Introduction to Mellin-Barnes Representations

for Feynman Integrals

Tord Riemann, DESY, Zeuthen

CALC, July 2006, Dubna

- **Introduction: 2-loop QED contributions to Bhabha scattering**
- **Barnes' contour integrals for the hypergeometric function**
- **Loop momentum integrations with Feynman parameters for L-loop n-point functions**
- **Representation by Mellin-Barnes integrals**
- **Treatment of divergencies in $d = 4 - 2\epsilon$ (MB package)**
- **Numerical evaluations, nested infinite series, approximations, and all that**
- **Summary**

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Introduction: 2-loop QED contributions to Bhabha scattering

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \to e^+e^-)$$

We are using a scheme with

(1) $m_e \neq 0$ **(good with the MC's)**

(2) $m_\gamma = 0$ **(bad with the MC's;** $\to$ **Mastrolia, Remiddi 2003)**

(3) dim.reg. for UV and IR divergences

Also:

Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij: all but many 2-boxes

G. Heinrich, V. Smirnov: Calculation of selected complicated Feynman integrals

.

There are few topologies only:

- **1-loop: 5**
- **2-loop self energies: 5 (3 for external legs)**
- **2-loop vertices: 5**
- **2-loop boxes: 6** $\longrightarrow$ **next slides**

The many Feynman integrals may be reduced to 'few' master integrals by sophisticated methods (Remiddi-Laporta algorithm, 1996/2000 $\rightarrow$ IdSolver (Czakon 2003)).

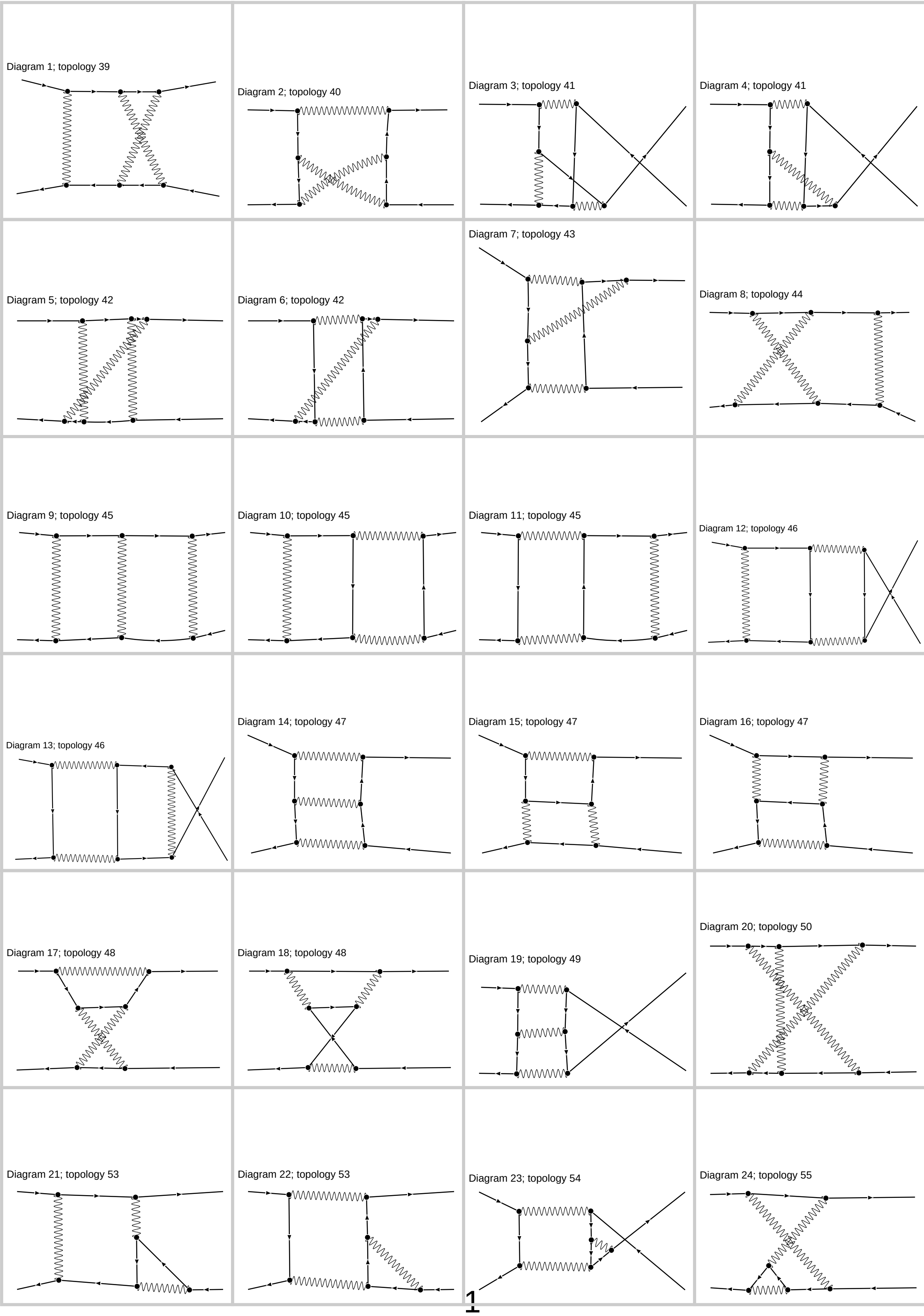
The massive diagrams (See also http://www-zeuthen.desy.de/theory/research/bhabha)

Assume 3 leptonic flavors, do not look at loops in external legs.

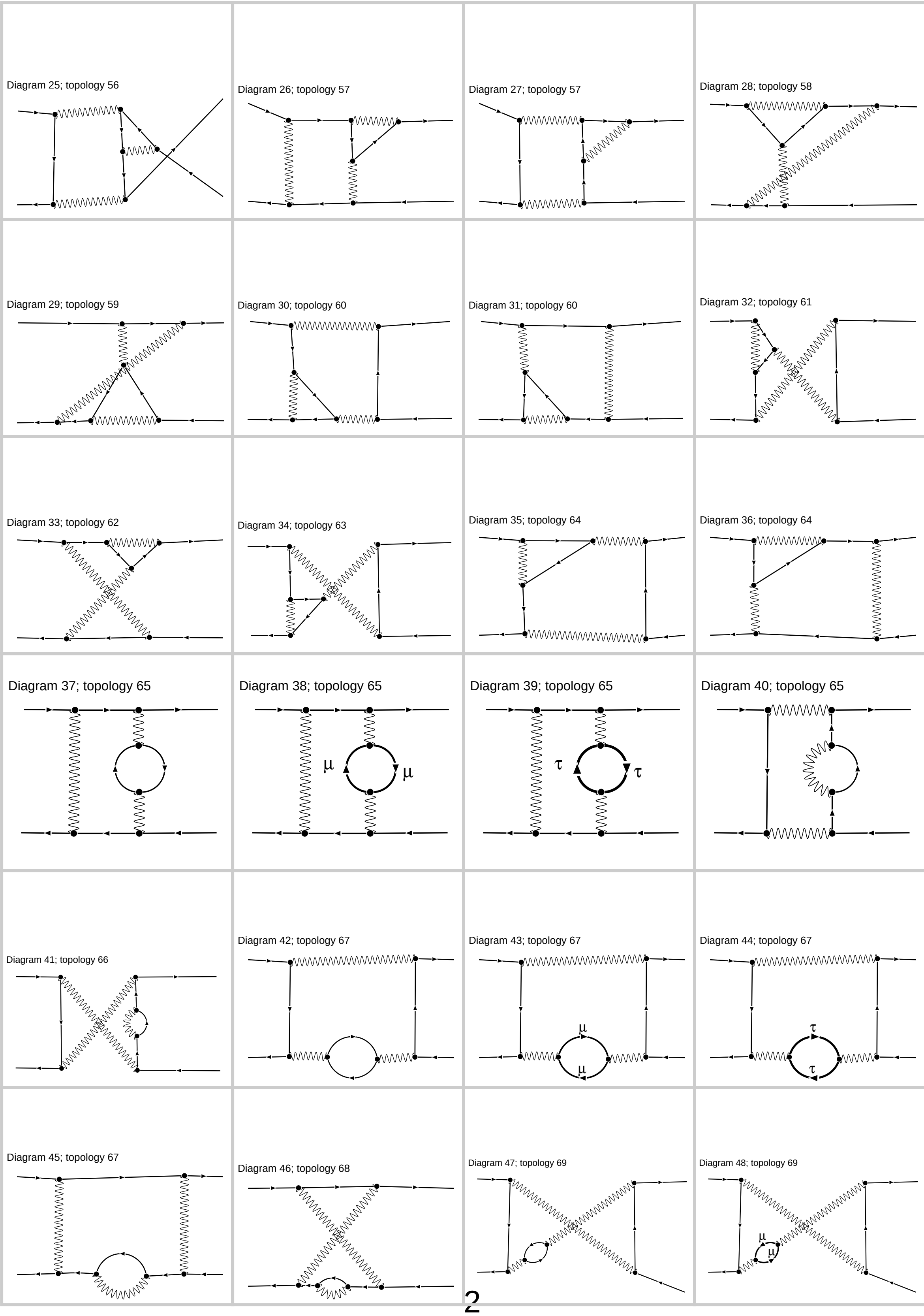
Not too many QED diagrams:

