Effective Field Theories

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(or down to infinitely small distances)
All our theories are effective low-energy (or large-distance)
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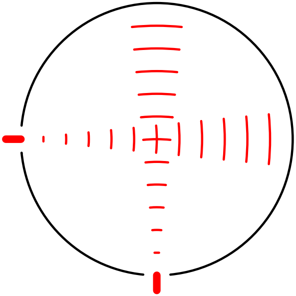
The Lagrangian contains all possible operators (allowed by symmetries). Coefficients of operators of dimension $n+4$ contain $1/M^n$. If M is much larger than energies we are interested in, we can retain only renormalizable terms (dimension 4), and, maybe, a power correction or two.

Photonia



Quantum PhotoDynamics (QPD)

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

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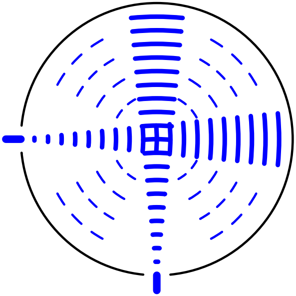
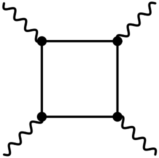


Quantum PhotoDynamics (QPD)

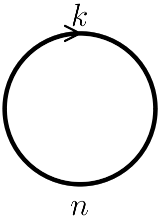
$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2 F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}$$

QED



$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{180 M^4} \left[-5 \left(F_{\mu\nu} F^{\mu\nu} \right)^2 + 14 F_{\mu\nu} F^{\nu\alpha} F_{\alpha\beta} F^{\beta\mu} \right]$$

Vacuum diagram

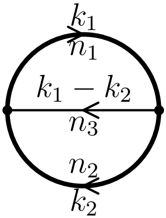


$$\int \frac{d^d k}{D^n} = i\pi^{d/2} M^{d-2n} V(n)$$

$$D = M^2 - k^2 - i0$$

$$V(n) = \frac{\Gamma(-d/2+n)}{\Gamma(n)}$$

2-loop vacuum diagram



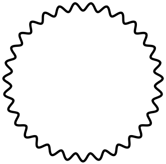
$$\int \frac{d^d k_1\, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = -\pi^d M^{2(d-n_1-n_2-n_3)} V(n_1, n_2, n_3)$$

$$D_1 = M^2 - k_1^2 \quad D_2 = M^2 - k_2^2 \quad D_3 = -(k_1 - k_2)^2$$

$$V(n_1, n_2, n_3) =$$

$$\frac{\Gamma\left(\frac{d}{2} - n_3\right) \Gamma\left(n_1 + n_3 - \frac{d}{2}\right) \Gamma\left(n_2 + n_3 - \frac{d}{2}\right) \Gamma(n_1 + n_2 + n_3 - d)}{\Gamma\left(\frac{d}{2}\right) \Gamma(n_1) \Gamma(n_2) \Gamma(n_1 + n_2 + 2n_3 - d)}$$

Thermal radiation $T \ll m$



$\sim T^4$

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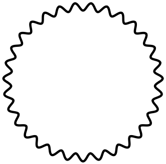

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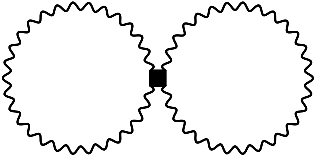

$\sim \frac{\alpha^2}{M^4} T^8$

Full theory (QED)

$$L = \bar{\psi}_0 \left(i \not{D} - M_0 \right) \psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}$$

$$D_\mu \psi_0 = \left(\partial_\mu - i e_0 A_{0\mu} \right) \psi_0$$

$$F_{0\mu\nu} = \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}$$

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Renormalization

$$\psi_0 = Z_\psi^{1/2} \psi \qquad A_0 = Z_A^{1/2} A$$

$$a_0 = Z_A a \qquad e_0 = Z_\alpha^{1/2} e \qquad M_0 = Z_m M$$

$\overline{\text{MS}}$ scheme

$$\frac{e_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\alpha(\mu)}{4\pi} Z_\alpha(\alpha(\mu)) e^{\gamma\varepsilon} \,.$$

$$Z_i = 1 + \frac{z_1}{\varepsilon} \frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \left(\frac{\alpha}{4\pi} \right)^2 + \cdots$$

Photon propagator

$$
\begin{aligned}
-iD_{\mu\nu}(p) &= -iD^0_{\mu\nu}(p) + (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\nu}(p)\\
&+ (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\gamma}(p)i\Pi^{\gamma\delta}(p)(-i)D^0_{\gamma\nu}(p) + \cdots
\end{aligned}
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Ward identity $\Pi_{\mu\nu}(p)p^\nu = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

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$$D_{\mu\nu}(p) = \frac{1}{p^2(1-\Pi(p^2))}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + a_0 \frac{p_\mu p_\nu}{(p^2)^2}$$

On-shell renormalization

$$A_0 = \left(Z_A^{\mathrm{os}}\right)^{1/2} A_{\mathrm{os}}$$

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Near the mass shell

$$D_\perp(p^2) = \frac{1}{1-\Pi(p^2)}\frac{1}{p^2} = \frac{1}{1-\Pi(0)}\frac{1}{p^2} + \cdots$$

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By definition, $D_\perp^{\mathrm{os}}(p^2)$ behaves as the free propagator $1/p^2$ near the mass shell

$$Z_A^{\mathrm{os}} = \frac{1}{1 - \Pi(0)}$$

Low-energy effective theory (QPD)

$$L' = -\frac{1}{4} F'_{0\mu\nu} F_0'^{\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0'^{\mu}\right)^2$$

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No renormalization

$$A'_0 = A'(\mu) = A'_{\mathrm{os}} \qquad a'_0 = a'(\mu) = a'_{\mathrm{os}}$$

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$$A(\mu) = \left(\zeta_A(\mu) \right)^{1/2} A'(\mu) \qquad \zeta_A(\mu) = \frac{\zeta_A^0}{Z_A} = \frac{Z_A^{\text{os}}}{Z_A}$$

