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Geometrical approach to the evaluation of Feynman diagrams

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partly based on work with **R. Delbourgo** and **M. Yu. Kalmykov**

Earlier Papers: Singularities, Reduction, etc.

L.D. Landau, Nucl. Phys. **13** (1959) 181

G. Källén and A. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, No.6 (1958) 1

S. Mandelstam, Phys. Rev. **115** (1959) 1742

R.E. Cutkosky, J. Math. Phys. **1** (1960) 429

J.C. Taylor, Phys. Rev. **117** (1960) 261

Yu.A. Simonov, Sov. Phys. JETP **16** (1963) 1599

R. Karplus, C.M. Sommerfield, E.H. Wichmann, Phys. Rev. **114** (1959) 376

A.C.T. Wu, Mat. Fys. Medd. Dan. Vid. Selsk. **33**, No.3 (1961) 1

L.M. Brown, Nuovo Cim. **22** (1961) 178

F.R. Halpern, Phys. Rev. Lett. **10** (1963) 310

B. Petersson, J. Math. Phys. **6** (1965) 1955

D.B. Melrose, Nuovo Cim. **40A** (1965) 181

G. Källén and J. Toll, J. Math. Phys. **6** (1965) 299 (1965)

B.G. Nickel, J. Math. Phys. **19** (1978) 542

A. Denner, U. Nierste, R. Scharf, Nucl. Phys. **B367** (1991) 637

N. Ortner and P. Wagner, Ann. Inst. Henri Poincaré (Phys. Théor.) **63** (1995) 81

P. Wagner, Indag. Math. **7** (1996) 527

Dimensional regularization

One of the most powerful tools used in loop calculations is *dimensional regularization*: the idea is to use the space-time dimension n as a regulator, $n = 4 - 2\varepsilon$ $(\varepsilon \to 0)$,

$$\int \mathrm{d}^4 k \left\{ \ldots \right\} \quad \Rightarrow \quad \int \mathrm{d}^n k \left\{ \ldots \right\}$$

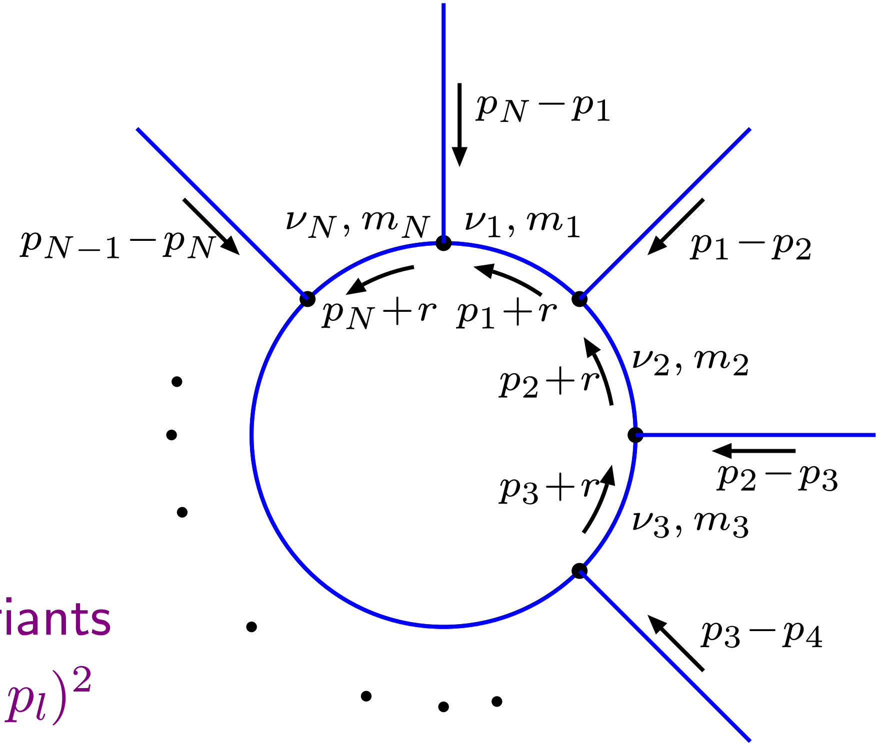
G. 'tHooft and M. Veltman, Nucl. Phys. **B44** (1972) 189;
C.G. Bollini and J.J. Giambiagi, Nuovo Cimento **12B** (1972) 20;
J.F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;
G.M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

Then singularities appear as $1/\varepsilon$ poles. Simple example:

$$\int \frac{\mathrm{d}^n k}{[(p+k)^2 - m^2]^N} = \pi^{n/2} \mathrm{i}^{1-2N} \frac{\Gamma(N - n/2)}{\Gamma(N)} (m^2)^{n/2-N}$$

For $N = 2$: $\Gamma(2 - n/2) = \Gamma(\varepsilon) \sim 1/\varepsilon$

One-loop N-point function $J^{(N)}(n;\nu_1,\ldots,\nu_N)$



Depends on
$\frac{1}{2}N(N-1)$ invariants
$k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) \equiv \int \frac{\mathrm{d}^n k}{[(p_1+k)^2 - m_1^2]^{\nu_1} \cdots [(p_N+k)^2 - m_N^2]^{\nu_N}}$$