- **Born diagrams: 2**
- **1-loop diagrams: 14**
- **2-loop diagrams: 154 (with 68 double-boxes)** **interfere with Born**

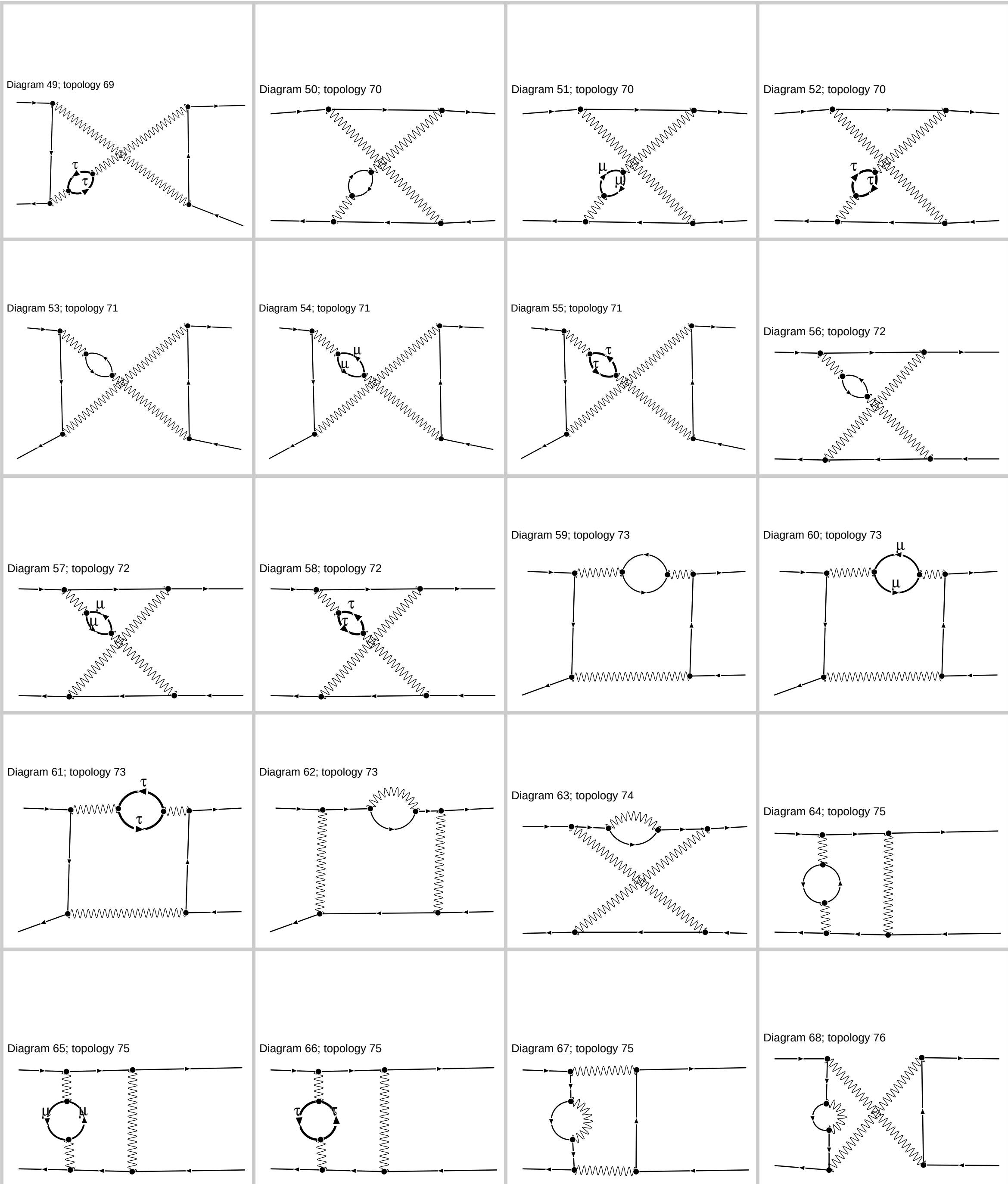
Irreducible two-loop diagrams: 1/3



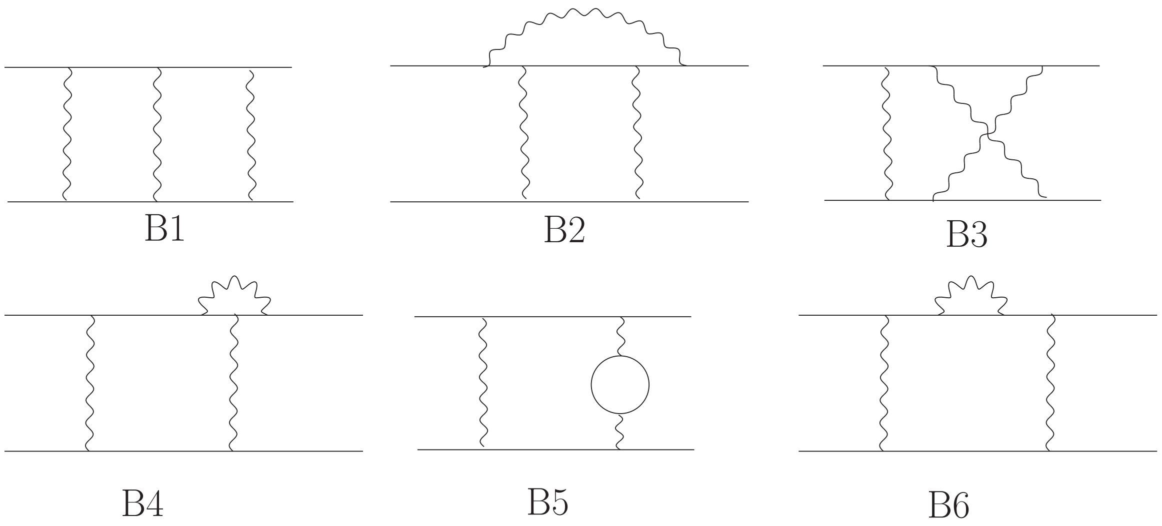
Irreducible two-loop diagrams: 2/3



Irreducible two-loop diagrams: 3/3

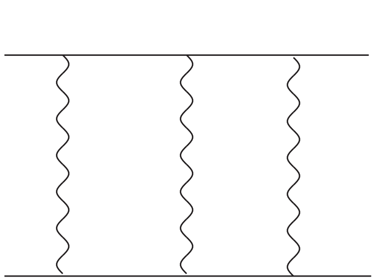


The two-loop box diagrams for massive Bhabha scattering

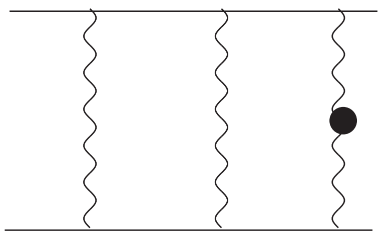


- B5: $\longrightarrow$ 5-line masters + simpler, completely known (2004)
  Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321
  Czakon, Gluza, Riemann: *http://www-zeuthen.desy.de/.../MastersBhabha.m* (unpubl.)
- B1–B3: $\longrightarrow$ 7-line masters + simpler, few masters known (Smirnov, Heinrich 2002,2004; for all planar masters the small mass limit: Czakon et al. 2006)
- B4, B6: $\longrightarrow$ planar 6,5-line masters + simpler small mass limit known (Czakon et al. 2006)