Photon self-energy

$$\Pi_\mu^\mu(p) = (d-1)p^2\Pi(p^2)$$

$$\frac{\partial}{\partial p_\nu}\frac{\partial}{\partial p^\nu}\Pi_\mu^\mu(p)\bigg|_{p=0} = 2d(d-1)\Pi(0)$$

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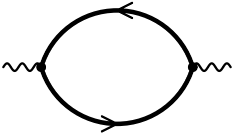

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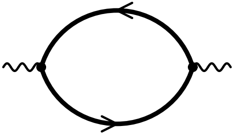

$$\Pi(0) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

$$\left(Z_A^{\text{os}}\right)^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3} \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots$$

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Re-expressing via renormalized quantities

$$\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) = e^{L\varepsilon} e^{\gamma\varepsilon}\Gamma(1+\varepsilon)\frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha Z_m^{-2\varepsilon}$$

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$\zeta_A^{-1} = Z_A / Z_A^{\mathrm{os}}$ must be finite at $\varepsilon \to 0$ (for example, at $L = 0$); $Z_A = 1 + z_1 \alpha(\mu)/(4\pi\varepsilon)$:

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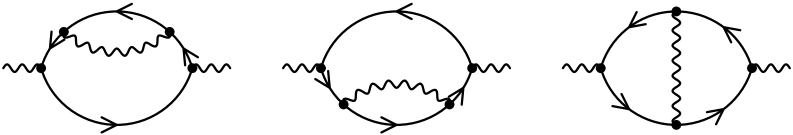
$$\zeta_A^{-1}(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \cdots$$

2 loops



$$\Pi(0) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) - \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

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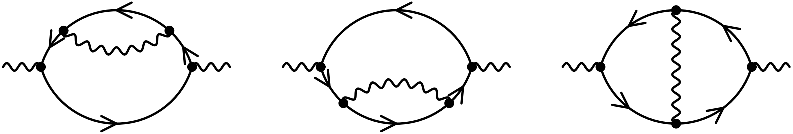

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$$\left(Z_A^{\text{os}}\right)^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3}e^{L\varepsilon}\frac{\alpha(\mu)}{4\pi\varepsilon}Z_\alpha Z_m^{-2\varepsilon} - \varepsilon\left(6 - \frac{13}{3}\varepsilon + \cdots\right)e^{2L\varepsilon}\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 + \cdots$$

$$Z_\alpha = Z_A^{-1} = 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots \qquad Z_m = 1 - 3\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots$$

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$$\zeta_A^{-1}(\mu) = 1 + \frac{4}{3}L\frac{\alpha(\mu)}{4\pi} + \left(-4L + \frac{13}{3}\right)\left(\frac{\alpha(\mu)}{4\pi}\right)^2 + \cdots$$

Charge

Full theory (QED)

$$Z_\alpha = Z_A^{-1} \qquad Z_\alpha^{\rm os} = \left(Z_A^{\rm os}\right)^{-1}$$

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Effective low-energy theory (QPD) — no renormalization

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On-shell charge (measured at large distances) is the same

$$\alpha_{\mathrm{os}} = \alpha'$$

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$$\alpha_{\text{os}} = \alpha'$$

$$\alpha(\mu) = \zeta_\alpha(\mu)\alpha' \qquad \zeta_\alpha(\mu) = \frac{Z_\alpha^{\text{os}}}{Z_\alpha} = \zeta_A^{-1}(\mu)$$

Popular choice: $\mu_0 = M(\mu_0)$

$$\zeta_\alpha(\mu_0) = 1 + \frac{13}{3}\left(\frac{\alpha(\mu_0)}{4\pi}\right)^2 + \cdots$$

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Another popular choice: $\mu = M_{\text{os}}$

$$\frac{M(\mu)}{M_{\text{os}}} = 1 - 6\left(\log\frac{\mu}{M_{\text{os}}} + \frac{2}{3}\right)\frac{\alpha}{4\pi} + \cdots$$

$$\zeta_\alpha(M_{\text{os}}) = 1 + 15\left(\frac{\alpha(M_{\text{os}})}{4\pi}\right)^2 + \cdots$$

QED with massless electrons and heavy muons

$$\psi_0 = \left(\zeta_\psi^0\right)^{1/2} \psi_0' \qquad \psi(\mu) = \zeta_\psi^{1/2}(\mu)\psi'(\mu) \qquad \zeta_\psi(\mu) = \zeta_\psi^0 \frac{Z_\psi'}{Z_\psi}$$

$$\not{p} S(p) = \zeta_\psi^0 \not{p} S'(p) + \mathcal{O}\left(\frac{p^2}{M^2}\right)$$

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Near the mass shell $p \to 0$

$$S(p) = \frac{Z_\psi^{\text{os}}}{\not{p}} \qquad Z_\psi^{\text{os}} = \frac{1}{1 - \Sigma_V(0)}$$

$$S'(p) = \frac{Z_\psi'^{\text{os}}}{\not{p}} \qquad Z_\psi'^{\text{os}} = \frac{1}{1 - \Sigma_V'(0)} = 1$$

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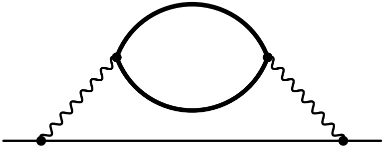
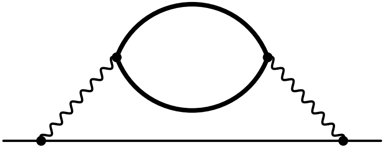
$$\zeta_\psi^0 = \frac{Z_\psi^{\text{os}}}{Z_\psi'^{\text{os}}} = \frac{1}{1 - \Sigma_V(0)}$$

2 loops



$$\zeta_\psi^0 = 1 + \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} \frac{e_0^4 M_0^{-4\varepsilon}}{(4\pi)^d} \Gamma^2(\varepsilon)$$