Feynman parameters

$$\frac{1}{\mathcal{A}_1 \ldots \mathcal{A}_N} = (N-1)! \int\limits_0^1 \ldots \int\limits_0^1 \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N \; \delta\left(\sum\limits_{i=1}^N \alpha_i - 1\right)}{(\alpha_1 \mathcal{A}_1 + \ldots + \alpha_N \mathcal{A}_N)^N}$$

where $\mathcal{A}_i \leftrightarrow (p_i + k)^2 - m_i^2$. Then the momentum integrations can be easily performed, and we arrive at a parametric integral representation

$$J^{(N)}(n; 1, \ldots, 1) = \mathrm{i}^{1-n} \pi^{n/2} \, \Gamma(N - n/2) \times \int\limits_0^1 \ldots \int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right)}{\left[\sum\sum\limits_{j<l} \alpha_j \alpha_l k_{jl}^2 - \sum \alpha_i m_i^2\right]^{N-n/2}}$$

Analytical and numerical techniques

In principle, one could calculate parametric integrals numerically (by Monte Carlo multidimensional integration, say). The number of integrations is $N-1$, where N is the number of *internal* lines.

Difficulties: usually, for physically relevant cases, there are (integrable) singularities in the integration region. This makes it difficult to obtain reliable results with reasonable precision. (Sometimes, subtraction of a simpler integral with similar singularity may help.)

Generally: it is desirable to take analytically as many integrations as possible.

In some cases, one can get through all integrations and obtain an exact answer, e.g., in terms of various hypergeometric functions (for an arbitrary space-time dimension $n = 4 - 2\varepsilon$). Expanded in ε, they usually yield polylogarithms, elliptic integrals and some other special functions.

Mellin–Barnes technique

Use contour integral representation:

$$\frac{1}{(p^2-m^2)^\nu} = \frac{1}{\Gamma(\nu)}\frac{1}{2\pi \mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s \frac{(-m^2)^s}{(p^2)^{\nu+s}}\,\Gamma(-s)\,\Gamma(\nu+s)$$

- All massive denominators are represented as contour integrals
- The resulting massless integral is evaluated
- Contour integrals are taken by using the *residue theorem*, usually in terms of the (generalized) hypergeometric functions
- For special cases of interest, these hypergeometric functions may be reduced to more familiar functions (not always!)

E.E. Boos and A.I.D., *Theor. Math. Phys.* **89** (1991) 1052

A.I.D., *J. Math. Phys.* **32** (1991) 1052; **33** (1992) 358

Geometrical Approach

To predict types of functions (and values of their arguments) which may appear in higher orders of ε-expansion, a geometrical approach happens to be very useful. It is summarized in the paper

A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299.

Using this approach, the results for *all* terms of the ε-expansion have been obtained for the one-loop two-point function with arbitrary masses, one-loop three-point integrals with massless internal lines and arbitary (off-shell) external momenta and two-loop vacuum diagrams with arbitrary masses.

A.I.D., Phys. Rev. **D61** (2000) 087701;

A.I.D. and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266

These results have been represented in terms of

$$\mathsf{Ls}_j(\theta) = -\int_0^\theta \mathrm{d}\phi \; \ln^{j-1}\left|2\sin\frac{\phi}{2}\right| ,$$

whose angular arguments have a rather transparent geometrical interpretation (angles of certain triangles).

In more complicated cases, generalizations of log-sine integrals (or more complex functions of angular variables) appear.

Analytic continuation $\Rightarrow$ (generalized) polylogarithms, etc.

Feynman parameters: trick #1

Back to the one-loop N-point function:

$$J^{(N)}(n;1,\ldots,1) \;=\; \mathrm{i}^{1-n}\pi^{n/2}\,\Gamma(N-n/2) \times \int\limits_0^1 \ldots \int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum \alpha_i - 1\right)}{\left[\sum\sum\limits_{j<l} \alpha_j\alpha_l k_{jl}^2 - \sum \alpha_i m_i^2\right]^{N-n/2}}$$

By using $\sum \alpha_i = 1$ we can make the quadratic form homogeneous in α_i:

$$\left[\sum\sum\limits_{j<l}\alpha_j\alpha_l k_{jl}^2 - \left(\sum \alpha_i\right)\left(\sum \alpha_i m_i^2\right)\right] \Rightarrow -\left[\sum \alpha_i^2 m_i^2 + 2\sum\sum\limits_{j<l}\alpha_j\alpha_l m_j m_l c_{jl}\right],$$

