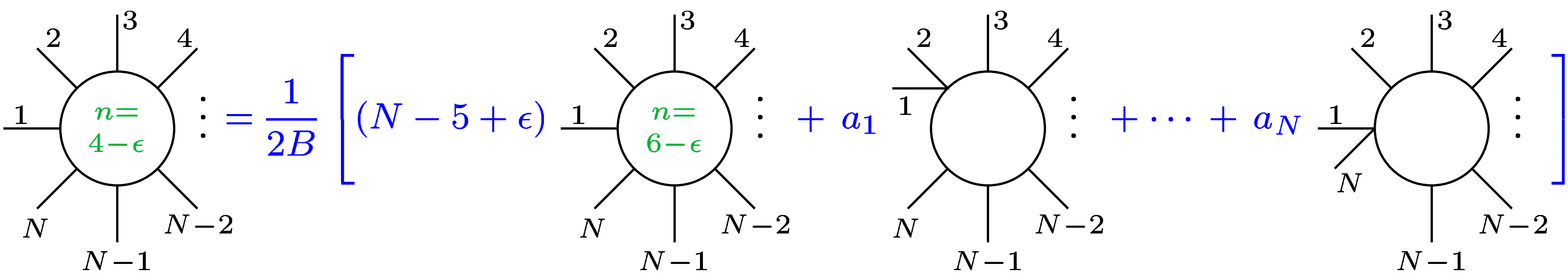
$$V^{\mu}(z) = \frac{1}{B}\left(1 + \frac{\mathcal{P}^t\,\partial_z}{\mu+1}\right)V^{\mu+1}(z), \qquad \mathcal{P} = -\frac{z-Z}{2}$$

$$V(z) = z^t H z + 2K^t z + L, \qquad Z = -K^t H^{-1}, \qquad B = L - K^t H^{-1} K$$

Procedure:

- Raise the power of V to a positive value by iterative application of the theorem and integration by parts
- ϵ-expansion and numerical integration of the smooth integrals

Example: scalar N-point function



A new BST relation

$$V(z) = z^t\, H\, z + 2\, K^t\, z + L = (z^t - Z^t)\, H\, (z - Z) + B = Q(z) + B$$

It can be easily proven that:

$$\mathcal{P}^t \partial_z\, Q(z) = -Q(z), \quad \mathcal{P} = -\frac{z-Z}{2}, \qquad V^\mu(z) = \left(\beta - \mathcal{P}^t\, \partial_z\right) \int_0^1 dy\, y^{\beta-1} \left[Q(z)\, y + B\right]^\mu$$

If $\mu = -1$ and $\beta = 1$

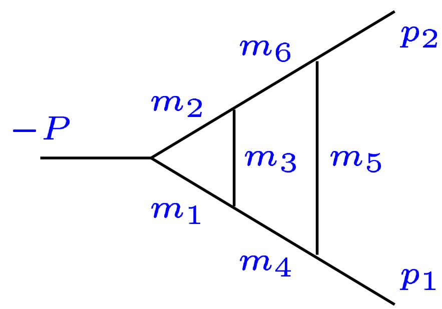
$$V^{-1} = \left(1 - \mathcal{P}^t\, \partial_z\right) \frac{1}{Q}\, \ln\left(1 + \frac{Q}{B}\right)$$

Example: scalar 3-point function

$$C_0 = \sum_{i=1}^{3} \frac{a_i}{2} \int_0^1 dx_1\, \frac{1}{V[i](x_1) - B}\, \ln \frac{V[i](x_1)}{B}, \qquad a_1 = 1 - Z_1, \quad a_2 = Z_1 - Z_2, \quad a_3 = Z_2.$$

$V[i]$ is the polynomial of the two-point function obtained by shrinking to a point the i^{th} propagator.