2 loops



$$\zeta_\psi^0 = 1 + \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} \frac{e_0^4 M_0^{-4\varepsilon}}{(4\pi)^d} \Gamma^2(\varepsilon)$$

$$\zeta_\psi(M) = 1 - \frac{5}{6} \left(\frac{\alpha(M)}{4\pi} \right)^2 + \cdots$$

Electron mass

On-shell mass is the same

$$m_{\text{os}} = m'_{\text{os}}$$

$$m_0 = \zeta_m^0 m'_0 \qquad \zeta_m^0 = \frac{Z_m^{\text{os}}}{Z_m'^{\text{os}}}$$

$$m(\mu) = \zeta_m(\mu) m'(\mu) \qquad \zeta_m = \zeta_m^0 \frac{Z'_m}{Z_m} = \frac{Z_m^{\text{os}}}{Z_m'^{\text{os}}} \frac{Z'_m}{Z_m}$$

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When calculating Z_m^{os} we may set $m = 0$, then $Z_m'^{\text{os}} = 1$, Z_m^{os} is given by diagrams with a muon loop

2 approaches

QED processes with $p_i \sim m \ll M$

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1. Low-energy effective theory

$$L' = \bar{\psi}_0' i \not{D}' \psi_0' - \frac{1}{4} F'_{0\mu\nu} F_0'^{\mu\nu} + \frac{c_0}{M} \bar{\psi}_0' F'_{0\mu\nu} \sigma^{\mu\nu} \psi_0' + \mathcal{O}\left(\frac{1}{M^2}\right)$$

2 approaches

QED processes with $p_i \sim m \ll M$

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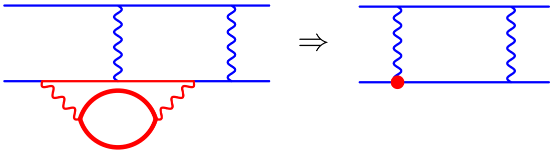
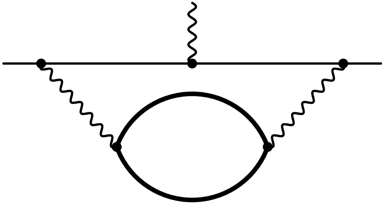
2. Expansion by regions (hard and soft)





$c \sim e^5$

Two ways to search for "new physics":

- To raise energies of our accelerators in the hope to produce real new particles (e.g., muons);
- To measure low-energy quantities (such as the electron magnetic moment) with a high precision in the hope to find effects of higher terms in the effective Lagrangian caused by loops of virtual new particles

Power counting

$$\lambda = \frac{m}{M}$$

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When acting on soft fields,

$$\partial_\mu \sim \lambda$$

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Soft photon

$$<0|T\{A_\mu(x)A_\nu(0)\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{1}{p^2}\left[g_{\mu\nu} - (1-a)\frac{p_\mu p_\nu}{p^2}\right]$$

$p \sim \lambda$, $x \sim 1/\lambda$

$$A \sim \lambda$$

$D_\mu \sim \lambda$

Soft electron

$$<0|T\left\{\psi(x)\bar{\psi}(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{1}{\not{p} - m}$$

$$\psi \sim \lambda^{3/2}$$

Soft electron

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Lagrangian

$$F_{\mu\nu}F^{\mu\nu} \sim \lambda^4 \qquad \bar{\psi}\left(i\not{D} - m\right)\psi \sim \lambda^4 \qquad \bar{\psi}F_{\mu\nu}\sigma^{\mu\nu}\psi \sim \lambda^5$$

Action: $1,\ \lambda\ \ldots$

QCD

$$L = -\frac{1}{4} G^{a}_{0\mu\nu} G^{a\mu\nu}_{0} + \sum_{i} \bar{q}_{0i} i \not{D} q_{0i} + \bar{Q}_{0} \left(i \not{D} - M_{0} \right) Q_{0}$$

Low-energy effective theory

$$
\begin{aligned}
L' = & -\frac{1}{4} G'^{a}_{0\mu\nu} G'^{a\mu\nu}_{0} + \sum_i \bar{q}'_{0i} i \not{D}' q'_{0i} \\
& + \frac{c_G}{M^2} f^{abc} G'^{a\ \mu}_{0\lambda} G'^{b\ \nu}_{0\mu} G'^{c\ \lambda}_{0\nu} \\
& + \frac{c_V}{M^2} \left(\sum_i \bar{q}'_{0i} \gamma^\mu q'_{0i} \right) \left(\sum_j \bar{q}'_{0j} \gamma_\mu q'_{0j} \right) \\
& + \frac{c_{Vc}}{M^2} \left(\sum_i \bar{q}'_{0i} \gamma^\mu t^a q'_{0i} \right) \left(\sum_j \bar{q}'_{0j} \gamma_\mu t^a q'_{0j} \right) \\
& + \frac{c_A}{M^2} \left(\sum_i \bar{q}'_{0i} \gamma^{[\lambda} \gamma^\mu \gamma^{\nu]} q'_{0i} \right) \left(\sum_j \bar{q}'_{0j} \gamma_{[\lambda} \gamma_\mu \gamma_{\nu]} q'_{0j} \right) \\
& + \frac{c_{Ac}}{M^2} \left(\sum_i \bar{q}'_{0i} \gamma^{[\lambda} \gamma^\mu \gamma^{\nu]} t^a q'_{0i} \right) \left(\sum_j \bar{q}'_{0j} \gamma_{[\lambda} \gamma_\mu \gamma_{\nu]} t^a q'_{0j} \right) + \mathcal{O}\left(\frac{1}{M^4} \right)
\end{aligned}
$$