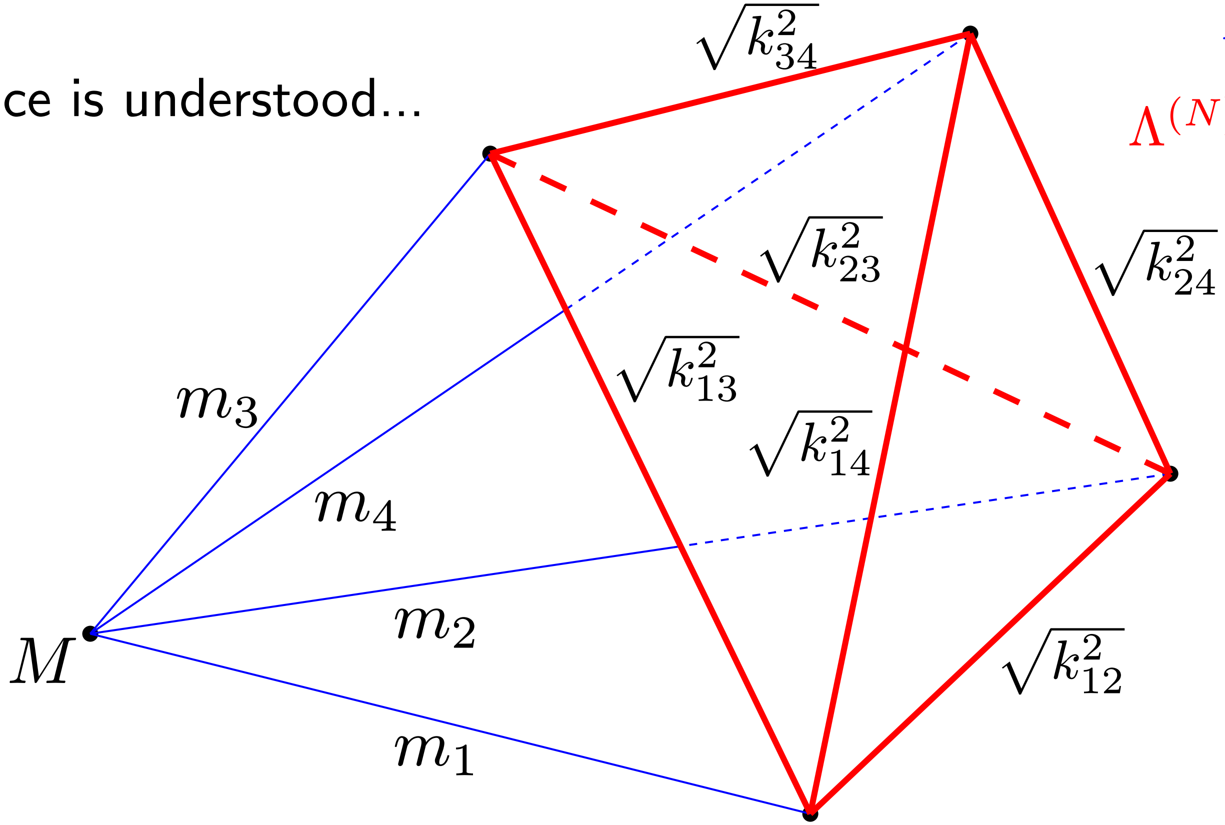
$$c_{jl} \equiv \frac{m_j^2+m_l^2-k_{jl}^2}{2m_jm_l}, \qquad c_{jl} = \cos\tau_{jl} = \begin{cases} 1, & k_{jl}^2 = (m_j-m_l)^2 \quad \text{pseudothreshold} \\ -1, & k_{jl}^2 = (m_j+m_l)^2 \quad \text{threshold} \end{cases}$$

The basic simplex for $N = 4$

$$D^{(N)} = \det \|c_{jl}\|$$

$$\Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\|$$

4-dim. space is understood...



$$V^{(N)} = \frac{(\Pi m_i)}{N!}\sqrt{D^{(N)}}, \qquad \overline{V}_0^{(N-1)} = \frac{1}{(N-1)!}\sqrt{\Lambda^{(N)}}, \qquad m_0 = (\Pi m_i)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

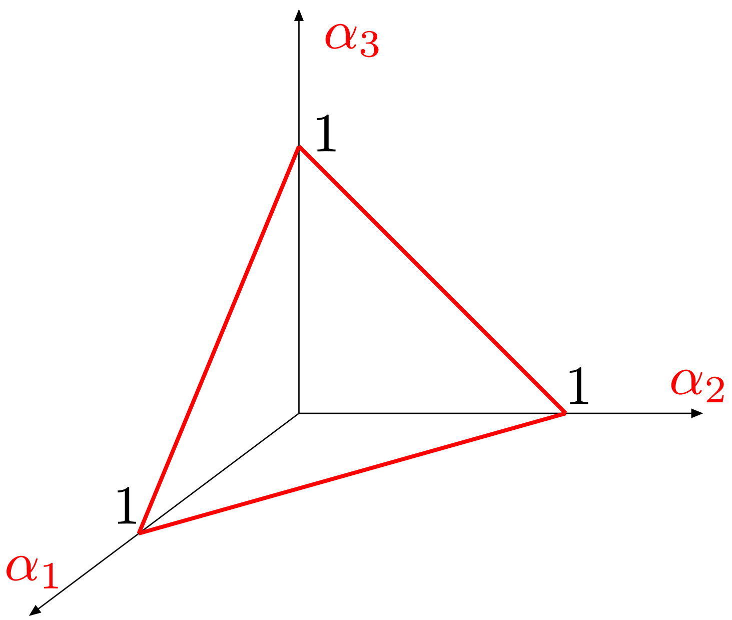
When we wish to proceed from three to more dimensions, however, we run up against a conceptual barrier so difficult that many people cannot surmount it. Just how far certain gifted individuals can visulaize, say, a space of four dimensions is an arguable matter which it would be wise to leave to the psychologists. My own opinion is that nobody can. $< \ldots >$ But these difficulties of "seeing the situation" do not prevent us from setting up a geometry of n dimensions in the classical sense and of reasoning about it.