The V^{231} diagram



After Feynman parametrisation we get:

$$V^{231} = C_\epsilon \int dS_2(x_1, x_2) \int dS_3(y_1, y_2, y_3)\, [x_2\,(1-x_2)]^{-1-\epsilon/2}\, y_3^{\epsilon/2}\, U^{-2-\epsilon}$$

$$\begin{aligned} U \quad = \quad & -[p_2\, y_1 - P\,(y_2 - X\, y_3) + p_1]^2 + (P^2 - p_1^2 + m_6^2 - m_5^2)\, y_1 \\ & -(P^2 + m_6^2 - m_4^2)\, y_2 + (M_x^2 - m_4^2)\, y_3 + p_1^2 + m_5^2 \end{aligned}$$

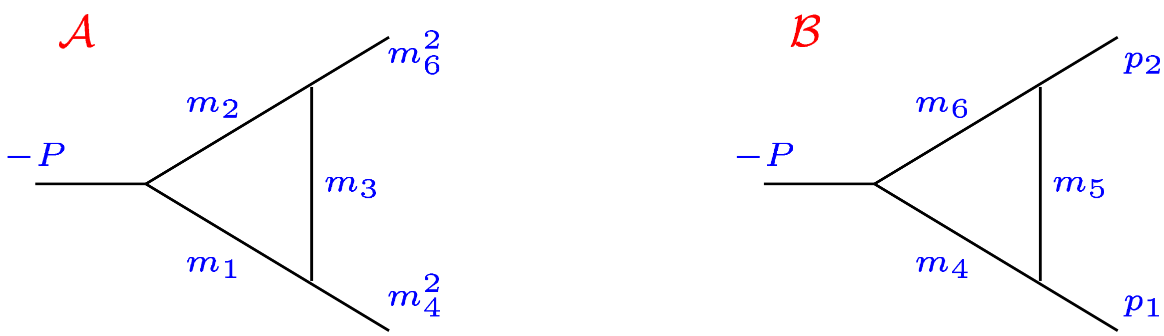
$$X = \frac{1-x_1}{1-x_2}, \qquad M_x^2 = \frac{-P^2\, x_1^2 + x_1\,(P^2 + m_1^2 - m_2^2) + x_2\,(m_3^2 - m_1^2) + m_2^2}{x_2\,(1-x_2)}.$$

- Transform $y_2 \to y_2 + X\, y_3$
- Set $\epsilon = 0$
- Perform the y_3 integration.

We obtain

$$V^{231} = \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^1 dy_1 \frac{1}{\mathcal{A}(x)} \left\{ \int_0^{\bar{X}\,y_1} dy_2 \left[\frac{x_2\,(x_1 - x_2)}{y_2\,\mathcal{A}(x) + x_2\,(x_1 - x_2)\,\mathcal{B}(y)} - \frac{1}{\mathcal{B}(y)} \right] + \int_{\bar{X}\,y_1}^{y_1} dy_2 \left[\frac{x_2\,(1 - x_1)}{(y_1 - y_2)\,\mathcal{A}(x) + x_2\,(1 - x_1)\,\mathcal{B}(y)} - \frac{1}{\mathcal{B}(y)} \right] \right\}$$

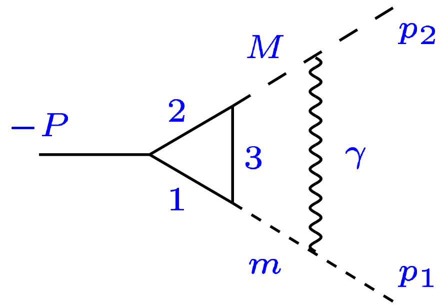
$\mathcal{A}$ and $\mathcal{B}$ are quadratics of one-loop triangles:



- Each integral in y_1 and y_2 is a C_0 function, ...
- ... whose arguments dipend on x_1 and x_2.
- Numerical computation of C_0 functions.

Infrared singularities

- Endpoint singularites in parametric space
- The B of the BST relations vanishes
  - Ex.: $V_{\rm IR}^{231}$