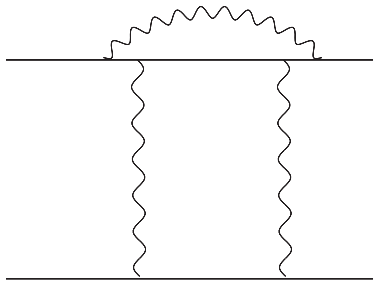
The basic planar 2-box master of B1, B7l4m, was a breakthrough



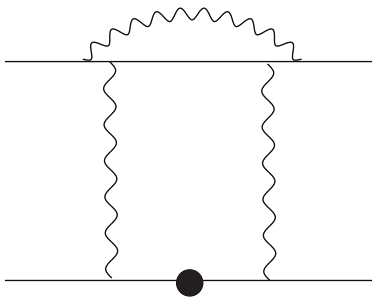
B7l4m1



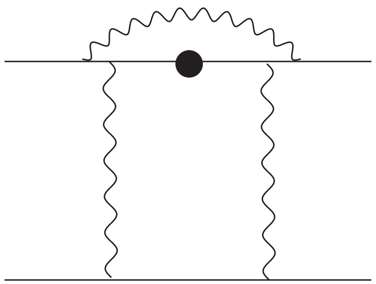
B7l4m1d



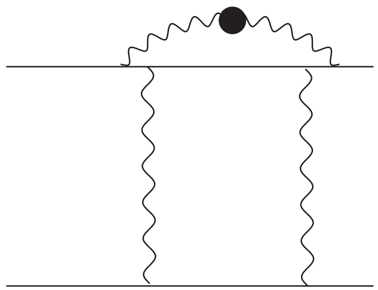
B7l4m2



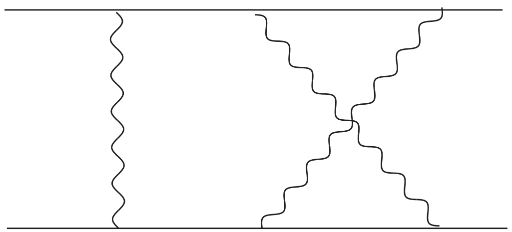
B7l4m2d1



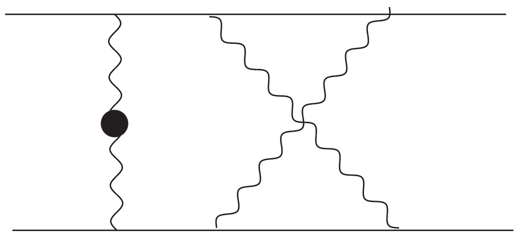
B7l4m2d2



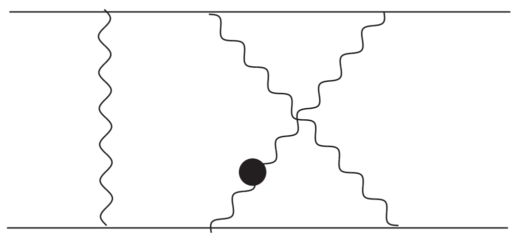
B7l4m2d3



B7l4m3

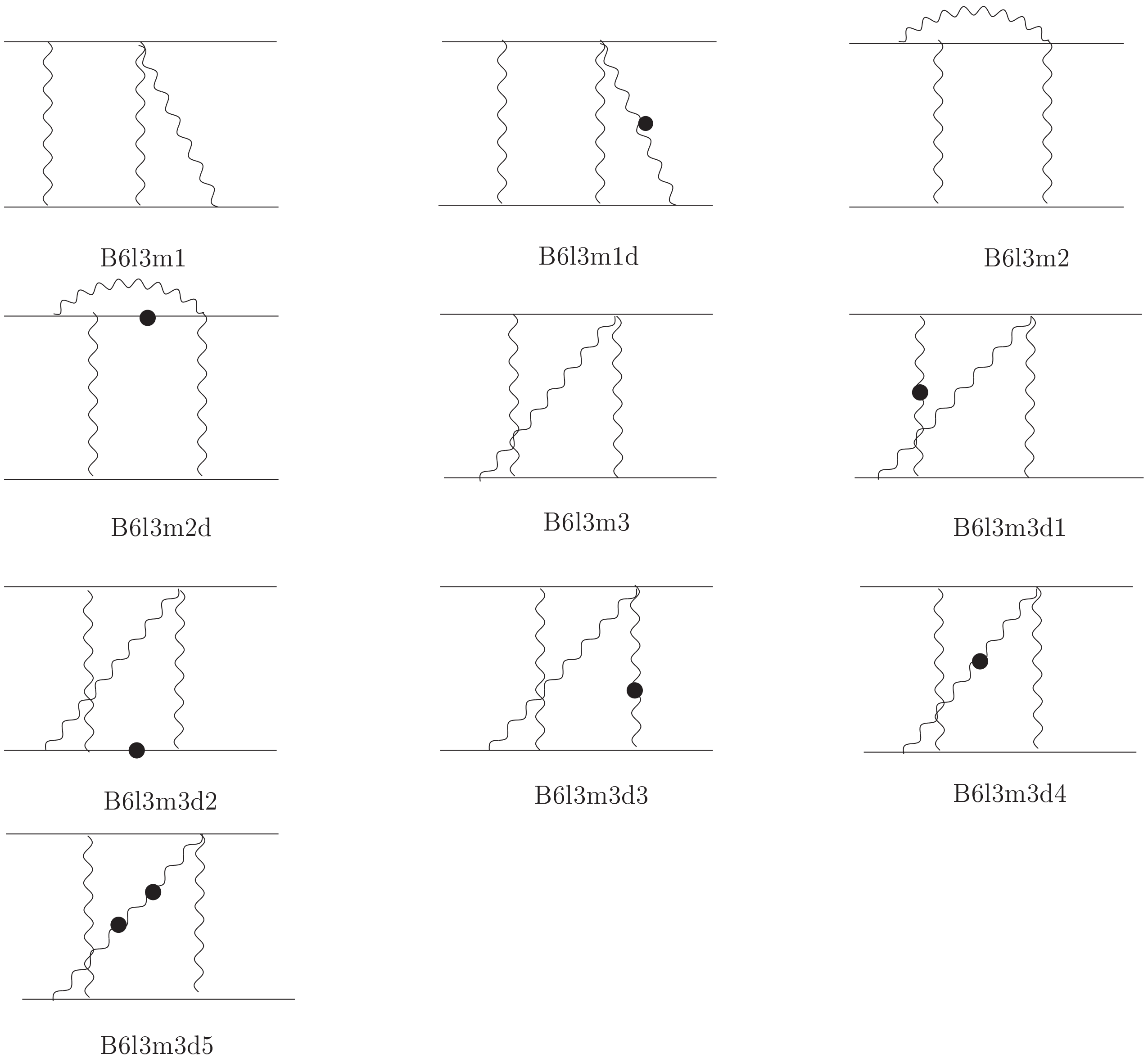


B7l4lm3d1

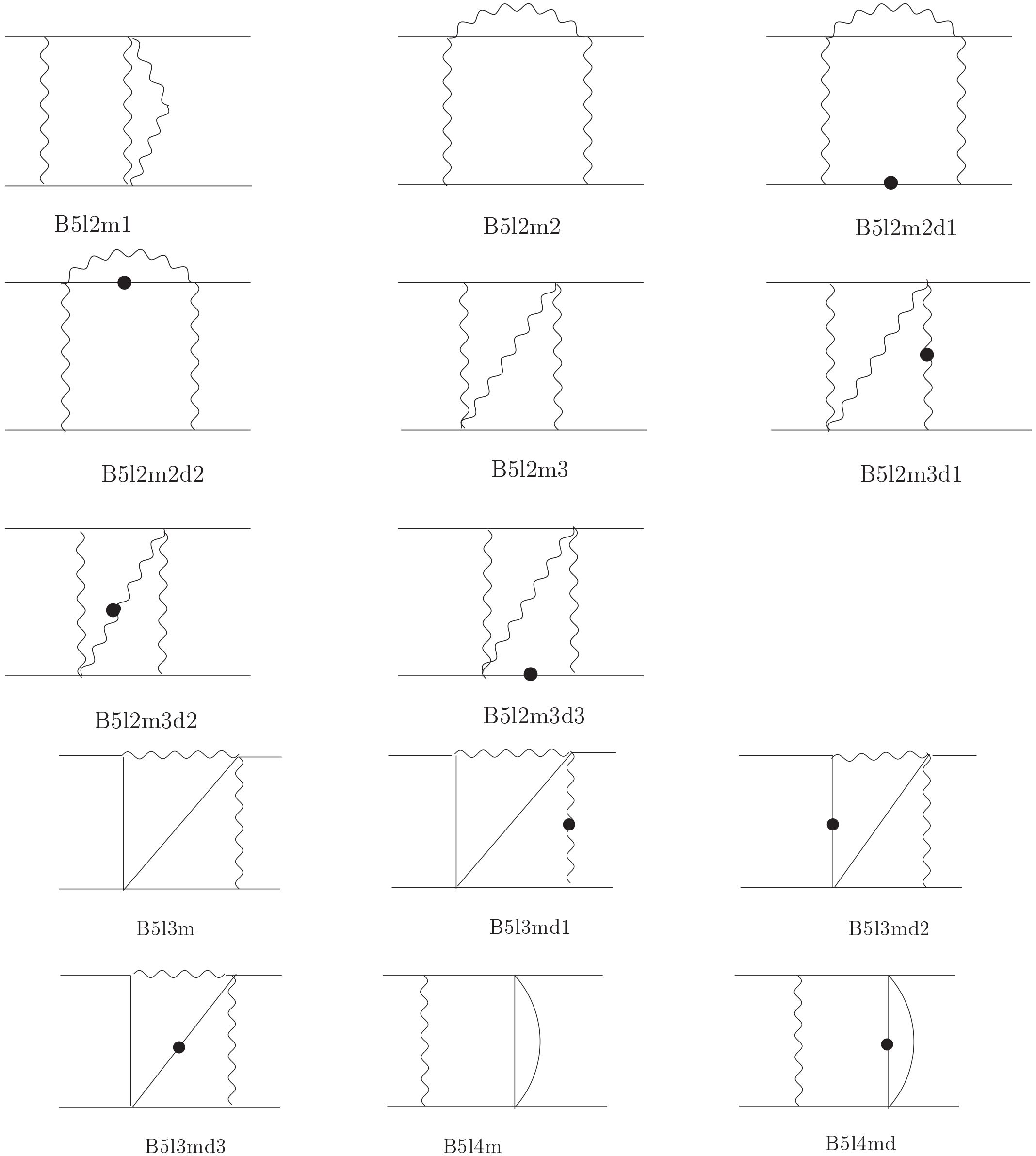


B7l4m3d2

The nine two-loop box MIs with seven internal lines.



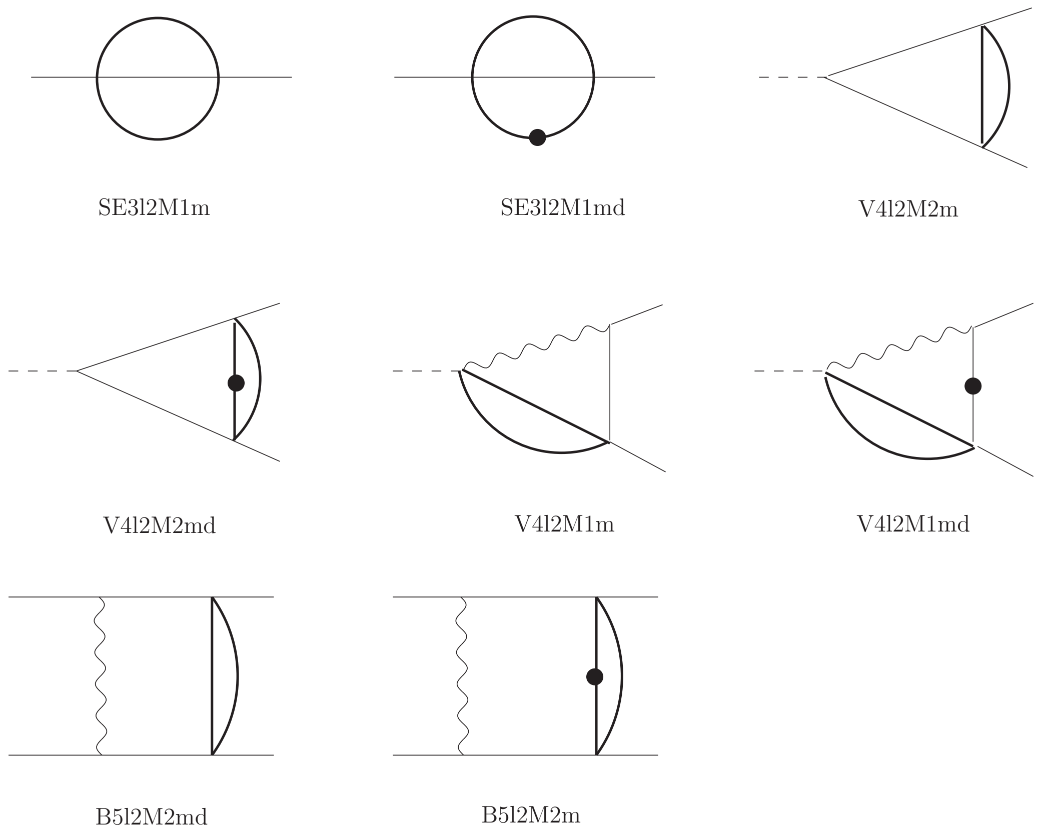
The ten two-loop box MIs with six internal lines.



The 15 two-loop box MIs with five internal lines.

The $N_f > 1$ contributions

Actis, Czakon, Gluza, TR, under study



The eight additional master integrals with two different mass scales.

These 2-box-diagrams represent a three-scale problem: $s/m^2, t/m^2, M^2/m^2$

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Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) \quad = \quad \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)}$$

where $|\mathrm{arg(-z)}| < \pi$ **(i.e.** $(-z)$ **is not on the neg. real axis) and the path is such that it separates the poles of** $\Gamma(a+\sigma)\Gamma(b+\sigma)$ **from the poles of** $\Gamma(-\sigma)$**.**
$1/\Gamma(c+\sigma)$ **has no pole.**
Assume $a \neq -n$ **and** $b \neq -n, n = 0, 1, 2, \cdots$ **so that the contour can be drawn.**

The poles of $\Gamma(\sigma)$ **are at** $\sigma = -n, n = 1, 2, \cdots$**, and it is:**

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ **the**

hypergeometric function ${}_2F_1(a,b,c,z)$ **(for proof see textbook):**

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma(-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N\to\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {}_2F_1(a,b,c,z)$$

The continuation of the hypergeometric series for $|z|>1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n} + \sum_{n=0}^{\infty}\frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {}_2F_1(a,b,c,z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}\ {}_2F_1(a,1-c+a,1-b+ac,z^{-1}) + \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b}\ {}_2F_1(b,1-c+b,1-a+b,z^{-1})$$

Corollary I

Putting $b = c$, we see that

$$
{}_2F_1(a,b,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!}
$$

$$
= \frac{1}{(1-z)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma\ (-z)^\sigma\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{B^{-a}}{(1-(-A/B))^{-a}} = \frac{B^{-\nu}}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma\ B^{-a-\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$ lie on the right of the path and the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ lie on the left, then

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) \quad = \quad \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

It is supposed that a,b,c,d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a+b+c+d-1)<0$. Take sum of residues of the integrand at poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of sin, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a+b+c+d-1)<0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a,b,c,d for which none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$.