QCD

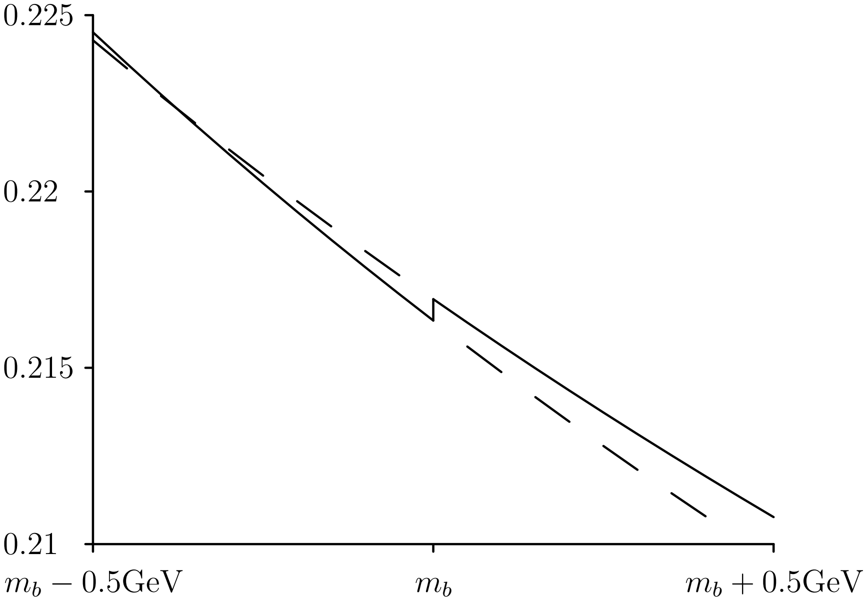
$$\alpha_s^{(n_l+1)}(\mu) = \zeta_\alpha(\mu)\alpha_s^{(n_l)}(\mu)$$

$$\zeta_\alpha(M_{\rm os}) = 1 + \left(15C_F - \frac{32}{9}C_A\right) T_F \left(\frac{\alpha_s(M)}{4\pi}\right)^2 + \cdots$$

RG equation

$$\frac{d\log\zeta_\alpha(\mu)}{d\log\mu} + 2\beta^{(n_l+1)}(\alpha_s^{(n_l+1)}(\mu)) - 2\beta^{(n_l)}(\alpha_s^{(n_l)}(\mu)) = 0$$

QCD



Photonia

Imported a single electron and study how it interacts with soft photons (both real and virtual)

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \varphi^{+}\left[iD_0 + \frac{c_k}{2M}\vec{D}^{\,2} - \frac{c_m}{2M}e\vec{B}\cdot\vec{\sigma} - \frac{c_d}{8M^2}e\left(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}\right) - i\frac{c_s}{8M^2}e\left(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}\right)\cdot\vec{\sigma} + \cdots\right]\varphi$$

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& - \frac{c_d}{8M^2}e\left(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}\right) \\
& - i\frac{c_s}{8M^2}e\left(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}\right)\cdot\vec{\sigma} + \cdots\Bigg]\varphi
\end{aligned}
$$

Leading order
mass shell: $E = 0$ independently of $\vec{p}$
$SU(2)$ spin symmetry: electron spin does not interact with electromagnetic field (and can be rotated at will) because the electron magnetic moment $\sim e/M$

Feynman rules

Propagator

$$S(k) = \frac{1}{k_0 + i0} \qquad S(x) = -i\theta(x_0)\delta(\vec{x}\,)$$

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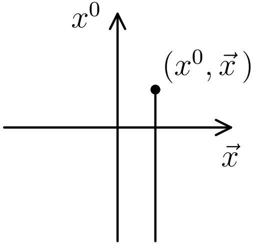
Vertex $ie_0 v^\mu$ where $v^\mu = (1, \vec{0}\,)$

Wilson line

Propagator in an external field

$$S(x) = -i\theta(x_0)\delta(\vec{x})W(x)$$

Wilson line $D_0 W(x)\varphi(x) = W(x)\partial_0\varphi(x)$



$$W(x^0, \vec{x}) = P\exp\left(i\int_{-\infty}^{x^0} A_0(x^{0\prime}, \vec{x})\, dx^{0\prime}\right)$$

$W^{-1}(x)D_0W(x) = \partial_0$

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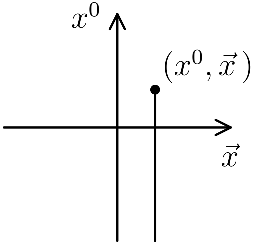

$$W(x^0, \vec{x}) = P\exp\left(i\int_{-\infty}^{x^0} A_0(x^{0\prime}, \vec{x})\, dx^{0\prime}\right)$$

$W^{-1}(x)D_0W(x) = \partial_0$ Transformation $\varphi(x) = W(x)\varphi^{(0)}(x)$:
the leading-order Lagrangian becomes free

$$L = \varphi^{(0)+} i\partial_0\varphi^{(0)}$$

Covariant notation

$$p = Mv + k$$

$$\not{v} h_v = h_v$$

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$$L = \bar{h}_v \Bigg[iv \cdot D - \frac{c_k}{2M} D_\perp^2 - \frac{c_m}{4M} F_{\mu\nu}\sigma^{\mu\nu}$$

$$- \frac{c_d}{8M^2} v^\mu [D_\perp^\nu, F_{\mu\nu}] + i \frac{c_s}{8M^2} [D_\perp^\mu, F^{\lambda\nu}]_+ v_\lambda \sigma_{\mu\nu} + \cdots \Bigg] h_v$$

$$D_\perp = D - v(v \cdot D)$$

Feynman rules

Tree level

Propagator

$$\frac{\not{p}+M}{p^2-M^2+i0}=\frac{M(1+\not{v})+\not{k}}{2Mv\cdot k+k^2+i0}=\frac{1+\not{v}}{2}\frac{1}{k\cdot v+i0}+\mathcal{O}\left(\frac{k}{M}\right)$$