$\longrightarrow$ B for $\mathcal{B}(y_1,y_2)$ is 0

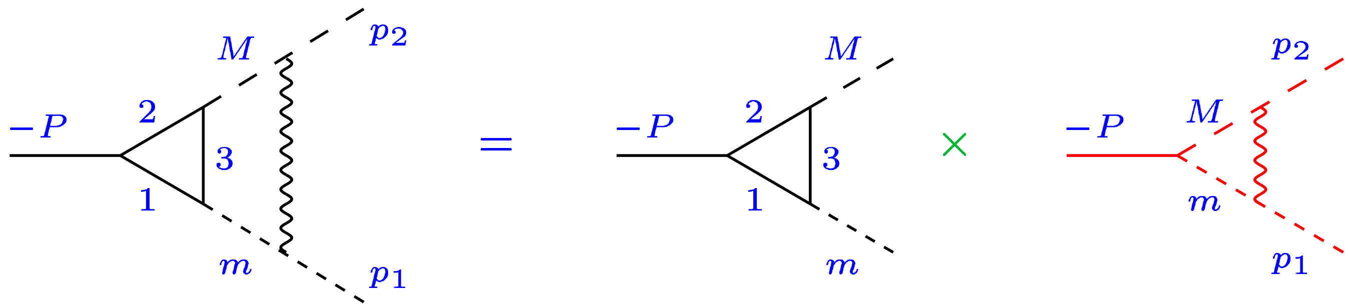
  - Keep $\epsilon \neq 0$
  - Use hypergeometric functions

$$V_{\rm IR}^{231} = C_\epsilon \sum_{i=1}^{2} \int dS_2(x_1,x_2) \int_0^1 dy_1 \int_{\alpha_i}^{\beta_i} dy_2 \left[\frac{a_i}{x_2(1-x_2)\,\mathcal{B}(y_1,y_2)^2} \right]^{1+\epsilon/2}$$

$${}_2F_1(2+\epsilon\,,\,1+\frac{\epsilon}{2}\,;\,2+\frac{\epsilon}{2}\,;\,-\frac{a_i\,\mathcal{A}(x_1,x_2)}{x_2(1-x_2)\mathcal{B}(y_1,y_2)})$$

$$\alpha_1 = 0, \quad \beta_1 = \bar{X} y_1, \quad a_1 = \frac{y_2}{\bar{X}}, \qquad \alpha_2 = \bar{X} y_1, \quad \beta_2 = y_1, \quad a_2 = \frac{y_1 - y_2}{X}$$

- Using the properties of ${}_2F_1$, we get:



$$+ \sum_{i=1}^{2} \int dS_2(x) \int_0^{X_i} dy_2 \, \frac{1}{\chi_i \mathcal{A}} \ln\left[1 + \frac{x_2(1-x_2) X_i \chi_i(y_2)}{y_2\, \mathcal{A}(x)}\right]$$

$$X_1 = \bar{X}, \quad X_2 = X, \qquad \chi_1(y) = P^2 y(1-y) + M^2(1-y) + m^2 y, \quad \chi_2(y) = \chi_1(1-y)$$

- Infrared pole contained in a 1-loop diagram
- Last terms: well-behaved for $\chi_i \to 0$, but not for $\mathcal{A} \to 0$.

$$\Downarrow$$

Another new BST relation (for $\mathcal{A}$)

$$\frac{1}{V} \ln\left(1 - \frac{P}{V}\right) = \frac{1}{B}\left[\frac{1}{P} \ln\left(1 - \frac{P}{V}\right) (1 + \mathcal{P}_1^t\, \partial_z)\, P + \mathcal{P}_1^t\, \partial_z \, \mathrm{Li}_2\left(\frac{P}{V}\right)\right]$$

Collinear divergencies

- They come from the coupling of light particles with massless particles
  Ex: V_{IR}^{231} with m and/or $M \ll |P|$.
- Sometimes the collinear logs can be extracted directly from the result
- Subtraction method

$$I = \int_0^1 dx \frac{1}{ax^2+bx+c} \ln\left(1 + \frac{ax^2+bx+c}{d\,x}\right), \qquad c \to 0$$

Add and subtract:

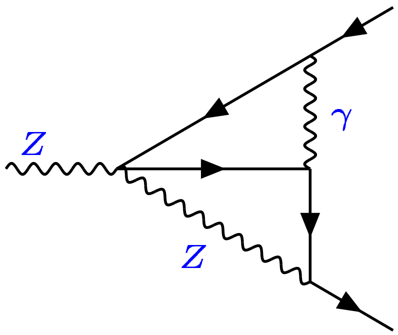
$$I_0 = \int_0^1 dx \frac{1}{bx+c} \ln\left(1 + \frac{bx+c}{d\,x}\right)$$