Corollary II Any real shift k: $\sigma+k, a-k, b-k, c+k, d+k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can this be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which may allow to perform the (massless) momentum integral (with index a of the line changed to $(a+\sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get a single monomial in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\ \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

Loop momentum integrations with Feynman parameters for L-loop n-point functions

Consider an arbitrary L-loop integral $G(X)$ with loop momenta k_l, with E external legs with momenta p_e, and with N internal lines with masses m_i and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \dots d^D k_L \; X(k_1, \dots, k_L)}{D_1^{\nu_1} \dots D_i^{\nu_i} \dots D_N^{\nu_N}}.$$

$$D_i = q_i^2 - m_i^2 = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e]^2 - m_i^2$$

The numerator may contain a tensor structure

$$X = (k_1^{\alpha_1} \cdots k_L^{\beta_L}) \; (p_{e_1}^{\alpha_1} \cdots p_{e_L}^{\beta_L})$$

Some numerators are reducible, i.e. one may divide them out against the numerators a la:

$$\frac{2kp_e}{D_1[(k+p_e)^2 - m^2] \dots D_N} \equiv \frac{[(k+p_e)^2 - m^2] - [k^2 - m_1^2] + (m^2 - m_e^2)]}{D_1[(k+p_e)^2 - m^2] \dots D_N}$$

$$= \frac{1}{D_1 \dots D_N} - \frac{1}{[(k+p_e)^2 - m^2] \dots D_N} + \frac{m^2 - m_e^2}{D_1[(k+p_e)^2 - m^2] \dots D_N}$$

For a two-loop QED box diagram, it is e.g. $L = 2$, $E = 4$, and we have as potential simplest numerators: $k_1^2, k_2^2, k_1 k_2$ and $2E$ products $k_1 p_e, k_2 p_e$ compared to N internal lines, $N = 5, 6, 7$.
This gives $I = L + L(L-1)/2 + 2E - N$ irreducible numerators of this type:
$I(N) = 9 - N = 4, 3, 2$ here.
This observation is of practical importance: imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals.
Which momenta combinations are irreducible is partly dependent on the choice of momenta conventions (and fixed by that) and partly dependent on choice.
Message: When evaluating all F.I. by MB-integrals, one should consider numerator integrals, and it is not too complicated compared to scalar ones.
Now introduce Feynman parameters:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}},$$

with $N_\nu = n_1 + \ldots n_N$.
The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(n_1+\ldots+n_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.

The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1})Q, \\ m^2 &= \bar{k}M\bar{k} - QM^{-1}Q + J. \end{aligned}$$

Remember:

$$M^{-1} = \frac{1}{(\text{det } M)} \ \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$I_k(1) = \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$I_k(1) \to (i)^L \int \frac{Dk_1^E \ldots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) = -(J - QM^{-1}Q)$$

and get

$$I_k(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals - will use only those - we are ready. For L-loops go on and now diagonalize the matrix M by a rotation:

$$\begin{aligned} k \to k'(x) &= V(x)\, k, && (1) \\ kMk &= k' M_{diag} k' && (2) \\ &\to \sum \alpha_i(x) k_i^2(x), && (3) \\ M_{diag}(x) &= (V^{-1})^+ M V^{-1} = (\alpha_1, \ldots, \alpha_L). \end{aligned}$$

This leaves both the integration measure and the integral invariant:

$$I_k(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(\sum_i \alpha_i k_i^2 + \mu^2)^{N_\nu}}.$$

Rescale now the k_i,

$$\bar{k}_i = \sqrt{\alpha_i} k_i,$$

with

$$\begin{aligned} d^D k_i &= (\alpha_i)^{-D/2} d^D \bar{k}_i, && (4) \\ \prod_{i=1}^{L} \alpha_i &= \det M, \end{aligned}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \to k$):

$$I_k(1) = (-1)^{N_\nu} (i)^L (\det M)^{-D/2} \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$i^L \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} = \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - DL/2}}, \tag{5}$$

$$i^L \int \frac{Dk_1 \ldots Dk_L \; k_1^2}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} = \frac{d}{2} \frac{\Gamma\left(N_\nu - \frac{D}{2}L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - DL/2 - 1}}.$$

These formulae follow for $L = 1$ immediately from any textbook.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Finally, one gets:

$$\begin{aligned} G(1) &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{(\det\, M)^{-D/2}}{(\mu^2)^{N_\nu - DL/2}}, \\ &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2}}{F(x)^{N_\nu - DL/2}} \end{aligned} \tag{6}$$

with

$$\begin{aligned} U(x) &= (\det\, M) \; (\rightarrow 1 \text{ for } L = 1) \\ F(x) &= (\det\, M)\, \mu^2 = -(\det\, M)\, J + Q\, \tilde{M}\, Q \; (\rightarrow -J + Q^2 \text{ for } L = 1) = \sum A_{ij} x_i x_j. \end{aligned}$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand, $k \to \bar{k} + U(x)\tilde{M}Q$, the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu-1-D(L+1)/2}}{F(x)^{N_\nu-DL/2}} \left[\sum_l \tilde{M}_{1l} Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha}k_{2\beta}) \quad = \quad & (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu-2-D(L+1)/2}}{F(x)^{N_\nu-DL/2}} \\ & \times \sum_l \left[[\tilde{M}_{1l}Q_l]_\alpha [\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{D}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{D}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x)F(x) \frac{(V_{1l}^{-1})^+(V_{2l}^{-1})}{\alpha_l} \right]. \end{aligned}$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu-D/2}}$$

Come back to the evaluation of Bhabha boxes . . .

. . . and look at B7l4m2, the second planar double box:

Perform the momentum and Feynman parameter integrations first for the subloop with k_1, then for the second.

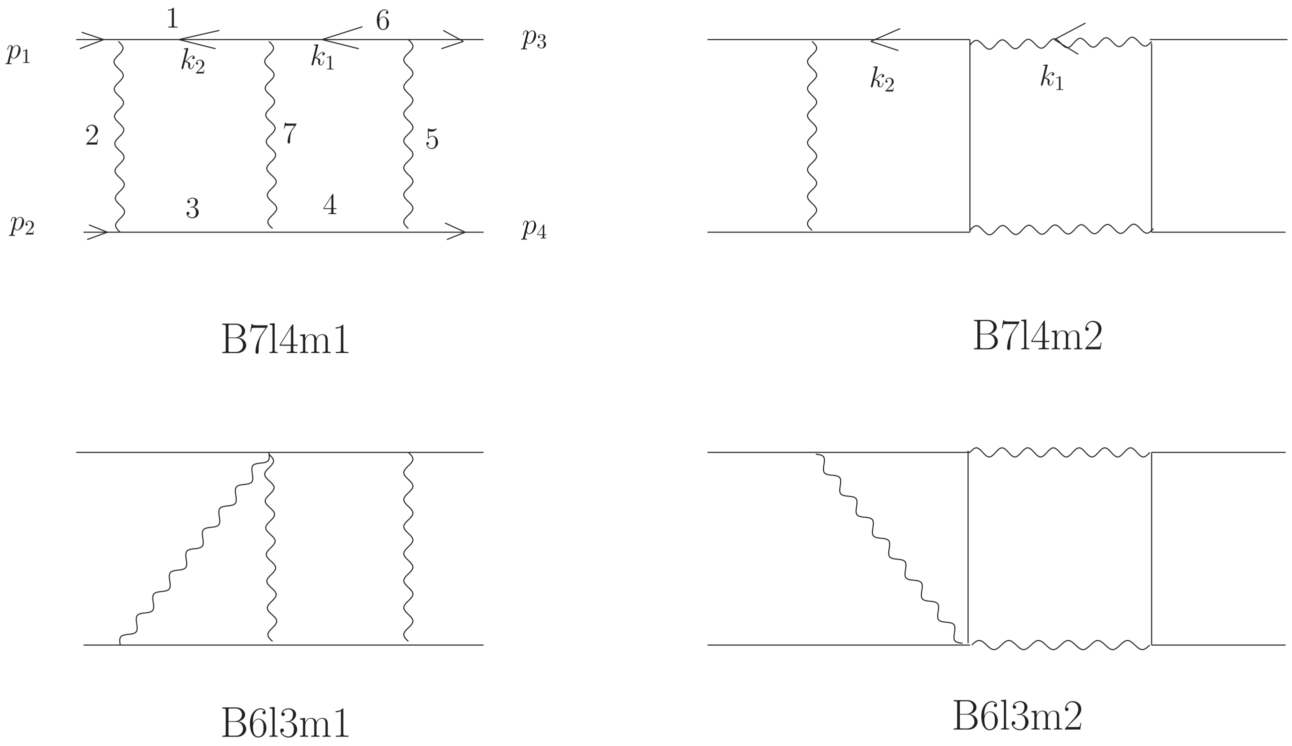

Figure 1: The planar 6- and 7-line topologies.

Integrating the Feynman parameters – get MB-Integrals

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box (for momenta see next page):

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \Pi_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \\ +(m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

We want to apply now:

$$\int_0^1 \prod_{j=4}^{7} dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-x_4-x_5-x_6-x_7\right) = \frac{\Gamma(\alpha_4)\Gamma(\alpha_5)\Gamma(\alpha_6)\Gamma(\alpha_7)}{\Gamma\left(\alpha_4+\alpha_5+\alpha_6++\alpha_7\right)}$$

with coefficients α_i dependent on a_i and on F

For this, we have to apply several MB-integrals here.