M.G. Kendall, *A course in the geometry of n dimensions*
(Griffin, London, 1961), p. 2

Feynman parameters: trick #2

Limits of integration:

$$\int_0^1 \ldots \int_0^1 \left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\} = \int_0^\infty \ldots \int_0^\infty \left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\}$$



Feynman parameters: trick #3

Linear substitution of α variables:

$$\alpha_i = \frac{\alpha'_i}{m_i}, \qquad \delta\left(\sum \alpha_i - 1\right) \Rightarrow \delta\left(\sum \frac{\alpha'_i}{m_i} - 1\right)$$

To restore the argument of the δ function in its original form, substitute

$$\alpha'_i = \mathcal{F}(\alpha''_1, \ldots, \alpha''_N)\alpha''_i, \qquad \text{with} \quad \mathcal{F}(\alpha_1, \ldots, \alpha_N) = \frac{\sum \alpha_i}{\sum \frac{\alpha_i}{m_i}}$$

The Jacobian of such substitution is $\mathcal{F}^N$. The integral becomes ($\alpha'' \to \alpha$):

$$J^{(N)}(n; 1, \ldots, 1) = \frac{\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N - \frac{n}{2}\right)}{\prod m_i} \int\limits_0^\infty \ldots \int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right)}{\left(\sum \frac{\alpha_i}{m_i}\right)^{n-N} \left(\alpha^T \|c\| \alpha\right)^{N-n/2}}$$

Feynman parameters: trick #4

Quadratic substitution of α variables:

$$\alpha_i = \mathcal{G}(\alpha'_1, \ldots, \alpha'_N)\alpha'_i, \qquad \text{with} \quad \mathcal{G}(\alpha_1, \ldots, \alpha_N) = \frac{\left(\alpha^T \|c\| \alpha\right)}{\sum \alpha_i}$$

Matrix notation:

$$\left(\alpha^T \|c\| \alpha\right) \equiv \sum_{j=1}^{N} \sum_{l=1}^{N} c_{jl}\alpha_j\alpha_l = \sum \alpha_i^2 + 2 \sum_{j<l}\sum \alpha_j\alpha_l c_{jl}$$

The Jacobian of such substitution is $2\mathcal{G}^N$. The integral becomes ($\alpha' \to \alpha$):

$$J^{(N)}(n; 1, \ldots, 1) = \frac{2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N - \frac{n}{2}\right)}{\prod m_i} \int_0^\infty \ldots \int_0^\infty \frac{\left(\prod \mathsf{d}\alpha_i\right) \cdot \delta\left(\alpha^T \|c\| \alpha - 1\right)}{\left(\sum \frac{\alpha_i}{m_i}\right)^{n-N}}$$

Feynman parameters: trick #5, etc.

Whenever a quadratic form occurs, an obvious idea is to *diagonalize* it: Eigenvectors, Eigenvalues, and all that...

Modified matrix: $C_{jl} = \left(\sqrt{F_j^{(N)}}c_{jl}\sqrt{F_l^{(N)}}\right)$, with $F_i^{(N)} = \frac{\partial}{\partial m_i^2}\left(m_i^2 D^{(N)}\right)$.

obeying $\sum\limits_{l=1}^{N} c_{jl}F_l^{(N)}\frac{1}{m_l} = D^{(N)}\frac{1}{m_j} \Rightarrow \sum\limits_{l=1}^{N} C_{jl}\frac{\sqrt{F_l^{(N)}}}{m_l} = D^{(N)}\frac{\sqrt{F_j^{(N)}}}{m_j} \Rightarrow$

Eigenvector: $f_i = \frac{\sqrt{F_i}}{m_i}$, Eigenvalue: $D^{(N)} = \det\|c_{jl}\|$ (Gram determinant),
Repeating tricks #3 and #4 we get

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N-\frac{n}{2}\right)\left(\Pi f_i\right)\int\limits_0^\infty\ldots\int\limits_0^\infty\frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum\alpha_i f_i\right)^{n-N}}$$

Diagonalization: see next page...

Feynman parameters: trick #5, etc. (continued)

Diagonalization: "rotate" variables $\alpha_i \to \beta_i$ so that $\alpha^T \|C\| \alpha = \sum_{i=1}^N \lambda_i \beta_i^2$
One of the β's (say β_N) is directed along f_i, so that $\lambda_N = D^{(N)}$.
Denominator $(\sum \alpha_i f_i)$ is proportional to β_N.