- Set $c = 0$ in $I - I_0$ → well behaved

$$I - I_0 = \int_0^1 dx \frac{1}{x}\left[\frac{1}{ax+b} \ln\left(1 + \frac{ax+b}{d}\right) - \frac{1}{b} \ln\left(1 + \frac{b}{d}\right)\right] + \mathcal{O}(c)$$

- Remaining I_0 → integrate explicity → $\ln(c)$

If the collinear divergency is connected to more variables, as for:



$\longrightarrow$ Collinear logs in the fermion mass

Start from the expression $\int dS(x,y)\, V(x,y)^{-1}$

$$\int_0^1 dx\,dy \frac{1}{xy\,a(x,y)+\lambda\,b(x,y)} = \int_0^1 dx\,dy \left\{ \left.\frac{1}{xy\,a(x,y)+\lambda\,b(x,y)}\right|_{x,y} + \left.\frac{1}{xy\,a(x,0)+\lambda\,b(x,0)}\right|_{x} + \left.\frac{1}{xy\,a(0,y)+\lambda\,b(0,y)}\right|_{y} + \frac{1}{xy\,a(0,0)+\lambda\,b(0,0)} \right\}, \qquad \lambda \to 0$$

$$f(z)|_z = f(z) - f(z)|_{z^2=\lambda z=0}$$

- First term $\rightarrow$ set $\lambda = 0$
- Second (third) term $\rightarrow$ integrate in y (x) $\rightarrow$ $\ln(\lambda)$
- Last term $\rightarrow$ integrate in x and y $\rightarrow$ $\ln^2(\lambda)$

Two-loop corrections to $\sin^2\theta_{\rm eff}^{\rm lept}$

- Two-loop electroweak fermionic corrections

  M. Awramik, M. Czakon, A. Freitas and G. Weiglein, hep-ph/0407317 (July 2004)

  W. Hollik, U. Meier and S.U., hep-ph/0507158 (July 2005)

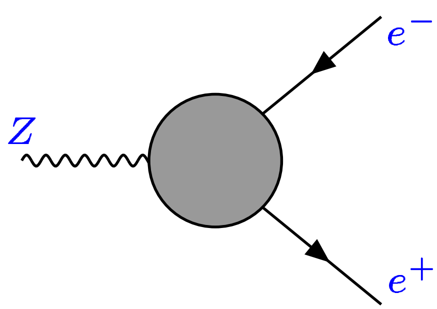
- Two-loop electroweak corrections with Higgs-mass dependence

  W. Hollik, U. Meier and S.U., hep-ph/0509302 (Sept. 2005)

- Two-loop electroweak bosonic corrections

  M. Awramik, M. Czakon and A. Freitas, hep-ph/0605339 (June 2006)

$$\sin^2\theta_{\rm eff}^{\rm lept} = \frac{1}{4}\left[1 - {\rm Re}\left(\frac{g_v}{g_a}\right)\right] = \left(1 - \frac{m_W^2}{m_Z^2}\right)\kappa \qquad s = m_Z^2$$



$$= \quad \bar{u}_l\,\gamma_\mu\,(g_v + g_a\,\gamma_5)\,v_l\,\epsilon_Z^\mu$$

$$\sin^2\theta^l_{eff} = \left(1 - \frac{M_W^2}{M_Z^2}\right)\kappa,$$

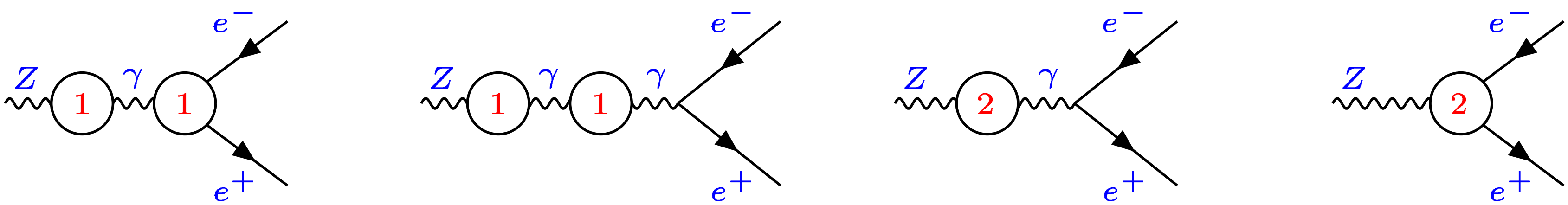
- Corrections on M_W^2

$$M_W^2\left(1 - \frac{M_W^2}{M_Z^2}\right) = \frac{\pi\alpha}{\sqrt{2}G_F}\left(1 + \Delta r\right),$$