And repeat the procedure for the 2nd subloop.

On-shell example: B412m, the 1-loop on-shell box

but here use of another sequence of MB-integrals than in Smirnov's book

```
den = (x4 d4 + x5 d5 + x6 d6 + x7 d7 // Expand) /. kinBha /. m^2 -> 1 //  Expand

Q = -Coefficient[den, k]/2 // Simplify
    =  p3 x4 + p2 x5 - p1 (x4 + x6)

M = Coefficient[den, k^2] // Simplify
    = x4 + x5 + x6 + x7 -> 1

J = den /. k -> 0 // Simplify
    = t x4

F[x] = (Q^2 - J  M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4 // Expand
    = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7

B412ma = mb[(x5+x6)^2, -tx7x4 - sx5x6, nu, ga]

B412mb = B412ma /. (-sx5x6 - tx4x7)^(-ga - nu) ->
            mb[(-s)x5x6, (-t)x7x4, nu+ga, de]
            /.((-s)x5x6)^de_ -> (-s)^de x5^de x6^de
            /.((x56^2)^ga    -> (x5 + x6)^(2ga)
```

```
=
   (inv2piI^2(-s)^de x5^de x6^de ((x5 + x6)^(2ga)((-t)x4x7)^(-de-ga-nu)
        Gamma[-de] Gamma[-ga] Gamma[de + ga + nu] /Gamma[nu]

B4l2mc = B4l2mb /. (x5 + x6)^(2ga) ->
            mb[x5, x6, -2ga, si]
            /. ((-t)x4x7)^si_ -> (-t)^si x4^si x7^si // ExpandAll
=
   1/(Gamma[-2ga] Gamma[nu])
inv2piI^3  (-s)^de (-t)^(-de - ga - nu)
            x4^(-de - ga - nu) x5^(de + si) x6^(de + 2 ga - si) x7^(-de - ga - nu)
            Gamma[-de] Gamma[-ga] Gamma[ de + ga + nu] Gamma[-si] Gamma[-2ga + si]

B4l2md =  xfactor4[a4, x4, a5, x5, a6, x6, a7, x7] B4l2mc
=
.... (-s)^de (-t)^(-de-ga-nu)
x4^(-1+a4-de-ga-nu) x5^(-1+a5+de+si) x6^(-1+a6+de+2ga-si) x7^(-1+a7- de-ga-nu)

B4l2me =
  B4l2md /.
      x4^B4_  x5^B5_  x6^B6_  x7^B7_ -> xint4[x4^B4 x5^B5 x6^B6 x7^B7]
= ...
```

```
B4l2mf =  B4l2me /.
      Gamma[a6 + de + 2 ga - si]Gamma[-si]Gamma[ a5 + de + si] Gamma[-2 ga + si]
    ->  barne1[si, a5 + de, -2 ga, a6 + de + 2 ga, 0]
```

This finishes the evaluation of the MB-representation for `B4l2m`.

Some routines in mathematica which were used:

```
(* Barnes' first lemma: \int d(si) Gamma(si1p+si)Gamma(si2p+si)Gamma(si1m-si)Gamma(si2m-si)
                        with 1/inv2piI = 2 Pi I *)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
  1/inv2piI  Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
      si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m]

 (* Mellin-Barnes integral: (A+B)^(-nu) = 1/(2 Pi I) \int d(si) a^si b^(-nu - si)
                            Gamma[-si]Gamma[nu+si]/Gamma[nu]   *)

mb[a_, b_, nu_, si_]:=inv2piI a^si b^(-nu-si)Gamma[-si]Gamma[nu+si]/Gamma[nu]

 (* After the k-integration, the integrand for \int\prod(dxi xi^(ai -  1))\delta(1-\sum xi)
              will be (L=1 loop) : xfactorn F^(-nu) Q(xi).pe with  nu = a1 + .. + an - d/2 *)

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
  I Pi^(d/2) (-1)^(a1 + a2 + a3) x1^(a1 - 1) x2^(a2 - 1)x3^(a3 - 1)Gamma[
        a1 + a2 + a3 - d/2]/(Gamma[a1]Gamma[a2]Gamma[a3])

 (* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 *)

xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_)  ] :=
  Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1]  / Gamma[a1 + a2 + a3 + 3]
```

Two different 2-dim. MB-representations for the massive 1-loop QED box:

```
(* s=(PP1+PP2)^2, t=(PP1-PP3)^2, s+t+u=4, scalar QED box *)
(* PRDfact = E^(ep EulerGamma) 1/(I Pi^(d/2)) *)
(* B4l2m = PRDfact * K4l2m[k-PP1, 0, k, 1, k-PP3, 0, k-PP1 - PP2, 1] *)
(* B4l2mINPUT with (\int dr1 dr3)  is the on - shell QED box *)

B4l2mINPUT[s_,t_,m1_,m2_,m3_,m4_] =
 ((-1)^(m1 + m2 + m3 + m4)*E^(ep*EulerGamma)*inv2piI^2*Pi^(2 - d/2 - ep)*
  (-s)^(2 - ep - m1 - m2 - m3 - m4 - r1 - r3)*(-t)^r3*Gamma[-r1]*
  Gamma[4 - 2*ep - 2*m1 - m2 - 2*m3 - m4 - 2*r3]*Gamma[2 - ep - m1 - m2 - m3 - r1 - r3]*
  Gamma[2 - ep - m1 - m3 - m4 - r1 - r3]*Gamma[-r3]*Gamma[m1 + r3]*Gamma[m3 + r3]*
  Gamma[-2 + ep + m1 + m2 + m3 + m4 + r1 + r3])
/(Gamma[m1]*Gamma[m2]*Gamma[m3] * Gamma[4 - 2*ep - m1 - m2 - m3 - m4]*Gamma[m4]*
  Gamma[4 - 2*ep - 2*m1 - m2 - 2*m3 - m4 - 2*r1 - 2*r3])

(* s=(PP1+PP2)^2, t=(PP1-PP3)^2, s+t+u=4, scalar QED box *)
(* PRDfact = E^(ep EulerGamma) 1/(I Pi^(d/2)) *)
(* B4l2m = PRDfact * K4l2m[k-PP1, 0, k, 1, k-PP3, 0, k-PP1 - PP2, 1] *)
(* B4l2mINPUT with (\int dr1 dr3)  is the on - shell QED box *)
(* the MB-sequence deviates from e.g. Smirnov book, done for B5l2m3 *)

B4l2mINPUTvar[s_, t_, m1_, m2_, m3_, m4_] = ((-1)^(m1 + m2 + m3 +m4)
       *E^(ep*EulerGamma)*inv2piI^2 * (-s)^r1 * (-t)^(d/2-m1-m2-m3-m4-r1-r3)
 *Gamma[-r1]*Gamma[m2 + r1]*
  Gamma[m4 + r1]*Gamma[d/2 - m1 - m2 - m4 - r1 - r3]*
  Gamma[d/2 - m2 - m3 - m4 - r1 - r3]*Gamma[-r3]*
  Gamma[-d/2 + m1 + m2 + m3 + m4 + r1 + r3]*Gamma[m2 + m4 + 2*r1 + 2*r3])
/ (Gamma[m1]*Gamma[m2]*Gamma[m3]*Gamma[d - m1 - m2 - m3 - m4]*Gamma[m4]*
  Gamma[m2 + m4 + 2*r1])
```

The 2-dim. MB-representation for the 1-loop QED box with numerator $k.pe$ depends on the choice of momentum flow

```
B4l2mnumINPUTvar[pe_, s_, t_, m1_, m2_, m3_, m4_] = -(((-1)^(m1 + m2 + m3 + m4)*E^(ep*EulerGamma)*inv2piI^2
     *pe*(-s)^r1* (-t)^((d - 2*(m1 + m2 + m3 + m4 + r1 + r3))/2)
   *Gamma[-r1]*Gamma[-r3]* Gamma[-d/2 + m1 + m2 + m3 + m4 + r1 + r3]*
   (p1*Gamma[1 + m2 + r1]*Gamma[m4 + r1]*Gamma[m2 + m4 + 2*r1]*
     Gamma[1 + m2 + m4 + 2*r1 + 2*r3]*Gamma[(d - 2*(m1 + m2 + m4 + r1 + r3))/
       2] + Gamma[m2 + r1]*((p1 - p3)*Gamma[m4 + r1]*
       Gamma[1 + m2 + m4 + 2*r1]*Gamma[m2 + m4 + 2*(r1 + r3)]*
       Gamma[(d - 2*(-1 + m1 + m2 + m4 + r1 + r3))/2] -
      p2*Gamma[1 + m4 + r1]*Gamma[m2 + m4 + 2*r1]*
       Gamma[1 + m2 + m4 + 2*r1 + 2*r3]*
       Gamma[(d - 2*(m1 + m2 + m4 + r1 + r3))/2]))*
   Gamma[(d - 2*(m2 + m3 + m4 + r1 + r3))/2])/(Gamma[m1]*Gamma[m2]*Gamma[m3]*
   Gamma[1 + d - m1 - m2 - m3 - m4]*Gamma[m4]*Gamma[m2 + m4 + 2*r1]*
   Gamma[1 + m2 + m4 + 2*r1]))
```