Tree level

Propagator

$$\frac{\not{p} + M}{p^2 - M^2 + i0} = \frac{M(1 + \not{v}) + \not{k}}{2Mv \cdot k + k^2 + i0} = \frac{1 + \not{v}}{2} \frac{1}{k \cdot v + i0} + \mathcal{O}\left(\frac{k}{M}\right)$$

Vertex $ie_0\gamma^\mu \to ie_0 v^\mu$

$$\frac{1 + \not{v}}{2} \gamma^\mu \frac{1 + \not{v}}{2} = \frac{1 + \not{v}}{2} v^\mu \frac{1 + \not{v}}{2}$$

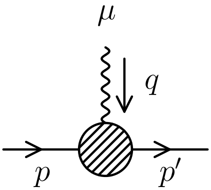
Mass shell

Propagator at $E \to 0$, $\vec{p} \to 0$

$$\frac{1}{E - \frac{c_k^0}{2M}\vec{p}^{\,2}}$$

$$c_k^0 = 1 \qquad c_k(\mu) = 1$$

Scattering in external field in QED



$$\bar{u}(p')\left[F_1(q^2)\frac{(p+p')^\mu}{2M} + F_M(q^2)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(p)$$

$$F_1(q^2) = 1 + F_1'(0)\frac{q^2}{M^2} + \cdots \qquad F_M(q^2) = F_M(0) + \cdots$$

Foldy–Wouthuysen transformation

$p = Mv + k$

$$u(p) = \left[1 + \frac{\not{k}}{2M} + \frac{k^2}{4M^2} + \cdots\right] u_v(k)$$

$\not{v} u_v = u_v$

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$$\bar{u}_v(k') \Bigg[F_1(q^2) \left(v^\mu + \frac{(k+k')^\mu}{2M} - \frac{q^2 + [\not{k}, \not{q}]}{8M^2} v^\mu + \cdots \right) + F_M(q^2) \left(\frac{[\not{q}, \gamma^\mu]}{4M} + \frac{q^2 + [\not{k}, \not{q}]}{4M^2} v^\mu + \cdots \right) \Bigg] u_v(k)$$

Scattering in the effective theory

Loop corrections vanish

$$\bar{u}_v(k')\Bigg[v^\mu + c_k\frac{(k+k')^\mu}{2M} + c_m\frac{[\not{q},\gamma^\mu]}{4M} + c_d\frac{q^2}{8M^2}v^\mu + c_s\frac{[\not{k},\not{q}]}{8M^2}v^\mu + \cdots\Bigg]u_v(k)$$

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$$\left. + c_d\frac{q^2}{8M^2}v^\mu + c_s\frac{[\not{k},\not{q}]}{8M^2}v^\mu + \cdots\right]u_v(k)$$

$$c_k = 1 \qquad c_m = F_M(0)$$
$$c_d = 8F_1'(0) + 2F_M(0) - 1 \qquad c_s = 2F_M(0) - 1$$

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Reparametrization invariance $v \to v + \delta v$, $\delta v \sim k/M$

$$c_k = 1 \qquad c_s = 2c_m - 1$$

Regions

Power counting

Small parameter

$$\lambda \sim \frac{k}{M}$$

Soft fields: $\partial \sim \lambda$, $A \sim \lambda$, $D \sim \lambda$

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$$<T\{\varphi(x)\varphi^+(0)\}> \sim \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{1}{k\cdot v + i0}$$

$$\varphi \sim \lambda^{3/2}$$

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$$\varphi \sim \lambda^{3/2}$$

$$\varphi^+ iD_0\varphi \sim \lambda^4$$

$$\varphi^+ \vec{D}^{\,2}\varphi \sim \lambda^5 \qquad \varphi^+ \vec{B}\cdot\vec{\sigma}\varphi \sim \lambda^5$$

Another derivation at tree level

$$\psi(x) = e^{-iMv\cdot x}\left(h_v(x) + H_v(x)\right)$$

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \qquad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x)$$

$$\not{v}h_v(x) = h_v(x) \qquad \not{v}H_v(x) = -H_v(x)$$

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$$\begin{aligned} L &= \bar{\psi}\left(i\not{D} - M\right)\psi \\ &= \bar{h}_v iv\cdot D h_v + \bar{H}_v\left(-iv\cdot D - 2M\right)H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v \end{aligned}$$

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$$= \bar{h}_v iv\cdot D h_v + \bar{H}_v\left(-iv\cdot D - 2M\right)H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v$$

Solution of the equation of motion

$$H_v = \frac{1}{2M + iv\cdot D}\not{D}_\perp h_v = \frac{1}{2M}i\not{D}_\perp h_v - \frac{iv\cdot D}{(2M)^2}i\not{D}_\perp h_v + \cdots$$

Another derivation at tree level

$$\psi(x) = e^{-iMv\cdot x}\left(h_v(x) + H_v(x)\right)$$

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \qquad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x)$$

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Solution of the equation of motion

$$H_v = \frac{1}{2M + iv\cdot D}\not{D}_\perp h_v = \frac{1}{2M}i\not{D}_\perp h_v - \frac{iv\cdot D}{(2M)^2}i\not{D}_\perp h_v + \cdots$$

$$L = \bar{h}_v\left[iv\cdot D - \frac{D_\perp^2}{2M} + \frac{eF_{\mu\nu}\sigma^{\mu\nu}}{4M} + \cdots\right]h_v$$

NRQED

$$\begin{aligned} L = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \varphi^+ \Bigg[iD_0 + \frac{c_k}{2M}\vec{D}^{\,2} - \frac{c_m}{2M} e\vec{B}\cdot\vec{\sigma} \\ & \qquad - \frac{c_d}{8M^2} e\left(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}\right) \\ & \qquad - i\frac{c_s}{8M^2} e\left(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}\right)\cdot\vec{\sigma} + \cdots \Bigg]\varphi \\ & + \chi^+ [e \to -e]\chi + \frac{1}{M^2} O_c + \cdots \end{aligned}$$