Assume (for a moment) that all $\lambda_i > 0$ and rescale $\beta_i = \frac{\gamma_i}{\sqrt{\lambda_i}} \Rightarrow$

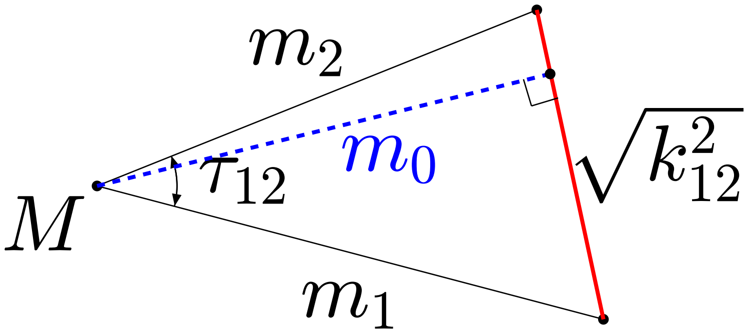
$$J^{(N)}(n; 1, \ldots, 1) = 2\mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int_{\Omega^{(N)}} \!\!\ldots \int \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{n-N}}\, \delta\left(\sum \gamma_i^2 - 1\right)$$

Remarkably: the same N-dim. solid angle $\Omega^{(N)}$ as in the basic simplex!

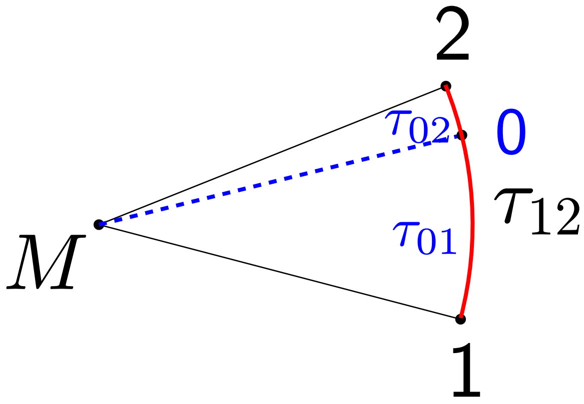
Special case: $N = n$ ($N = 2$ in 2d, $N = 3$ in 3d, $N = 4$ in 4d, etc.)

If some of λ_i are negative – hyperbolic surface (instead of spherical)
$\leftrightarrow$ analytical continuation.

General two-point function, geometrical approach



the basic triangle



the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

$$m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

General two-point function, geometrical approach

$$J^{(2)}(4-2\varepsilon;1,1) = \mathrm{i}\pi^{2-\varepsilon}\Gamma(\varepsilon)\,\frac{m_0^{1-2\varepsilon}}{\sqrt{k_{12}^2}}\left\{\Omega_1^{(2;4-2\varepsilon)} + \Omega_2^{(2;4-2\varepsilon)}\right\}$$

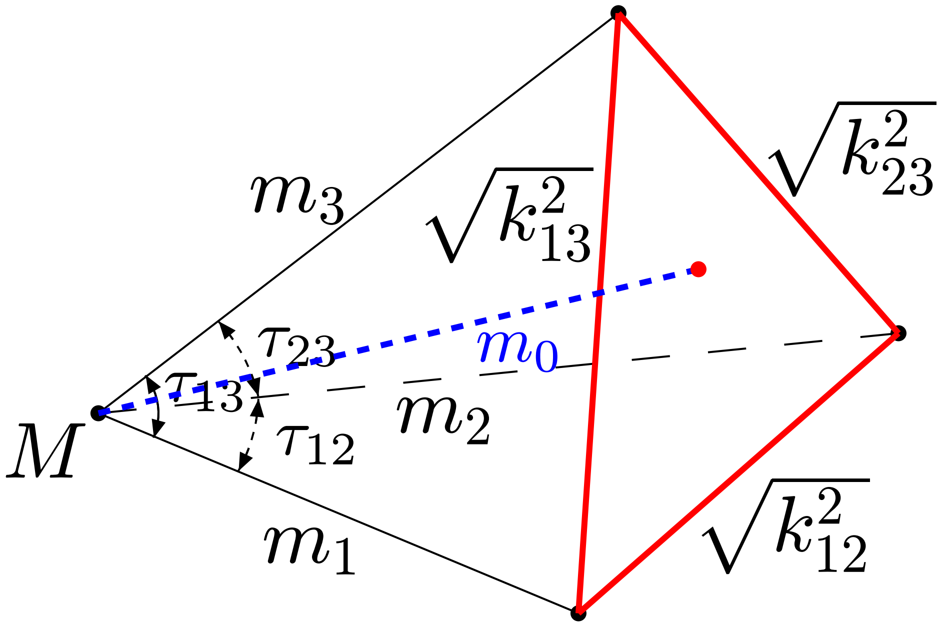
with

$$\Omega_i^{(2;4-2\varepsilon)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{2-2\varepsilon}\theta} = \tan\tau_{0i}\;{}_2F_1\left(\begin{array}{c} 1/2, \varepsilon \\ 3/2 \end{array}\middle| -\tan^2\tau_{0i}\right)$$