A 3-dim. representation which is not derived by shrinking lines from 7-line box:

```
b5l2m2 = InputForm[((-1)^(a1 + a2 + a3 + a4 + a5)*E^(2*ep*EulerGamma)*inv2piI^3*
   (-s)^(2 - a2 - a4 - a5 - ep - r1 - r3 + si)*(-t)^r3*Gamma[-r1]*
   Gamma[2 - a2 - a4 - a5 - ep - r1 - r3]*Gamma[-r3]*Gamma[a2 + r3]*
   Gamma[a4 + r3]*Gamma[4 - 2*a1 - a3 - 2*ep - si]*
   Gamma[-2 + a2 + a4 + a5 + ep + r1 + r3 - si]*Gamma[a1 + si]*
   Gamma[-2 + a1 + a3 + ep + si]*Gamma[4 - 2*a2 - 2*a4 - a5 - 2*ep - 2*r3 +
     si]*Gamma[2 - a2 - a4 - ep - r1 - r3 + si])/
  (Gamma[a1]*Gamma[a2]*Gamma[a3]*Gamma[a4]*Gamma[a5]*
   Gamma[4 - a1 - a3 - 2*ep]*Gamma[4 - a2 - a4 - a5 - 2*ep + si]*
   Gamma[4 - 2*a2 - 2*a4 - a5 - 2*ep - 2*r1 - 2*r3 + si])]
```

.

Another nice box with numerator, B5l3m($p_e.k_1$)

We used it for the determination if the small mass expansion.

$$\mathtt{B5l3m}(p_e\cdot k_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\; e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\,\epsilon)-a_{12345}-\alpha-\beta-\delta}\;(-t)^{\delta}$$

$$\frac{\Gamma[-4+2\,\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\,\epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\,\Gamma[-\beta]}{\Gamma[7-3\,\epsilon-a_{12345}-\alpha]\,\Gamma[5-2\,\epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\,\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\frac{\Gamma[4-2\,\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\Big\{(p_e\cdot p_3)\,\Gamma[1+a_4+\delta]\,\Gamma[6-3\,\epsilon-$$

$$\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\,\Gamma[3-\epsilon-a_{12}-\alpha]\,\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2$$

$$\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\,\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\,\Gamma[a_1+\gamma]\,\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\,\Gamma[7-3\,\epsilon-a$$

$$\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\,\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\,\Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\,\Gamma[4-2\,\epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\,\Gamma[1+a_1+\gamma]\Big]\,\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\,\epsilon-a_{12345}-\alpha]\,\Gamma[3-\epsilon-a_{12}-\alpha]$$

$$\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\,\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\,\Gamma[5-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\,\epsilon-a_{11223}$$

$$\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\,\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

.

B5l2m2

$$\text{B5l2m2} \;=\; \frac{m^{4\epsilon}(-1)^{a_{12345}}e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[4-2\epsilon-a_{13}](2\pi i)^3}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma(-s)^{2-\epsilon-a_{245}-\gamma-\alpha+\beta}(-t)^{\alpha}$$
$$\Gamma[-2+\epsilon+a_{13}+\beta]\Gamma[-\gamma]\Gamma[2-\epsilon-a_{245}-\gamma-\alpha]\Gamma[-\alpha]$$
$$\Gamma[a_2+\alpha]\Gamma[a_4+\alpha]\Gamma[4-2\epsilon-a_{113}-\beta]\Gamma[-2+\epsilon+a_{245}+\gamma+\alpha-\beta]\Gamma[a_1+\beta]$$
$$\frac{\Gamma[4-2\epsilon-a_{2245}-2\alpha+\beta]\Gamma[2-\epsilon-a_{24}-\gamma-\alpha+\beta]}{\Gamma[4-2\epsilon-a_{245}+\beta]\Gamma[4-2\epsilon-a_{22445}-2\gamma-2\alpha+\beta]}$$

This kind of expression now has to be evaluated:

- Check special cases of indices, set lines to 1 (by setting $a_i \to 0$ if possible)
- Extract the ϵ-dependence related to UV and IR singularities (see next pages)
- After that: may set $s < 0,\ t < 0$ and evaluate numerically Euclidean case
- Use sector decomposition for a numerical comparison - if you have a program for that
- Try to go Minkowskian in a numerical way (if you like this)
- Go on analytically, e.g. by taking residua $\to$ get nested infinite sums from the residua
- Try to sum them up

B4l2m2

[Fleischer, Gluza, Lorca, TR 2006] `B4l2m`, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite ϵ:

$$\begin{aligned}\texttt{B4l2m} = \texttt{Box}(t,s) &= \frac{e^{\epsilon\gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \\ &\frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}}\Gamma[2+\epsilon+z_1+z_2]\Gamma^2[1+z_1]\Gamma[-z_1]\Gamma[-z_2] \\ &\Gamma^2[-1-\epsilon-z_1-z_2]\frac{\Gamma[-2-2\epsilon-2z_1]}{\Gamma[-2-2\epsilon-2z_1-2z_2]}\end{aligned} \tag{7}$$

`Mathematica` package `MB` used for analytical expansion $\epsilon \to 0$:

[Czakon:2005rk]

$$\texttt{B412m} = -\frac{1}{\epsilon}\texttt{I1} + \ln(-s)\texttt{I1} + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]\texttt{I1} - 2\texttt{I2}\right). \quad (8)$$

with I1 being also the divergent part of the vertex function $C_0(t; m, 0, m)/s = \texttt{V312m}/s$ (as is well-known):

$$\texttt{I1} = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1} \frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y) \quad (9)$$

with $y = (\sqrt{1-4m^2/t} - 1)/(\sqrt{1-4m^2/t} + 1)$: close contour to left, take residua at $(1+z_1) = -n$, sum up with Mathematica:

```
Residue[F[x]Gamma[1 + x], {x, -n}] // InputForm  =   -(-1)^n F[-n]/n!

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), {n, 0, Infinity}] // InputForm
=  (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])
```

The I2 is more complicated:

$$\texttt{I2} = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2} \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{-s}{-t}\right)^{z_1} \Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1] \quad (10)$$

$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(\frac{m^2}{-t}\right)^{z_2} \Gamma[-z_2]\frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]}\Gamma[2+z_1+z_2].$$

The expansion of `B412m` at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \quad (11)$$

$$r = \frac{s}{t}, \quad (12)$$

Look, under the integral, at $(-m^2/t)^{z_2}$,

and close the path to the right.

Seek the residua from the poles of Γ-functions with the smallest powers in m^2 and try to sum the resulting series.