NRQED

$$
\begin{aligned}
L = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \varphi^{+}\Bigg[iD_0 + \frac{c_k}{2M}\vec{D}^{\,2} - \frac{c_m}{2M}e\vec{B}\cdot\vec{\sigma} \\
& \qquad - \frac{c_d}{8M^2}e\left(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}\right) \\
& \qquad - i\frac{c_s}{8M^2}e\left(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}\right)\cdot\vec{\sigma} + \cdots\Bigg]\varphi \\
& + \chi^{+}\left[e \to -e\right]\chi + \frac{1}{M^2}O_c + \cdots
\end{aligned}
$$

$$
\begin{aligned}
O_c &= d_s(\psi^{+}\psi)(\chi^{+}\chi) + d_v(\psi^{+}\vec{\sigma}\psi)\cdot(\chi^{+}\vec{\sigma}\chi) \\
&= \bar{d}_s(\psi^{+}\chi)(\chi^{+}\psi) + \bar{d}_v(\psi^{+}\vec{\sigma}\chi)\cdot(\chi^{+}\vec{\sigma}\psi)
\end{aligned}
$$

$$
d_s = -\frac{1}{2}\bar{d}_s - \frac{3}{2}\bar{d}_v \qquad d_v = -\frac{1}{2}\bar{d}_s + \frac{1}{2}\bar{d}_v
$$

Contact interaction

$$\bar{d}_v = -\pi\alpha + \mathcal{O}(\alpha^2) \qquad \bar{d}_s = \mathcal{O}(\alpha^2)$$

Regions

Region	E	$\vec{p}$	Electron	Photon
			$\frac{1}{(M+E)^2-\vec{p}^{\,2}-M^2}$	$\frac{1}{E^2-\vec{p}^{\,2}}$
Hard	$\sim M$	$\sim M$	$\frac{1}{2ME+E^2-\vec{p}^{\,2}}$	$\frac{1}{E^2-\vec{p}^{\,2}}$
Soft	$\sim Mv$	$\sim Mv$	$\frac{1}{2ME+E^2-\vec{p}^{\,2}}$	$\frac{1}{E^2-\vec{p}^{\,2}}$
Potential	$\sim Mv^2$	$\sim Mv$	$\frac{1}{2ME-\vec{p}^{\,2}+E^2}$	$\frac{1}{-\vec{p}^{\,2}+E^2}$
Ultrasoft	$\sim Mv^2$	$\sim Mv^2$	$\frac{1}{2ME+E^2-\vec{p}^{\,2}}$	$\frac{1}{E^2-\vec{p}^{\,2}}$

QED

Regions

Region	E	$\vec{p}$	Electron $\dfrac{1}{(M+E)^2-\vec{p}^{\,2}-M^2}$	Photon $\dfrac{1}{E^2-\vec{p}^{\,2}}$
Hard	$\sim M$	$\sim M$	$\dfrac{1}{2ME+E^2-\vec{p}^{\,2}}$	$\dfrac{1}{E^2-\vec{p}^{\,2}}$
Soft	$\sim Mv$	$\sim Mv$	$\dfrac{1}{2ME+E^2-\vec{p}^{\,2}}$	$\dfrac{1}{E^2-\vec{p}^{\,2}}$
Potential	$\sim Mv^2$	$\sim Mv$	$\dfrac{1}{2ME-\vec{p}^{\,2}+E^2}$	$\dfrac{1}{-\vec{p}^{\,2}+E^2}$
Ultrasoft	$\sim Mv^2$	$\sim Mv^2$	$\dfrac{1}{2ME+E^2-\vec{p}^{\,2}}$	$\dfrac{1}{E^2-\vec{p}^{\,2}}$

NRQED

Regions

Region	E	$\vec{p}$	Electron $\frac{1}{(M+E)^2-\vec{p}^2-M^2}$	Photon $\frac{1}{E^2-\vec{p}^2}$
Hard	$\sim M$	$\sim M$	$\frac{1}{2ME+E^2-\vec{p}^2}$	$\frac{1}{E^2-\vec{p}^2}$
Soft	$\sim Mv$	$\sim Mv$	$\frac{1}{2ME+E^2-\vec{p}^2}$	$\frac{1}{E^2-\vec{p}^2}$
Potential	$\sim Mv^2$	$\sim Mv$	$\frac{1}{2ME-\vec{p}^2+E^2}$	$\frac{1}{-\vec{p}^2+E^2}$
Ultrasoft	$\sim Mv^2$	$\sim Mv^2$	$\frac{1}{2ME+E^2-\vec{p}^2}$	$\frac{1}{E^2-\vec{p}^2}$

pNRQED

pNRQED

$$L = \int d^3\vec{r} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \varphi^+ \left(iD_0 + \frac{\vec{D}^{\,2}}{2M} + \cdots \right) \varphi \right.$$

$$\left. + \chi^+ (e \to -e) \chi \right]$$

$$- \int d^3\vec{r}_1 \, d^3\vec{r}_2 \, \varphi^+(t, \vec{r}_1) \chi^+(t, \vec{r}_2) V(\vec{r}_1 - \vec{r}_2) \chi(t, \vec{r}_2) \varphi(t, \vec{r}_1)$$

Breit potential

$$
\begin{aligned}
V(\vec{r}) = \; & -\frac{\alpha}{r} - \frac{\alpha}{2M^2 r}\left[\vec{p}^{\,2} + \frac{1}{r^2}\vec{r}\cdot(\vec{r}\cdot\vec{p})\vec{p}\right] \\
& + \frac{3\alpha}{2M^2 r^3}\vec{l}\cdot\vec{s} - \frac{\alpha}{4M^2 r^3}\left[\vec{\sigma}_1\cdot\vec{\sigma}_2 - \frac{3}{r^2}(\vec{r}\cdot\vec{\sigma}_1)(\vec{r}\cdot\vec{\sigma}_2)\right] \\
& + \frac{\pi\alpha}{2M^2}\left[5 + \frac{7}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\right]\delta(\vec{r})
\end{aligned}
$$