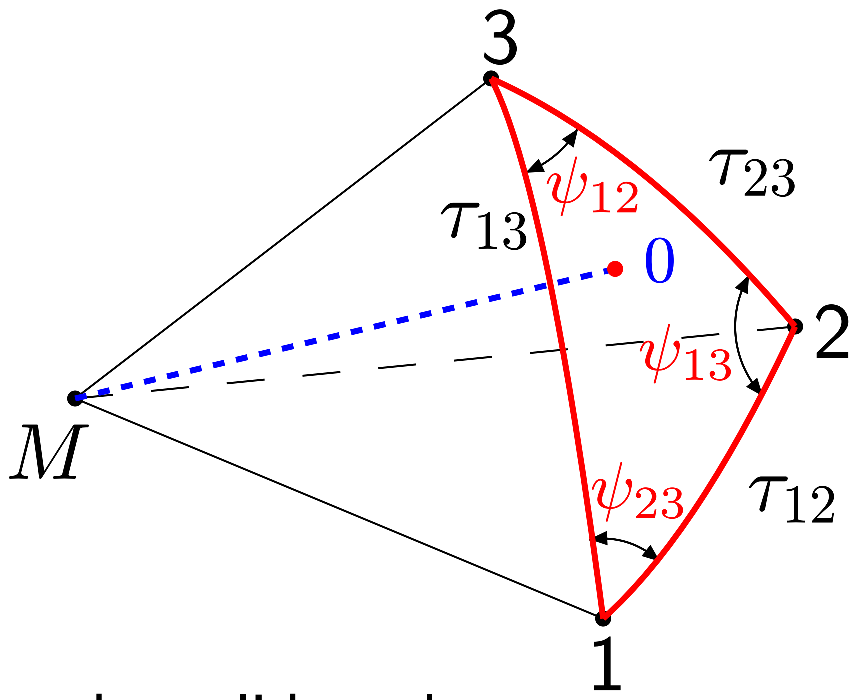
$$c_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1-c_{12}^2 = \sin^2\tau_{12}, \quad m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{k_{12}^2}},$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01}+\tau_{02} = \tau_{12}.$$

General three-point function: geometrical approach



the basic tetrahedron



the solid angle

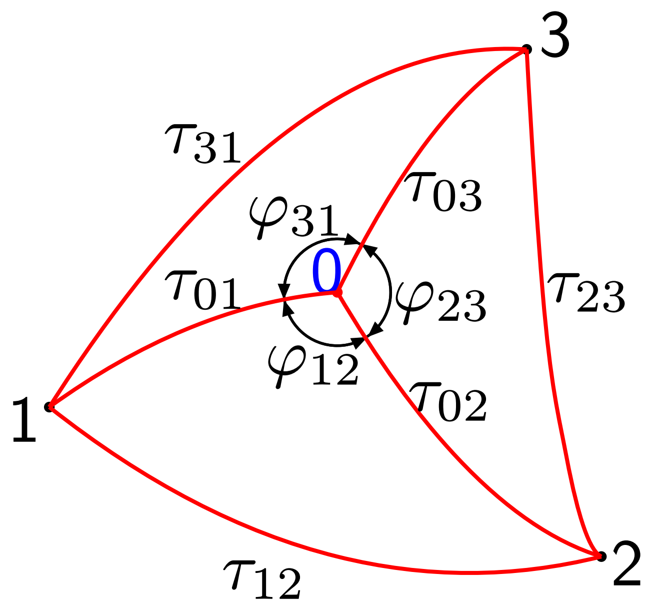
Special case $n = 3 \Rightarrow$ the area of spherical triangle (“spherical excess”):

$$\Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \ .$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

General three-point function: geometrical approach

Relation to the angles accociated with a spherical (or hyperbolic) triangle:



$$\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi$$

$$\cos\tau_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \text{ etc.}$$

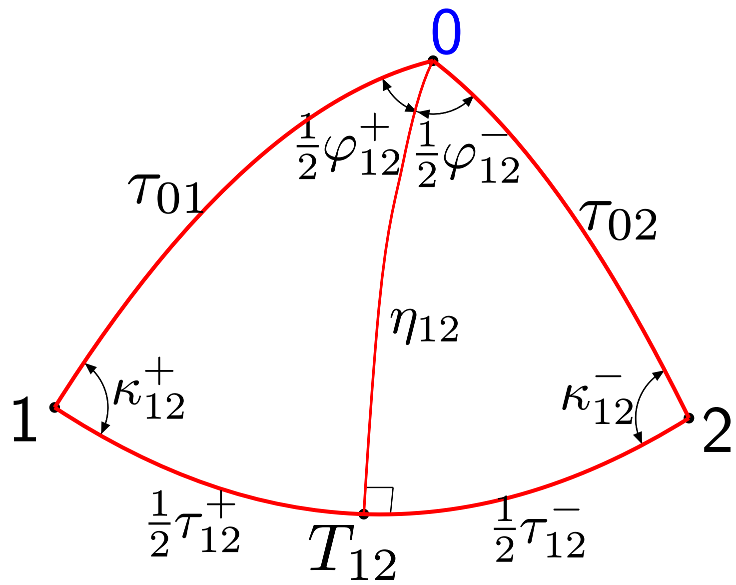
$$\cos\tau_{0i} = \frac{m_0}{m_i} \qquad (i=1,2,3)$$

$$m_0 = m_1m_2m_3\sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}}$$

$$D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}, \quad \Lambda^{(3)} = \tfrac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right]$$