Automatize this, it is not too easy.

we have obtained a compact answer for `I2` with the additional aid of `XSUMMER`

[Moch:2005uc]

. The box contribution of order ϵ in this limit becomes:

$$\begin{aligned} \mathtt{B412m}[t,s,m^2;+1] = & \frac{1}{st}\Big\{4\zeta_3 - 9\zeta_2 \ln(m_t) + \frac{2}{3}\ln^3(m_t) + 6\zeta_2\ln(r) - \ln^2(m_t)\ln(r) \\ & + \frac{1}{3}\ln^3(r) - 6\zeta_2\ln(1+r) + 2\ln(-r)\ln(r)\ln(1+r) - \ln^2(r)\ln(1+r) \\ & + 2\ln(r)\mathrm{Li}_2(1+r) + 2\mathrm{Li}_3(-r)\Big\} + \mathcal{O}(m_t). \end{aligned} \quad (13)$$

Shrinking of lines; seek the ϵ-expansion

Go on with some study of the 2nd planar 2-box, B7l4m2 (see also Smirnov book 4.73):

$$B_{\mathrm{pl},2} = \frac{\mathrm{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{-t}{-s}\right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})}$$

with $a = a_1 + \ldots + a_7$ and

$$\begin{aligned} z_i &= \mathrm{const} + i\, \Im m(z_i) \\ d &= 4 - 2\epsilon \\ \mathrm{const} &= \frac{(i\pi^{d/2})^2(-1)^a(-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d-a_{4567})} \end{aligned}$$

The integrand includes e.g.:

$$\begin{aligned} \Gamma_2 &= \Gamma(-z_2) \\ \Gamma_4 &= \Gamma(-z_4) \\ \Gamma_7 &= \Gamma(a_4 + z_2 + z_4) \\ \Gamma_8 &= \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \\ &\ldots \end{aligned}$$

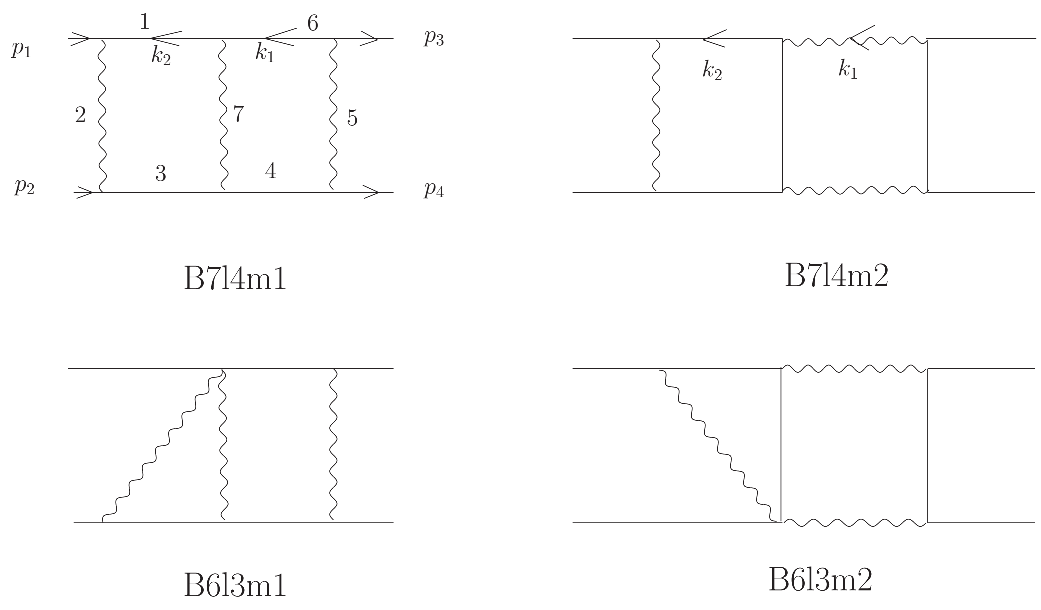

Figure 2: The planar 6- and 7-line topologies.

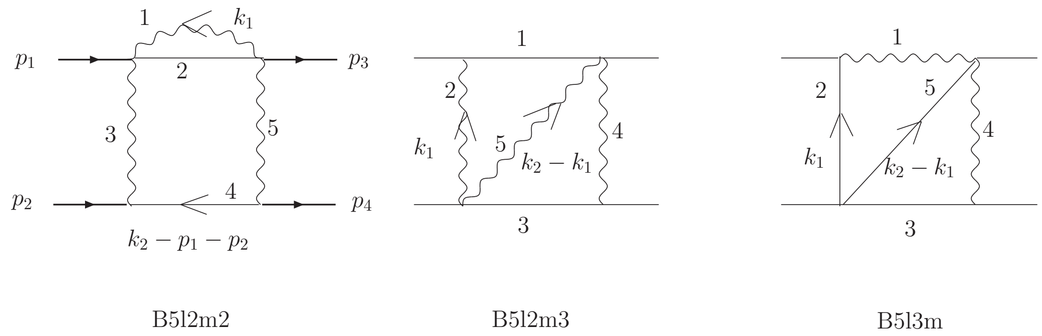

Figure 3: The 5-line topologies. B7l4m2: shrink line 1 get B6l3m2, then line 4 get B5l3m

Example:

derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and then setting $a_4 = 0$.

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

$\to$ Start with the z_2, z_4 integrations by

taking the residues for closing the integration contours to the right:

$$\begin{aligned}
I_{2,4} &= \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4+z_2+z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \\
&= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,..} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4+z_2+n)}{\Gamma(a_4)} R(z_i) \\
&= \sum_{n,m=0,1,..} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4+n+m)}{\Gamma(a_4)} R(z_i) \to_{a_4=0} 1 \times R(z_i)
\end{aligned}$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$

we got a 4-fold Mellin-Barnes integral for topology B5l3m (by "shrinking of lines")

with $24 - 3 = 21$ z_i-dependent Γ-functions which may yield residua within four-fold sums.

The MB-representation has to be calculated explicitely at fixed indices, e.g.

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0$$

General Tasks, first two steps automated by `MB.m`:

- Find a region of definiteness of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

An important tool is the command FindInstance of Mathematica 5:
It allows to solve a system of inequalities.
Here an example for B7l4m3, the non-planar massive double box:

```
sol = FindInstance[
    Cases[B7l4m3 ... Gamma[x_] -> x > 0 /. ep -> -1/10, {z1, z2, z3, z4, z5, z6, z7, z8}]
```

The result is:

```
 {z1 ->  -1/20, z2 ->  -1/40,  z3 ->   -1/20,  z4 -> -29/32,
  z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 ->  -5/64}
```

Really, all arguments are positive:

```
      G1[11/160] G10[1/320] G11[3/40] G12[3/40] G13[41/80] G14[37/40] G15[1/20] G16[1/40]
      G17[1/20] G18[29/32] G19[67/80] G2[7/160] G20[83/160] G21[273/320] G3[7/80] G4[139/160]
      G5[143/160] G6[1/320] G7[41/80] G8[1/80] G9[43/80]
```

Now set $\epsilon = 0$:

```
      G1[11/160] G1
 {z1 ->  -1/20, z2 ->  -1/40,  z3 ->   -1/20,  z4 -> -29/32,
  z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 ->  -5/64,   ep -> 0}
```

Determine again the arguments of the Gamma-functions; observe:
2 arguments are negative now: those for G3 and G8

```
      G1[11/160] G10[33/320] G11[3/40] G12[7/40] G13[57/80] G14[37/40] G15[1/20] G16[1/40]
      G17[1/20] G18[29/32] G19[67/80] G2[7/160] G20[83/160] G21[273/320] G4[123/160]
      G5[127/160] G6[1/320] G7[5/16] G9[27/80] G3[-9/80] G8[-31/80]
```

Perform the corresponding shifts of integration curve, add the residua and again perform the test for the arguments of the new, lower-dimensional MB-integrals.

We derived an algorithmic solution for isolating the singularities in $1/\epsilon$

The automatization of that: MB.m (M. Czakon)

$$\begin{aligned} B5l3md2 \quad &\rightarrow \quad MB(\text{4-dim,fin}) + MB_3(\text{3-dim,fin}) \\ &+ \quad MB_{36}(\text{2-dim}, \epsilon^{-1}, \text{fin}) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1}, \text{fin}) \\ &+ \quad MB_5(\text{3-dim,fin}) \end{aligned}$$

After these preparations e.g.:

$$\begin{aligned} MB_{365}(\text{1-dim}, \epsilon^{-2}) \quad &\sim \quad \frac{1}{\epsilon^2}\frac{1}{2\pi i}\int dz_6 \frac{(-s)^{(-z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} \\ &= \quad \frac{1}{\epsilon^2}\sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} \\ &= \quad -\frac{1}{\epsilon^2}\frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \\ &= \quad \frac{1}{\epsilon^2}\frac{-x}{4(1-x^2)}H[0,x] \end{aligned}$$

Here residua were taken at $z_6 = -n-1, n = 0, 1, ..$, and $H[0,x] = \ln(x)$ and $x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$.

Summary

- We have introduced to the representation of L-loop N-point Feynman integrals of general type
- The determination of the ϵ-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, ... so you have something to do if you like to ...

Problem: Determine the small mass limit of B5l2m2 or of any other of the 2-loop boxes for Bhabha scattering. Stefano Actis may check your solution. He leaves soon.