- Corrections on κ

$$\kappa = 1 + \Delta\kappa, \qquad g_{v,a} = g^0_{v,a}\left(1 + g^1_{v,a} + g^2_{v,a}\right)$$

$$\Delta\kappa^{(1)} = \frac{g^0_v}{g^0_v - g^0_a}\left(g^1_v - g^1_a\right), \qquad \Delta\kappa^{(2)} = \frac{g^0_v}{g^0_v - g^0_a}\left[-g^1_a\left(g^1_v - g^1_a\right) + g^2_v - g^2_a\right]$$

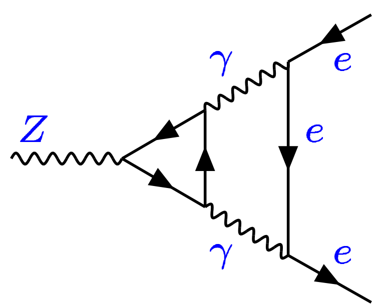


The γ_5 problem

- In 4 dimensions: $\{\gamma_\mu, \gamma_5\} = 0, \qquad \mathrm{Tr}\{\gamma_\mu\,\gamma_\nu\,\gamma_\rho\,\gamma_\sigma\,\gamma_5\} = 4\,\epsilon_{\mu\nu\rho\sigma}$
- In D dimensions:
  naive prescription $\rightarrow \quad \{\gamma_\mu, \gamma_5\} = 0, \qquad \mathrm{Tr}\{\gamma_\mu\,\gamma_\nu\,\gamma_\rho\,\gamma_\sigma\,\gamma_5\} = 0$

The difference between naive and consistent treatement is finite and can be evalueted in 4 dimension (Freitas-Hollik-Walter-Weiglein, hep-ph/0202131).

We keep the electron mass as physical regulator for collinear divergency in diagram such as:



$m_H[GeV]$	$\Delta\kappa^{(1)}[\times 10^{-4}]$	$\Delta\kappa^{(2)}_{fer}[\times 10^{-4}]$	$\Delta\kappa^{(2)}_{fer,sub}[\times 10^{-4}]$	$\Delta\kappa^{(2)}_{bos,sub}[\times 10^{-4}]$
100	438.937	-0.637(1)	0	0
200	419.599	-2.165(1)	-0.528	0.265
600	379.560	-5.012(1)	-4.375	0.914
1000	358.619	-4.737(1)	-4.100	1.849

$$\Delta\kappa_{sub}(m_H) = \Delta\kappa(m_H) - \Delta\kappa(m_H = 100\,GeV)$$

$m_H[GeV]$	$\Delta m^{(\alpha^2)}_{W,bos}[MeV]$	$\sin^2\theta^{sub}_{\rm eff}(\Delta m_{W,bos})[\times 10^{-5}]$	$\sin^2\theta^{sub}_{\rm eff}(\Delta\kappa_{bos})[\times 10^{-5}]$
100	-1.0	0	0
200	-0.5	-0.97	0.59
600	-0.1	-1.74	2.03
1000	0.6	-3.10	4.11

$$\sin^2\theta^{sub}_{\rm eff}(m_H) = \sin^2\theta_{\rm eff}(m_H) - \sin^2\theta_{\rm eff}(m_H = 100\,GeV)$$

State of the art

- Algebraic manipulation implemented in the FORM code **Graphshot** (S.Actis, G.Passarino).
- Fortran codes to compute 2-loop self-energies and vertices (IR and not IR) $\rightarrow$ available (link to Graphshot to be done)
- Extraction of collinear logs $\rightarrow$ partially done (not always needed for numerical stability)
- Next process under examination $H \rightarrow \gamma\gamma$