General three-point function: geometrical approach

One of the three triangles ($\frac{1}{2}(\varphi_{12}^{+} + \varphi_{12}^{-}) = \varphi_{12}$):



Three-point function in $n = 4 - 2\varepsilon$ dimensions

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}}{\sqrt{\Lambda^{(3)}}}\,\Gamma\left(3-\frac{n}{2}\right)\; m_0^{n-4}\,\Omega^{(3;n)} ,$$

$$\Omega^{(3;n)} = \int\!\!\int_{\Omega^{(3)}} \frac{\sin^{n-2}\theta\;\mathrm{d}\theta\;\mathrm{d}\phi}{\cos^{n-3}\theta} \quad = \quad \omega\left(\tfrac{1}{2}\varphi_{12}^{+},\eta_{12}\right) + \omega\left(\tfrac{1}{2}\varphi_{12}^{-},\eta_{12}\right) + \omega\left(\tfrac{1}{2}\varphi_{23}^{+},\eta_{23}\right) + \omega\left(\tfrac{1}{2}\varphi_{23}^{-},\eta_{23}\right) + \omega\left(\tfrac{1}{2}\varphi_{31}^{+},\eta_{31}\right) + \omega\left(\tfrac{1}{2}\varphi_{31}^{-},\eta_{31}\right) ,$$

with

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right) = \frac{1}{2\varepsilon}\int_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}\right] = \frac{1}{2}\sum_{j=0}^{\infty}\frac{(-\varepsilon)^j}{(j+1)!}\int_0^{\varphi/2}\mathrm{d}\phi\;\ln^{j+1}\left(1+\frac{\tan^2\eta_{12}}{\cos^2\phi}\right)$$

A.I.D., hep-th/9908032, Nucl.Instr.Meth. A559 (2006) 293

The result for arbitrary ε can be presented in terms of Appell's hypergeometric function F_1,

$$\omega\left(\tfrac{1}{2}\varphi, \eta\right) = \frac{1}{2\varepsilon}\left[\frac{\varphi}{2} - \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\cos^{2\varepsilon}\tau_0\; F_1\left(1, 1, \varepsilon; \tfrac{3}{2}\middle|\ \sin^2\frac{\varphi}{2}, \sin^2\frac{\tau}{2}\right)\right],$$

with $\cos\tau_0 = \cos\eta\ \cos\frac{\tau}{2}$,

$$F_1(a, b, b', c|x, y) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a)_{j+l}\ (b)_j\ (b')_l}{(c)_{j+l}}\ \frac{x^j\ y^l}{j!\ l!}$$

Similar functions occurred in

O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Some special cases: L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081

Special value of n: $n=4$ $(\varepsilon \to 0)$:

$$\int\limits_0^{\varphi/2} \mathrm{d}\phi \; \ln\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right) = \tfrac{1}{2}\tau \; \ln\left(\frac{1+\sin\eta}{1-\sin\eta}\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi+\tau\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi-\tau\right) - \mathrm{Cl}_2\left(\varphi\right)$$

Compare with: P. Wagner, Indag. Math. 7 (1996) 527

After analytical continuation, corresponds to

G. 'tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365

Analytic Continuation: Arbitrary Dimension

Consider $\displaystyle \int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}$.

Substitute $z \Rightarrow e^{2\mathrm{i}\phi}$, so that $\cos^2\phi \Rightarrow \dfrac{(1+z)^2}{4z}$,

$$1 + \frac{\tan^2\eta}{\cos^2\phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad \text{with} \quad \rho \equiv \frac{1-\sin\eta}{1+\sin\eta}$$

In this way, $\displaystyle \int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} \Rightarrow \frac{\mathrm{i}}{2}\int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon}$,

with $z_0 \leftrightarrow e^{2\mathrm{i}\varphi_0}$.

Analytic Continuation: Expansion in ε

Expanding in ε, we get

$$Q_j \equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^j \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \; .$$

The first term, $\mathcal{O}(1)$:

$$\begin{aligned} Q_1 &\equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \\ &= \mathrm{Li}_2\left(-z_0\rho\right) + \mathrm{Li}_2\left(-z_0/\rho\right) - 2\mathrm{Li}_2\left(-z_0\right) + \tfrac{1}{2}\ln^2 \rho \end{aligned}$$

Analytic Continuation: Expansion in ε (continued)

$$
\begin{aligned}
Q_2 \quad &\equiv \quad \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^2 \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \\
&= \quad \ln\rho \Bigg[2\mathrm{Li}_2\left(\frac{1-\rho}{1+z_0\rho}\right) + 2\mathrm{Li}_2\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) - 2\mathrm{Li}_2\left(\frac{\rho-1}{z_0+\rho}\right) - 2\mathrm{Li}_2\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\
&\qquad -\mathrm{Li}_2\left(\frac{1-\rho^2}{1+z_0\rho}\right) - \mathrm{Li}_2\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) + \mathrm{Li}_2\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{Li}_2\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right) \Bigg] \\
&\qquad +4\mathrm{S}_{1,2}\left(\frac{1-\rho}{1+z_0\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) + 4\mathrm{S}_{1,2}\left(\frac{\rho-1}{z_0+\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\
&\qquad -\mathrm{S}_{1,2}\left(\frac{1-\rho^2}{1+z_0\rho}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) - \mathrm{S}_{1,2}\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right)
\end{aligned}
$$