Bilocal gauge-invariant field

$$\chi(t,\vec{r}_2)\varphi(t,\vec{r}_1) \to P\exp\left(ie\int_{\vec{r}_2}^{\vec{r}_1}\vec{A}(t,\vec{r})\cdot d\vec{r}\right)S(t,\vec{R},\vec{r})$$

$$\vec{R}=\frac{\vec{r}_1+\vec{r}_2}{2}\qquad \vec{r}=\vec{r}_1-\vec{r}_2$$

Multipole expansion

$$L = -\frac{1}{4}\int d^3\vec{r}\,F_{\mu\nu}F^{\mu\nu}+\int d^3\vec{R}\,d^3\vec{r}\,S^+(t,\vec{R},\vec{r})\Bigg[i\partial_0-\frac{\vec{p}^{\,2}}{M} + \frac{\vec{p}^{\,4}}{4M^3}-\frac{\vec{P}^{\,2}}{4M}-V(\vec{r})+e\vec{r}\cdot\vec{E}(t,\vec{R})\Bigg]S(t,\vec{R},\vec{r})$$

Bilocal gauge-invariant field

$$\chi(t,\vec{r}_2)\varphi(t,\vec{r}_1) \to P\exp\left(ie\int_{\vec{r}_2}^{\vec{r}_1}\vec{A}(t,\vec{r})\cdot d\vec{r}\right) S(t,\vec{R},\vec{r})$$

$$\vec{R} = \frac{\vec{r}_1+\vec{r}_2}{2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

Multipole expansion

$$L = -\frac{1}{4}\int d^3\vec{r}\, F_{\mu\nu}F^{\mu\nu} + \int d^3\vec{R}\, d^3\vec{r}\, S^+(t,\vec{R},\vec{r})\left[i\partial_0 - \frac{\vec{p}^{\,2}}{M} + \frac{\vec{p}^{\,4}}{4M^3} - \frac{\vec{P}^{\,2}}{4M} - V(\vec{r}) + e\vec{r}\cdot\vec{E}(t,\vec{R})\right] S(t,\vec{R},\vec{r})$$

Power counting

$$p \sim M\alpha \qquad r \sim \frac{1}{M\alpha} \qquad \partial_0 \sim M\alpha^2 \qquad P \sim M\alpha^2$$

$$A \sim M\alpha^2 \qquad \partial \sim M\alpha^2$$

Positronium energy levels

$$E_{nls} = -\frac{M\alpha^2}{4n^2} + M\alpha^4 \left[-\frac{1}{2n^3(2l+1)} + \frac{11}{64n^4} + \frac{7}{12}\frac{\delta_{l0}\delta_{s1}}{n^3} \right] + \cdots$$

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Vector current

$$\bar{\psi}\vec{\gamma}\psi = c_v\,\chi^+\vec{\sigma}\varphi + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$c_v = 1 - 2\frac{\alpha}{\pi} + \cdots$$

Soft–collinear effective theory

Inclusuve $B \to X_s\gamma$ (neglecting m_s and m_K)

$$M_X^2 = M_B(M_B - 2E_\gamma)$$

- $\frac{M_B}{2} - E_\gamma \sim \frac{\Lambda^2}{M_B}$
  Exclusive channels $M_X \sim \Lambda$
- $\frac{M_B}{2} - E_\gamma \sim \Lambda$
  Jet $M_X^2 \sim M_B\Lambda$
- $\frac{M_B}{2} - E_\gamma \sim M_B$
  $M_X \sim M_B$

Light-front components

$$n^\mu_\pm = (1, \mp 1, \vec{0})$$

$$n^2_+ = n^2_- = 0 \qquad n_+ \cdot n_- = 2$$

$$a_\pm = a \cdot n_\pm = a^0 \pm a^1$$

$$a^\mu = \frac{1}{2}\left(a_+ n^\mu_- + a_- n^\mu_+\right) + a^\mu_\perp$$

$$a \cdot b = \frac{1}{2}\left(a_+ b_- + a_- b_+\right) - \vec{a}_\perp \cdot \vec{b}_\perp$$

$$v^\mu = \frac{1}{2}\left(n^\mu_+ + n^\mu_-\right) \qquad v_+ = v_- = 1 \qquad \vec{v}_\perp = \vec{0}$$

$$\gamma_\pm = \gamma \cdot n_\pm = \not{n}_\pm$$

Regions

$$\lambda \sim \frac{\Lambda}{E}$$

Region	$p = (p_+, p_-, p_\perp)$	p^2
Hard	$(1, 1, 1)$	1
Hard–collinear	$(\lambda, 1, \lambda^{1/2})$	λ
Soft	$(\lambda, \lambda, \lambda)$	λ^2

Power counting

Soft fields

$$\partial \sim \lambda \qquad A_s \sim \lambda \qquad q_s \sim \lambda^{3/2} \qquad h_v \sim \lambda^{3/2}$$

Power counting

Soft fields

$$\partial \sim \lambda \qquad A_s \sim \lambda \qquad q_s \sim \lambda^{3/2} \qquad h_v \sim \lambda^{3/2}$$

Hard-collinear fields

$$\partial \sim (\lambda, 1, \lambda^{1/2})$$

Hard-collinear gluon

$$<0|T\left\{A^\mu_{hc}(x)A^\nu_{hc}(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\frac{1}{p^2}\left[g^{\mu\nu} - (1-a)\frac{p^\mu p^\nu}{p^2}\right]$$

$$A_{hc} \sim (\lambda, 1, \lambda^{1/2})$$

Hard-collinear quark

$$\psi_{hc} = \xi + \eta \qquad \gamma_+\xi = 0 \qquad \gamma_-\eta = 0$$

$$\xi = \frac{1}{4}\gamma_+\gamma_-\psi_{hc} \qquad \eta = \frac{1}{4}\gamma_-\gamma_+\psi_{hc}$$

Hard-collinear quark

$$\psi_{hc} = \xi + \eta \qquad \gamma_+\xi = 0 \qquad \gamma_-\eta = 0$$

$$\xi = \frac{1}{4}\gamma_+\gamma_-\psi_{hc} \qquad \eta = \frac{1}{4}\gamma_-\gamma_+\psi_{hc}$$

$$<0|T\left\{\psi_{hc}(x)\bar{\psi}_{hc}(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\frac{1}{2}\left(p_+\gamma_- + p_-\gamma_+\right) + \not{p}_\perp}{p^2}$$

$$<T\left\{\xi(x)\bar{\xi}(0)\right\}> \sim \lambda \qquad <T\left\{\eta(x)\bar{\eta}(0)\right\}> \sim \lambda^2$$

$$<T\left\{\xi(x)\bar{\eta}(0)\right\}> \sim <T\left\{\eta(x)\bar{\xi}(0)\right\}> \sim \lambda^{3/2}$$

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$$\psi_{hc} = \xi + \eta \qquad \gamma_+ \xi = 0 \qquad \gamma_- \eta = 0$$

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$$<0|T\left\{\psi_{hc}(x)\bar{\psi}_{hc}(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\frac{1}{2}(p_+\gamma_- + p_-\gamma_+) + \not{p}_\perp}{p^2}$$

$$<T\left\{\xi(x)\bar{\xi}(0)\right\}> \sim \lambda \qquad <T\left\{\eta(x)\bar{\eta}(0)\right\}> \sim \lambda^2$$

$$<T\left\{\xi(x)\bar{\eta}(0)\right\}> \sim <T\left\{\eta(x)\bar{\xi}(0)\right\}> \sim \lambda^{3/2}$$

$$\xi \sim \lambda^{1/2} \qquad \eta \sim \lambda$$

Hard-collinear Lagrangian

$$
\begin{aligned}
L &= \bar{\psi} i \not{D} \psi = (\bar{\xi} + \bar{\eta}) i \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta) \\
&= \frac{1}{2} \bar{\xi} D_+ \gamma_- \xi + \frac{1}{2} \bar{\eta} D_- \gamma_+ \eta + \bar{\xi} \not{D}_\perp \eta + \bar{\eta} \not{D}_\perp \xi
\end{aligned}
$$

Hard-collinear Lagrangian

$$L = \bar{\psi} i \not{D} \psi = (\bar{\xi} + \bar{\eta}) i \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta)$$
$$= \frac{1}{2} \bar{\xi} D_+ \gamma_- \xi + \frac{1}{2} \bar{\eta} D_- \gamma_+ \eta + \bar{\xi} \not{D}_\perp \eta + \bar{\eta} \not{D}_\perp \xi$$

$$\frac{1}{2} D_- \gamma_+ \eta + \not{D}_\perp \xi = 0 \qquad \eta = -\frac{1}{2} \gamma_- \frac{1}{iD_- + i0} i \not{D}_\perp \xi$$

Wilson line

$$W(x) = P\exp\left(ig\int_{-\infty}^{0} A_{-}(x+tn_{-})dt\right)$$

Wilson line

$$W(x) = P\exp\left(ig\int_{-\infty}^{0} A_-(x+tn_-)dt\right)$$

$$W^{-1}D_-W = \partial_- \qquad \frac{1}{iD_- + i0} = W\frac{1}{i\partial_- + i0}W^{-1}$$

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$$\begin{aligned}\eta(x) &= -\frac{1}{2}\gamma_- W(x)\frac{1}{i\partial_- + i0}\left(W^{-1}i\not{D}_\perp\xi\right)_x \\ &= \frac{i}{2}\gamma_- W(x)\int_{-\infty}^{0}\left(W^{-1}i\not{D}_\perp\xi\right)_{x+tn_-}dt\end{aligned}$$

Wilson line

$$W(x) = P\exp\left(ig\int_{-\infty}^{0} A_-(x+tn_-)dt\right)$$

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$$\begin{aligned}L &= \bar{\xi}(x)D_+\xi(x) \\ &\quad + \frac{i}{2}\left(\bar{\xi}i\not{D}_\perp W\right)_x\gamma_+\int_{-\infty}^{0}\left(W^{-1}i\not{D}_\perp\xi\right)_{x+tn_-}dt\end{aligned}$$

$$D^\mu = \partial^\mu - igA^\mu_{hc} - igA^\mu_s \sim (\lambda, 1, \lambda^{1/2}) + (\lambda, 1, \lambda^{1/2}) + (\lambda, \lambda, \lambda)$$

$$W = W_{hc} + \mathcal{O}(\lambda)$$

$$D^\mu = \partial^\mu - igA^\mu_{hc} - igA^\mu_s \sim (\lambda, 1, \lambda^{1/2}) + (\lambda, 1, \lambda^{1/2}) + (\lambda, \lambda, \lambda)$$

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Multipole expansion: when a hard-collinear field interacts with a soft field, $x \sim (1, \lambda^{-1}, \lambda^{-1/2})$, but the soft field varies at a scale $(\lambda^{-1}, \lambda^{-1}, \lambda^{-1})$

$$\phi_s(x) = \phi_s\left(\frac{1}{2}x_- n_+\right) + x_\perp \cdot \partial_\perp \phi_s\left(\frac{1}{2}x_- n_+\right) + \frac{1}{2}x_+ \partial_- \phi_s\left(\frac{1}{2}x_- n_+\right) +$$

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Leading-order Lagrangian

$$L = \bar{\xi}(x) D^{hc}_+ \xi(x) + \bar{\xi}(x) g A^s_+\left(\frac{1}{2}x_- n_+\right)\xi(x)$$
$$+ \frac{i}{2}\left(\bar{\xi} i \not{D}^{hc}_\perp W_{hc}\right)_x \gamma_+ \int_{-\infty}^{0} \left(W^{-1}_{hc} i \not{D}^{hc}_\perp \xi\right)_{x+tn_-} dt$$