Compare with: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

ε-expansion: higher terms

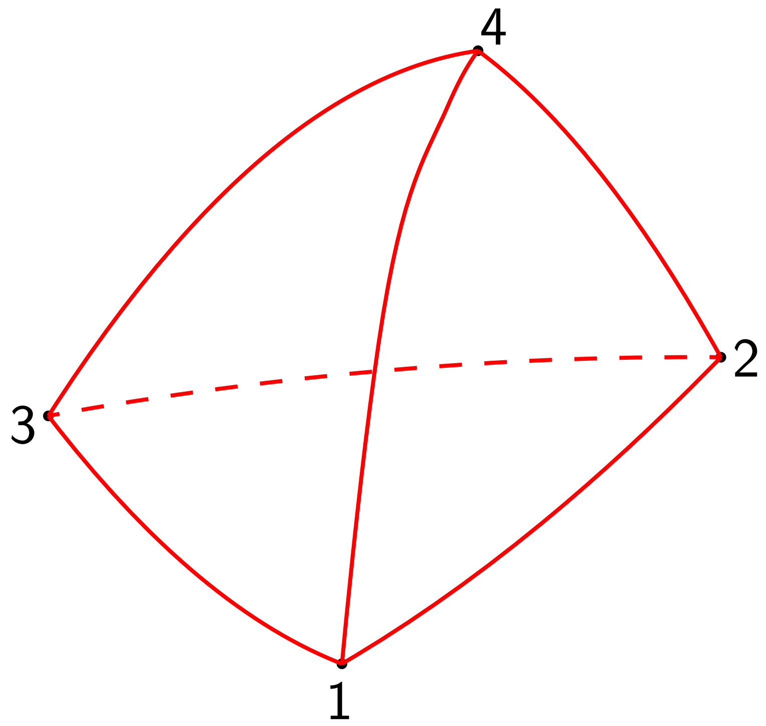
ε^2-term

$$Q_3 = \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^3 \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] ,$$

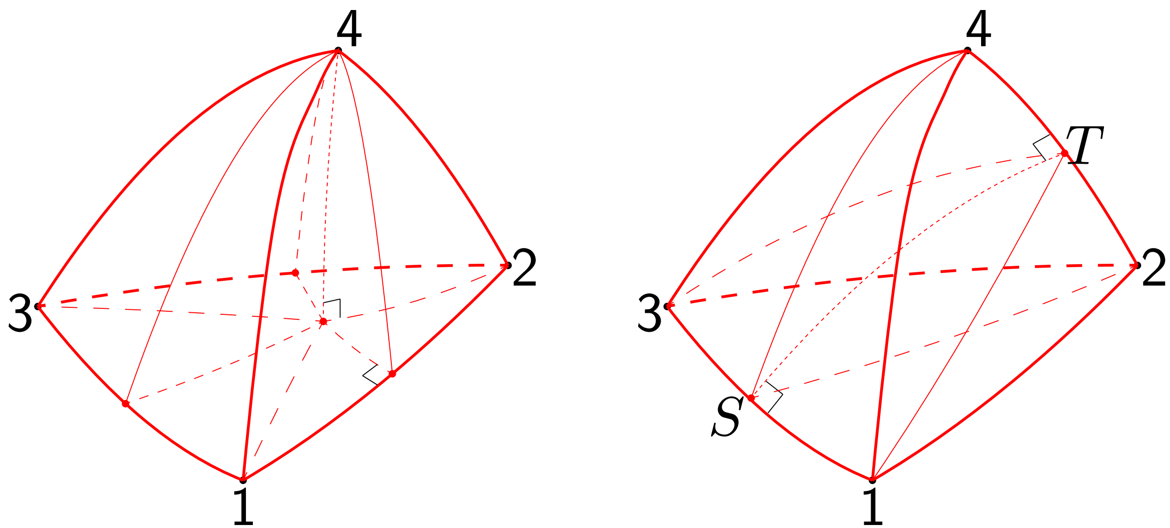
etc.

Some special cases considered in J.G. Körner, Z. Merebashvili, M. Rogal, Phys. Rev. D71 (2005) 054028

Geometrical approach: 4-point function



The spherical tetrahedon



Different ways of splitting a non-Euclidean tetrahedron

Summary

- A geometrical way to calculate dimensionally-regulated Feynman diagrams is reviewed.

- In the one-loop N-point case, the results can be related to certain volume integrals in non-Euclidean geometry. For example, the result for the four-point function can be associated with the content of a spherical or hyperbolic tetrahedron in three-dimensional spherical or hyperbolic space (Lobachevsky, Schläfli, ...)

- Analytical continuation of the results to other regions of kinematical variables (momenta and masses of the particles) is discussed. In a number of cases, analytic results can be presented in terms of the (generalized) polylogarithms and associated functions.